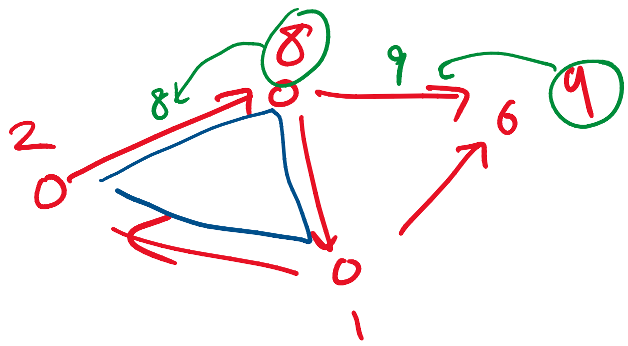
September 30, 2020 10:52 AM

HW2 available

APSP

Large int wts or real wts? OPEN

today: vertex wts



Warm-Up: Min-wt triangle for real vertex wts

Vassilevska, Williams, Yuster's Alg'm: ('06)

Lemma Given $n \times n$ Boolean matrices A, B, can compute min-witness product C:

$$c_{ij} = \min \{ k : a_{ik} \wedge b_{kj} \text{ is true} \}$$

in $O(n^{2.529})$ time.

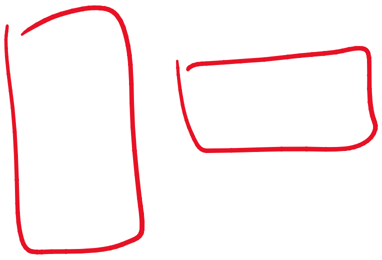
Pf: divide $[n]$ into r intervals $I_1, \ldots, I_r$ of length n/r.

$I_1 \quad I_2 \quad \ldots \quad I_r$

Step 1. for each interval I_g $(g = 1, \ldots, r)$,

Compute $d_{ij}^{(g)} = \bigvee (a_{ik} \wedge b_{kj})$

Compute $d_{ij}^{(g)} = \underbrace{\bigvee_{k \in I_g} (a_{ik} \wedge b_{kj})}_{\text{Boolean MM}}$



$\Rightarrow$ $O(\cancel{r\, n^{\omega}})$ time

$O(r\, M(n, \frac{n}{r}, n))$ brute force!

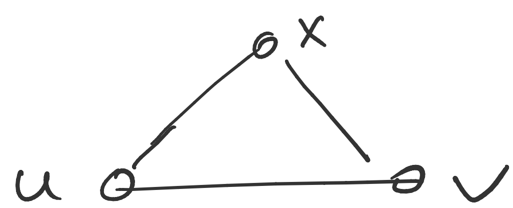
Step 2. compute c_{ij} by searching in I_g for smallest g s.t. $d_{ij}^{(g)}$ true.

$$O\left(n^2 \cdot \frac{n}{r}\right)$$

Total: $O\left(r\, \underline{M(n, \frac{n}{r}, n)} + \frac{n^3}{r}\right)$.

like Zwick's alg'm. $\square$

Alg'm:



for each $u, v \in V$,

Compute $c_{uv} = \min\{wt(x) : ux \in E \wedge xv \in E\}$

$\Rightarrow$ min-witness product !! $\leftarrow O(n^{2.529})$
(after sorting vertices by wt)

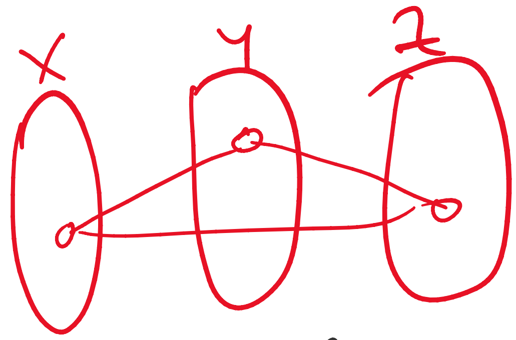
return $\min_{u,v \in V} \left(\underbrace{c_{uv} + wt(u) + wt(v)}\right)$ $\leftarrow O(n^2)$

$\Rightarrow$ $\boxed{O(n^{2.529})}$ time

Czumaj & Lingas' Alg'm ('10): near $O(n^{\omega})$ time

To compute min-wt triangle in $X \times Y \times Z$:

