in $\boxed{\tilde{O}(c\, n^{\omega})}$ time.

Pf: Let $a'_{ik} = k M^{a_{ik}}$, $b'_{kj} = M^{b_{kj}}$

Compute standard MM for A', B'

$$c'_{ij} = \sum_{k=1}^{n} a'_{ik} \cdot b'_{kj}$$

$$= \sum_{k=1}^{n} M^{a_{ik} + b_{kj}}$$

$$= \text{something} \cdot k \cdot M^{\max_k (a_{ik} + b_{kj})} + \underbrace{\cdots}_{\text{lower-order terms}}$$

$\Rightarrow$ get $(\max, +)$-MM.

$\Rightarrow$ get $(\min, +)$-MM $\quad$ $c - a_{ik}$, $c - b_{kj}$

Set $M = n+1$.

$\Rightarrow$ need to work with numbers in $[(n+1)^c]$

$O(c \log n)$-bit

each arithmetic op takes $\tilde{O}(c)$ time.

by FFT. $\square$

---

unfortunately, for long paths,
matrix entries get bigger

first idea - short vs. long paths

Case 1. short paths with $\leq L$ edges

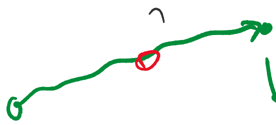
repeated squaring

repeated squaring

$\Rightarrow$ $O(\log n)$ $(\min,+)$-MMs with entries in $[cL]$

$\Rightarrow$ by Lemma, $\tilde{O}(cL \cdot n^{\omega})$

Case 2. long paths with $\geq L$ edges.



Hitting Set Lemma $\exists$ subset $R_L \subseteq V$ of size $\underline{O\left(\frac{n}{L}\log n\right)}$

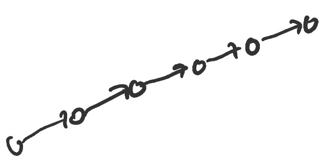
that hits all shortest paths with $\geq L$ edges.

Pf 1: just take a random subset R !

w/ samp. prob. $p = \frac{10}{L}\log n$

Fix s,t.

Let π_{st} be vertices in shortest path from s to t



$|\pi_{st}| \geq L$.

$$\begin{aligned}\text{Prob}[R \text{ not hit } \pi_{st}] &\leq (1-p)^L \\ &\leq e^{-pL} \\ &= e^{-10\log n} \\ &\leq n^{-10}.\end{aligned}$$

$\Rightarrow$ $\text{Prob}[\text{for some } s,t,\ R \text{ not hit } \pi_{st}]$

(union bd) $\leq n^2 \cdot n^{-10} = n^{-8}$. $\square$

Pf 2: by greedy alg'm for hitting set ... $\square$

single-source

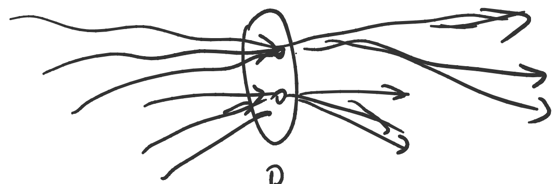
$\downarrow$

SSSP from each $r \in R_L$

Run SSSP from each $r \in R_L$

& SSSP to each $r \in R_L$ ← $\tilde{O}(m|R_L|) = \tilde{O}(n^2|R_L|) = \tilde{O}\left(\frac{n^3}{L}\right)$

single-sink



R_L

for each $s,t \in V$,

take $\min_{x \in R_L} \left(d(s,x) + d(x,t) \right)$

by brute force

← $O(|R_L| \cdot n^2) = O\left(\frac{n^3}{L}\right)$

$\Rightarrow \tilde{O}\left(\frac{n^3}{L}\right)$.

Total time: $\tilde{O}\left(cLn^\omega + \frac{n^3}{L}\right)$.

Set $L = \frac{n^{\frac{3-\omega}{2}}}{\sqrt{c}} \Rightarrow \boxed{\tilde{O}\left(\sqrt{c}\, n^{\frac{3+\omega}{2}}\right)}$

$= O(n^{2.687})$ if $c = O(1)$.

$cLn^\omega = \frac{n^3}{L}$

$L^2 = \frac{n^{3-\omega}}{c}$

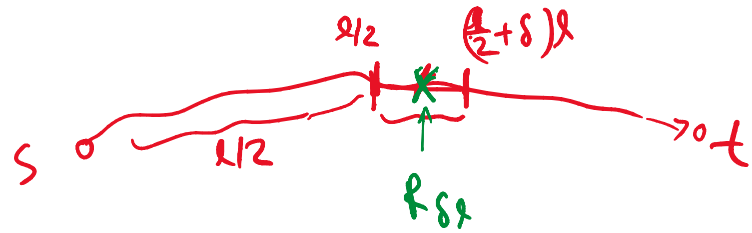
Alon, Galil, Margalit '97

Zwick's better idea. use hitting set also for short case!!

