$(\min,+)$-Convol has $O(n^{2-\delta})$ alg'm for some $\delta>0$

$\iff$ 0/1 Knapsack has $O(t^{2-\delta'})$ alg'm for some $\delta'>0$.

$\iff$ Unbdd Knapsack has $O(t^{2-\delta''})$ alg'm for some $\delta''>0$.

⋮

Pf: $(\Rightarrow)$ reduce Unbdd Knapsack $\rightarrow$ $(\max,+)$-Convol

by repeated squaring

$$f^{(\ell)}[j] = \max_{j'} \left(\underbrace{f^{(\ell/2)}[j'] + f^{(\ell/2)}[j-j']}_{} \right)$$

max profit for capacity j with $\le \ell$ items ($j=0,\dots,t$)

$(\max,+)$-Convol !! on input of size t

reduce 0/1 Knapsack to $(\max,+)$-Convol by modifying Bringman's subset sum alg.

$(\Leftarrow)$ Will reduce in other dir…

Reduction 1: $(\max,+)$-Convol $\rightarrow$ $(\max,+)$-Convol. Decis

Given $a_0,\dots,a_{n-1}$, $b_0,\dots,b_{n-1}$, $c_0,\dots,c_{2n-2}$

for each k, decide if $\max_i (a_i + b_{k-i}) > c_k$.

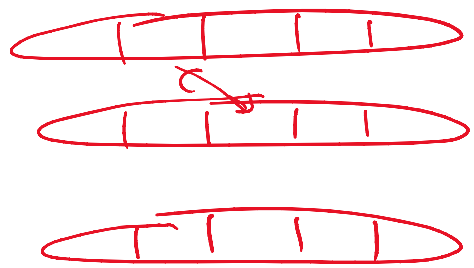
i.e. $\exists i,\ a_i + b_{k-i} > c_k$

by binary search

Reduction 2: $(\max,+)$-Convol Decis $\rightarrow$ Detect-One $(\max,+)$-Convol Decis

test if $\exists k,\ \exists i,\ a_i + b_{k-i} > c_k$

by same idea as APNT $\rightarrow$ 1NT $\rightarrow$ NT



test if

$\forall k, \forall i,\ a_i + b_{k-i} \le c_k$

i.e. $\forall i, \forall j,\ a_i + b_j \le c_{i+j}$

$(f(i+j) \ge f(i) + f(j))$

Problem: Super-additivity

Given $f_0, \dots, f_{n-1} \ge 0$,

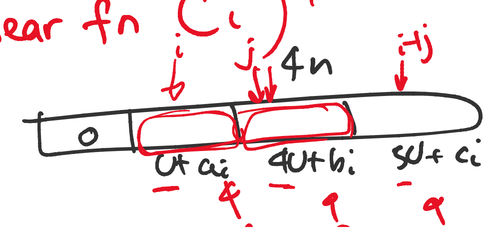
decide if $\forall i,j,\ f_i + f_j \le f_{i+j}$

Reduction 3: Detect-One $(\max,+)$-Convol. Decis $\rightarrow$ Super-additivity

need to combine 3 sequences into one

w.l.o.g. assume a_i, b_i, c_i increas. & are in $[0, U)$

(if not, add linear fn Ci)



for $i = 0, \dots, n-1$,

let $f_i = 0$

$U + a_i \quad 40U + b_i \quad 50U + c_i$

$f_i \quad f_j \quad f_{i+j}$

let $f_i = 0$

$f_{n+i} = U + a_i$

$f_{2n+i} = 40U + b_i$

$f_{3n+i} = 50U + c_i$

$f_i \quad f_i \quad f_{i+j}$

Reduction 4: Superadditivity $\longrightarrow$ Unbounded Knapsack

Assume unbdd knapsack could be solved in $O(t^{2-\delta})$ time.

To solve superadditivity, given $f_0, \ldots, f_{n-1} \in [0, \frac{U}{4})$: increas.

for each $i = 0, \ldots, n-1$

Create items $(\underset{\text{weight}}{i}, \underset{\text{profit (type 1)}}{f_i})$, $(\underset{\text{weight}}{t-i}, \underset{\text{profit (type 2)}}{U - f_i})$

with $\underset{\text{capacity}}{t} = 2n$

Solve unbdd knapsack in $O(n^{2-\delta})$ time

Claim f is superadditive

$\iff$ max profit $\le U$.

Pf: $(\Leftarrow)$ Suppose f is not superadditive.

Then $\exists i, j,\ f_i + f_j > f_{i+j}$.

Choose 3 items:

(i, f_i), (j, f_j), $(t-i-j, U - f_{i+j})$

profit $f_i + f_j + U - f_{i+j}$

$> U$.

$(\Rightarrow)$ Suppose f is superadditive.

Consider any feasible sol'n I.

