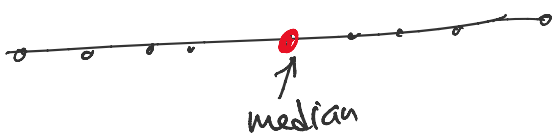
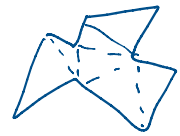
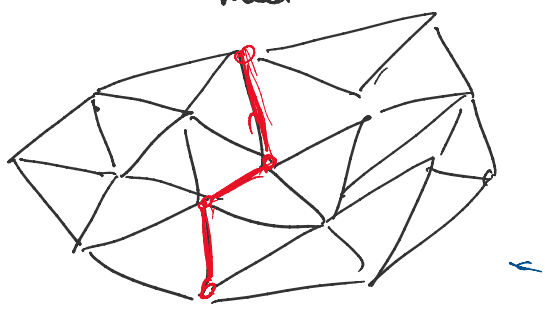
Method 5: Planar Separators (Lipton-Tarjan '77)

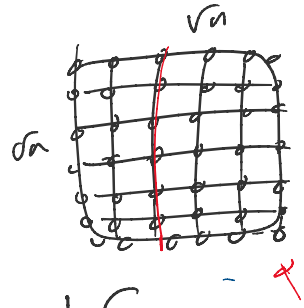
1D





2D



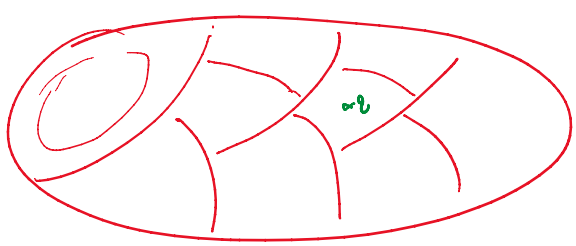


Thm Given a triangulated planar graph G with n faces, can find subset R of $\boxed{O(\sqrt{n})}$ edges that divide G into 2 regions each with $\leq \frac{2}{3}n$ faces.

Generalized Thm For any b, can find subset R of $O\left(\frac{n}{\sqrt{b}}\right)$ edges that divide G into $O\left(\frac{n}{b}\right)$ regions each with $\leq b$ faces each.

also called "r-division" (each region has $O(\sqrt{b})$ edges)

Pf: Apply Thm recursively... $\square$



Point Location DS:

retriangulation

known

Point Location DS:

build DS for separator R by some known method

recursively build DS for each region

$S_0(n)$ space
$Q_0(n)$ query time

Query alg'm:

first find region of R containing q

then recursively query in that region

$$\begin{cases} S(n) \le \sum_i S(n_i) + S_0\left(O\left(\frac{n}{\sqrt{b}}\right)\right) + O(n) \\ \quad \text{with } \sum_i n_i \le n, \quad \max_i n_i \le b \\ Q(n) \le Q(n_i) + Q_0\left(O\left(\frac{n}{\sqrt{b}}\right)\right) + O(1) \end{cases}$$

Option 1: start with $S_0(n) = O(n^2)$, $Q_0(n) = O(\log n)$ (method 0)

Set $b = n^{0.9}$: $S(n) \le \sum_i S(n_i) + \underbrace{\cancel{O\left(\left(n^{0.55}\right)^2\right)}}_{O(n^{1.1})} + O(n)$

$$\Rightarrow S(n) = O(n^{1.1})$$

$$Q(n) \le Q(n^{0.9}) + O(\log n)$$

$$\Rightarrow Q(n) = O(\log n + 0.9 \log n + 0.9^2 \log n + \dots)$$

$$= O(\log n)$$

Bootstrap! $S_0(n) = O(n^{1.1})$, $Q_0(n) = O(\log n)$

Set $b = n^{0.9}$: $S(n) \le \sum_i S(n_i) + \underbrace{O\left(\left(n^{0.55}\right)^{1.1}\right) + O(n)}_{O(n)}$

$$\Rightarrow S(n) = O(n \log\log n)$$

$$Q(n) = O(\log n)$$

$$Q(n) = O(\log n)$$

Bootstrap again!