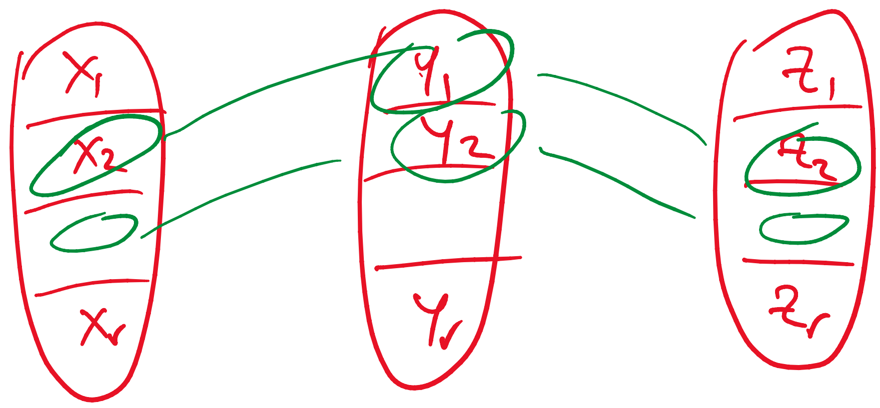
sort vertices by wt



divide X into r subintervals $X_1, \ldots, X_r$ of size n/r

Y into $Y_1, \ldots, Y_r$ " " "

Z into $Z_1, \ldots, Z_r$ " " "



for each $(i,j,k) \in [r]^3$

test if $\exists$ triangle in $X_i \times Y_j \times Z_k$ $\leftarrow$ $O((\frac{n}{r})^{\omega})$ time

if so, add (i,j,k) to T

for each $(i,j,k) \in T$ that is a minimal pt of T

i.e. there is no $(i',j',k') \in T$ with $i' < i,\ j' < j,\ k' < j$.

recurse in X_i, Y_j, Z_k.

• $(i,j,k) \in T$

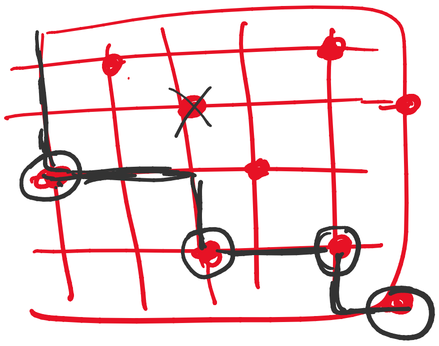
Fact for any set $T \subseteq [r]^3$,

$\circ\ (i,j,k) \in T$

$(i',j',k') \in T \quad i' < i,\ j' < j,\ k' < k$

Fact for any set $T \subseteq [r]^3$,
$\#$ minimal pts $\le O(r^2)$.

e.g. in 2D, in $[r]^2$,
$\#$ minimal pts $\le O(r)$.



staircase

$$\Rightarrow T(n) \le \underline{cr^2\, T\!\left(\tfrac{n}{r}\right)} + O\!\left(\cancel{r^3 \left(\tfrac{n}{r}\right)^{\omega}}\right) \quad \underline{O(n^{\omega})}$$

choose r to be large const

$$\Rightarrow T(n) = O\!\left(\max\left\{ n^{\log_r (cr^2)}, \; n^{\omega} \right\}\right)$$

$$\frac{\log(cr^2)}{\log r} = 2 + \frac{\log c}{\log r}$$

$$= \boxed{O\!\left(\max\left\{ n^{2+\varepsilon}, \; n^{\omega} \right\}\right)}$$

APSP for real vertex wts. ($\underline{C'07}$)

$n \times n$ real matrix A,

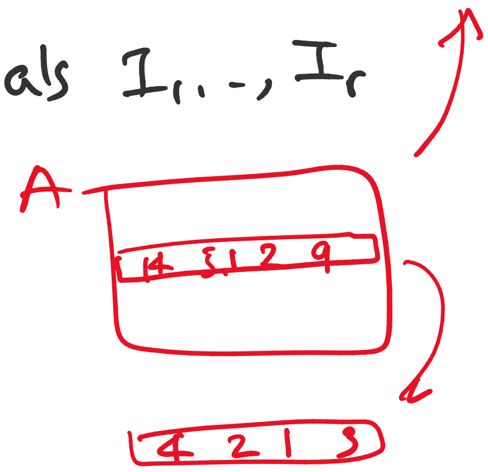
Lemma Given $n \times n$ real matrix A,
& $n \times n$ matrix B where all entries are either 0 or ∞,

can compute $(\min,+)$-product C

$$c_{ij} = \min_k (a_{ik} + b_{kj}) = \min_{k:\, b_{kj} \neq \infty} a_{ik}$$

in $O(n^{(\omega+3)/2})$ time
$= O(n^{2.687})$

Pf: Sort each row of A & map entries to $[n]$.
Divide $[n]$ into r intervals $I_1, \dots, I_r$



Step 1. for each I_g $(g = 1, \dots, r)$

Compute $$d_{ij}^{(g)} = \bigvee_k \left([a_{ik} \in I_g] \wedge [b_{kj} \neq \infty] \right)$$

$\Rightarrow$ Boolean MM

$O(r n^{\omega})$

Step 2. Compute c_{ij} by brute force search over all k with $a_{ik} \in I_g$ where g is smallest s.t. $d_{ij}^{(g)}$ true.