... (i, f_i) (j, f_j) with $i+j \le n$

Consider any feasible soln I.

If I uses items (i, f_i), (j, f_j), with $i+j \le$ ...
Can replace with $(i+j, f_{i+j})$
& profit can only increase.

(note: duplicates are ok for unbdd knapsack)

Case 1. I can use ~~only~~ one item of type 2, of the form $(t-k, U-f_k)$. $t-k \ge n$

$\Rightarrow$ type 1 items has total wt $\le n$.

So, $I = \{ (i, f_i), (t-k, U-f_k) \}$.

with $i + t - k \le t \Rightarrow i \le k$

profit $f_i + U - f_k \le U$. $\square$

Case 2. I uses no item of type 2

$\Rightarrow$ I uses ≤ 4 items of type 1 $\Rightarrow$ profit $\le U$.

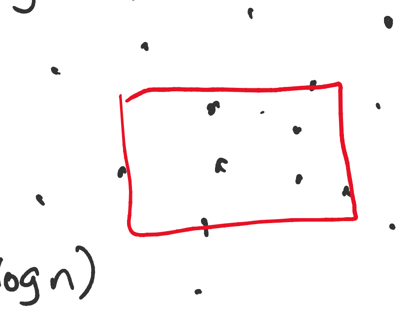
Reduction 5: Unbdd Knapsack $\to$ 0/1 Knapsack

idea - any # can be expressed as sum of distinct powers of 2.

To solve unbdd knapsack:
for each item (w_i, p_i),
Create new item $(2^\ell w_i, 2^\ell p_i)$
for $\ell = 0, \dots, \log U$.
Solve 0/1 Knapsack on these $O(n \log U)$ items with same t. $\square$

A geometric appl'n:

Problem Given n pts in 2D and k, find min area axis-aligned rectangle containing k pts.



C., Har-Peled '20: $O(n^2 \log n)$

Thm If min-k-enclos rect could be solved in $O(n^{2-\delta})$ time, then $(\min,+)$-Convol ...... $O(n^{2-\delta'})$ time.

Pf:

Reduction: Detect-One $(\min,+)$-Convol Decis $\longrightarrow$ min-k-enclos rect.

Given $a_0, .., a_{n-1}$, $b_0, .., b_{n-1}$, $c_0, .., c_{2n-2}$:

To decide whether $\exists i, j$, $a_i + b_j \le c_{i+j}$:

w.l.o.g. assume a_i, b_i, c_i increas. & in $[0, U)$.

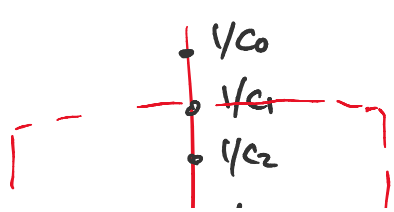
with $a_0 = 0$, $b_0 = 0$.

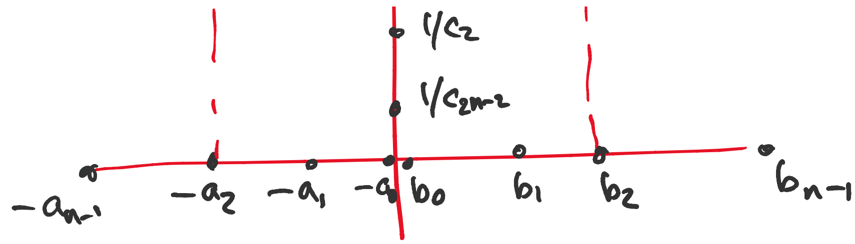
Create pts $(-a_i, 0)$ $\quad i = 0, .., n-1$

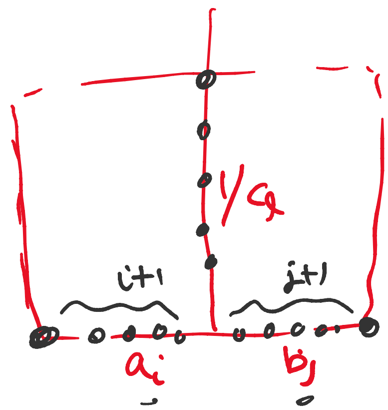
$(0, b_j)$ $\quad j = 0, .., n-1$

$(0, \frac{1}{c_\ell})$ $\quad \ell = 0, .., 2n-2$

Set $k = 2n+1$.







$\#$ pts enclosed

$$= i+1+j+1+2n-2-\ell+1$$
$$= i+j-\ell+2n+1$$
$$= k$$
$$\Longleftrightarrow i+j=\ell.$$

$$\text{area} = (a_i+b_j)\cdot\frac{1}{c_\ell} \leq 1$$

$$\Longleftrightarrow \quad a_i+b_j \leq c_\ell. \qquad \square$$