Set $b = (\log\log n)^2$

don't recurse

$$S_0(n) = O(n \log\log n), \quad Q_0(n) = O(\log n)$$

$$S(n) \le \sum_i n_i + \underbrace{\cancel{O\left(\frac{n}{\sqrt{b}} \log\log n\right)}}_{O(n)} + O(n)$$

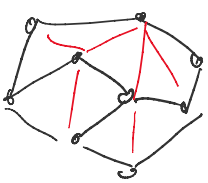
$$\Rightarrow S(n) = \boxed{O(n)}$$

$$Q(n) \le O((\log\log n)^2) + \underline{O(\log n)}$$

$$\Rightarrow Q(n) = \boxed{O(\log n)}$$

preproc time: $\boxed{O(n)}$ after triangulation

Chazelle's alg'm



Rmk: can be viewed as a general transformation to reduce space while preserving query time

Option 2: Set $b =$ large const

recurse in R instead!

$$S(n) \le S\left(O\left(\frac{n}{\sqrt{b}}\right)\right) + O(n)$$

$$\Rightarrow S(n) \le S\left(\frac{n}{2}\right) + O(n)$$

$$\Rightarrow \boxed{O(n)}$$

$$Q(n) \le Q\left(O\left(\frac{n}{\sqrt{b}}\right)\right) + O(1)$$

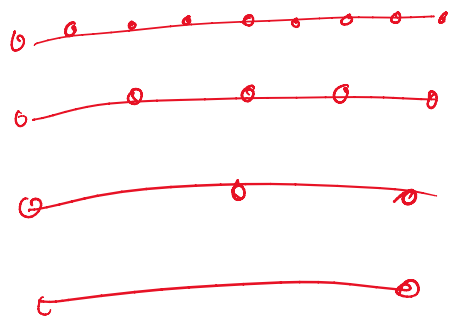
$$\Rightarrow Q(n) \le Q\left(\frac{n}{2}\right) + O(1)$$

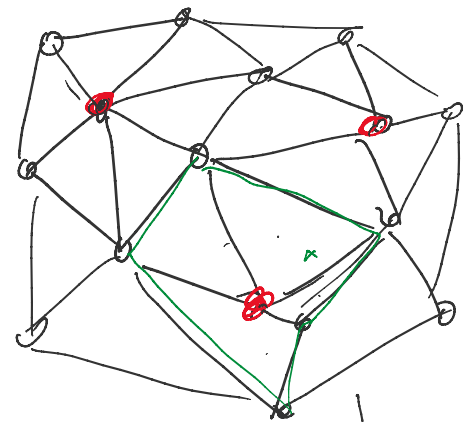
$$\Rightarrow \boxed{O(\log n)}$$

Method 6: Kirkpatrick's Hierarchy ('83)

Simplify option 2, without separators!

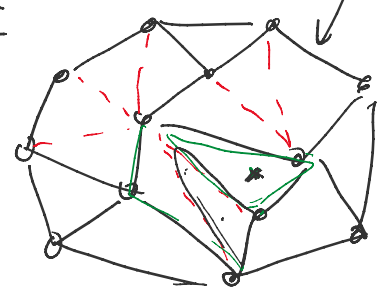
1D:





Lemma In any planar graph with n vertices, $\exists$ indep set I of size $\geq \frac{n}{24}$.

Furthermore, each vertex in I has deg ≤ 11.



Pf: by greedy alg'm.

total deg $= 2(\#\text{edges}) < 6n$

$\Rightarrow$ $\leq \frac{n}{2}$ vertices have deg ≥ 12

$\Rightarrow$ $\geq \frac{n}{2}$ vertices have deg ≤ 11. $\square$

Point Location DS:

remove I

now get a subdivision R with $\leq \frac{23n}{24}$ vertices

each region has size ≤ 11.

repeat ← retriangulate

$$S(n) \leq S\left(\frac{23}{24}n\right) + O(n) \Rightarrow \boxed{O(n)}$$

$$Q(n) \leq Q\left(\frac{23}{24}n\right) + O(1) \Rightarrow \boxed{O(\log n)}$$

$S(n) = S(\frac{23}{24}n)$

$$Q(n) \le Q\left(\frac{23}{24}n\right) + O(1) \Rightarrow \boxed{O(\log n)}$$

($O(n)$ preproc time after triang.)

Rmk. Dobkin-Kirkpatrick hierarchy extends to 3D convex polyhedra

Method 7: Random Sampling (Clarkson-Shor '89)

Cutting Lemma Given n disjoint line segments S, & r. can divide $\mathbb{R}^2$ into $O(r)$ disjoint cells s.t. each intersects $O\left(\frac{n}{r}\right)$ segs

triangles or trapezoids