$\Rightarrow$ $O(n^2 \cdot \frac{n}{r})$

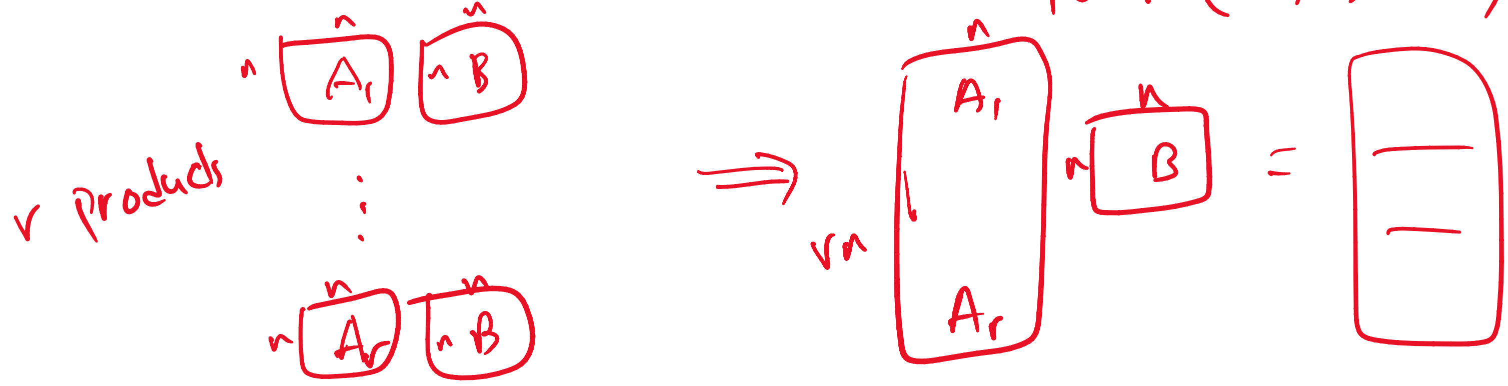
Total $O(rn^{\omega} + n^3)$

Total $O(rn^{\omega} + \frac{n^3}{r})$

Set $r = n^{\frac{3-\omega}{2}} \Rightarrow O(n^{\frac{3+\omega}{2}})$. □

Rmk - Slight improv w. rect MM $\Rightarrow O(n^{2.66})$

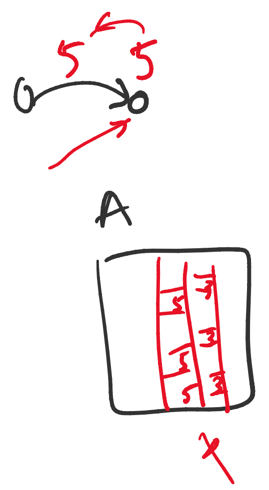
(Step 1: fewer rn^{ω} to $M(rn, n, n)$)



Alg'm:

Short case: Short paths with $\leq \ell$ edges

idea - forget repeated squaring
do ℓ iterations



$$d^{(\ell)}(u,v) = \min_{\substack{x \in V: \\ (x,v) \in E}} d^{(\ell-1)}(u,x) + w(v)$$

computed in prev iter.

Use Lemma!

$\Rightarrow O(\ell \cdot n^{\frac{\omega+3}{2}})$ time

$$\Rightarrow O\left(\ell \cdot n^{\frac{\omega+3}{2}}\right) \text{ time}$$

long case: paths with $\geq \ell$ edges

use hitting set!

$$\tilde{O}\left(\frac{n^3}{\ell}\right) \text{ time.}$$

Total: $\tilde{O}\left(\ell n^{\frac{\omega+3}{2}} + \frac{n^3}{\ell}\right)$

$\ell^2 = n^{3 - \frac{3+\omega}{2}}$

Set $\ell = n^{\frac{3-\omega}{4}}$

$$\Rightarrow \boxed{\tilde{O}\left(n^{3-\frac{3-\omega}{4}}\right)} = O\left(n^{2.844}\right)$$