Method 5: Planar Separators (Lipton-Tarjan '77)

1D

2D

Theorem (Thm): Given a triangulated planar graph G with n faces, can find subset R of $O(n)$ edges that divide G into 2 regions each with $\leq \frac{2}{3}n$ faces.

Generalized Thm: For any b, can find subset R of $O\left(\frac{n}{\sqrt{b}}\right)$ edges that divide G into $O\left(\frac{n}{b}\right)$ regions each with $\leq b$ faces each.

Proof (Pf): Apply Thm recursively.

Point Location DS: Verttriangulation

also called "V-division" (each region has $O\left(\sqrt{n}\right)$ edges)
Point Location DS:
build DS for separator $R$ by some known method
recursively build DS for each region

Query alg'n:
first find region of $R$ containing $q$
then recursively query in that region

$$S(n) \leq \sum S(n_i) + S_0\left(\frac{n}{b}\right) + O(n)$$

with $\sum n_i \leq n$, $\max n_i \leq b$

$$Q(n) \leq Q(n_i) + Q_0\left(\frac{m}{b}\right) + O(1)$$

Option 1: start with $S_0(n) = O(n^2)$
$Q_0(n) = O(\log n)$

Set $b = n^{0.9}$:

$$S(n) \leq \sum S(n_i) + O\left(n^{(0.55)^2}\right) + O(n)$$

$$= O(n^{1.1})$$

$$Q(n) \leq Q(n^{0.9}) + O(\log n)$$

$$= O(\log n + 0.9 \log n + 0.9^2 \log n + \ldots)$$

$$= O(\log n)$$

Bootstrap:
$S_0(n) = O(n^{1.1})$, $Q_0(n) = O(\log n)$

Set $b = n^{0.9}$:

$$S(n) \leq \sum S(n_i) + O\left(n^{(0.55)^2}\right) + O(n)$$

$$= O(n \log \log n)$$

$$Q(n) = O(\log n)$$
Bootstrap again!
Set \( b = (\log \log n)^2 \)
don't recurse

\[
Q(n) = O(\log n)
\]
\[
S_b(n) = O(n \log \log n), \quad S_b(n) = O(\log n)
\]
\[
S(n) \leq \sum_i n_i + O\left(\frac{n}{\sqrt{b}} \log \log n\right) + O(n)
\]
\[
\Rightarrow S(n) = O(n)
\]
\[
Q(n) \leq O((\log \log n)^2) + O(\log n)
\]
\[
\Rightarrow Q(n) = O(\log n)
\]

Preproc time: \( O(n) \) after triangulation

Chazelle's alg'nm

Rmk: can be viewed as a general transformation to reduce space while preserving query time

Option 2: Set \( b = \) large const
Recurse in \( R \) instead!

\[
S(n) \leq S\left(O\left(\frac{n}{\sqrt{b}}\right)\right) + O(n)
\]
\[
\Rightarrow S(n) \leq S(\frac{n}{2}) + O(n)
\]
\[
\Rightarrow O(n)
\]
\[
Q(n) \leq Q\left(O\left(\frac{n}{\sqrt{b}}\right)\right) + O(1)
\]
\[
\Rightarrow Q(n) \leq Q(\frac{n}{2}) + O(1)
\]
\[
\Rightarrow O(\log n)
\]
**Method 6:** Kirkpatrick's Hierarchy ('83)

Simplify Option 2, without separators!

**Lemma** In any planar graph with $n$ vertices, there exists an independent set $I$ of size $\geq \frac{n}{24}$.

Furthermore, each vertex in $I$ has degree $\leq 11$.

**Pf:** by greedy algorithm.

$$\text{total degree} = 2 \times \text{(number of edges)} \leq 6n$$

$$\Rightarrow \exists \frac{n}{2} \text{ vertices have degree } \geq 12$$

$$\Rightarrow \exists \frac{n}{2} \text{ vertices have degree } \leq 11.$$

**Point Location DS:**

remove $I$ now get a subdivision $R$ with $\leq \frac{23n}{24}$ vertices

each region has size $\leq 11$.

repeat retiangulate

$$S(n) \leq S\left(\frac{23}{24}n\right) + O(n) \Rightarrow O(n)$$

$\approx n < 2^{\frac{23}{24}n} + O(1) \Rightarrow O(\log n)$
\((\binom{n}{k}) = \frac{n!}{k!(n-k)!} \)

\[ Q(n) \leq Q\left(\frac{23}{24} n\right) + O(1) \Rightarrow O(\log n) \]

(O(n) preproc time after triang.)

Rmk.: Dobkin-Kirkpatrick hierarchy extends to 3D convex polyhedra

Method 7: Random Sampling (Clarkson-Shor '89)

Cutting Lemma: Given \( n \) disjoint line segments \( S \), such that \( \sum_{s \in S} |s| = 4r \), can divide \( \mathbb{R}^2 \) into \( O(r) \) disjoint cells, such that each intersects \( O\left(\frac{n}{r}\right) \) segs.

\[ \text{triangles or trapezoids} \]