Suppose we have computed all shortest paths

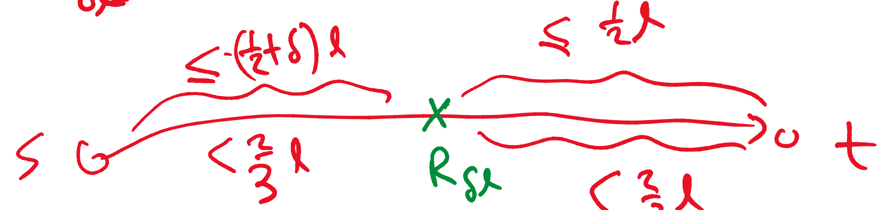
Suppose we have computed ...
with $\leq \frac{2}{3}\ell$ edges.

To compute shortest paths of length in $\left(\frac{2}{3}\ell, \ell\right]$:



$\delta = 0.1$

$R_{\delta\ell}$ to hit all shortest paths of length $\delta\ell$



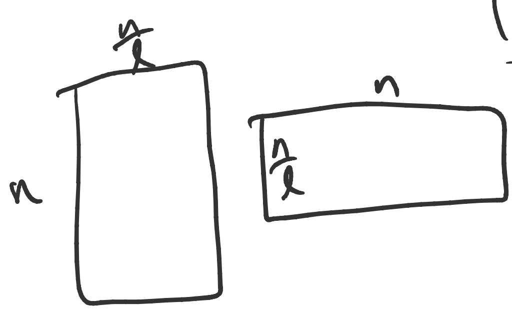
take $\min_{x \in R_{\delta\ell}} \left(\overset{(\leq \frac{2}{3}\ell)}{d(s,x)} + \overset{(\leq \frac{2}{3}\ell)}{d(x,t)} \right)$

already computed in prev iterations

$|R_{\delta\ell}| = \tilde{O}\left(\frac{n}{\ell}\right)$

$(\min,+)$-MM of $n \times |R_{\delta\ell}|$ and $|R_{\delta\ell}| \times n$ matrix.

with entries in $[c\ell]$.



in $\tilde{O}\left(c\ell \, M\left(n, \frac{n}{\ell}, n\right) \right)$ time.

Try $\ell = \left(\frac{3}{2}\right)^i \leq L$. $\quad$ ($O(\log n)$ iters)

Total time: $\tilde{O}\left(\max_{\ell \le L} \left(c\ell\, M(n, \frac{n}{\ell}, n) \right) + \frac{n^3}{L} \right).$

$$= \boxed{\tilde{O}\left(cL\, \underbrace{M(n, \tfrac{n}{L}, n)} + \frac{n^3}{L} \right)}$$

by quick upper bd:

$$\tilde{O}\left(cL \cdot L^2 \left(\frac{n}{L}\right)^{\omega} + \frac{n^3}{L} \right)$$

$$= \tilde{O}\left(c\, \underline{L^{3-\omega} n^{\omega}} + \frac{n^3}{L} \right)$$

$c L^{3-\omega} n^{\omega} = \frac{n^3}{L}$

$L^{4-\omega} = n^{3-\omega}/c$

Set $L = \dfrac{n^{\frac{3-\omega}{4-\omega}}}{c^{\frac{1}{4-\omega}}}$

$$= \tilde{O}\left(c^{\frac{1}{4-\omega}} n^{3 - \frac{3-\omega}{4-\omega}} \right)$$

$$= \tilde{O}\left(n^{2.615} \right) \quad \text{if } c = O(1)$$

by current best bd on rect MM,

Set $L = n^{0.471} \Rightarrow \boxed{\tilde{O}\left(n^{2.529}\right)}$ if $c = O(1)$.

Rmks - negative wts ok (if no neg-wt cycle)

- how to recover paths?

$\Rightarrow$ "witness finding" by rand....

- cond. LB ... later ...

Undirected case is easier ...

because $d(\cdot,\cdot)$ is a metric

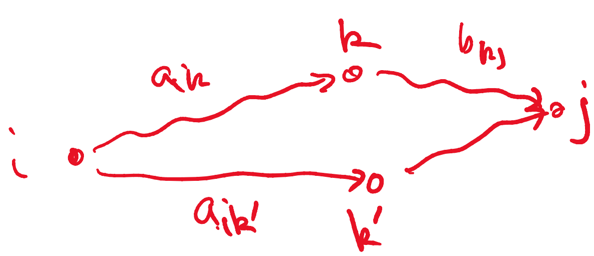
$$d(u,v) = d(v,u) \quad \text{(symmetry)}$$
$$d(u,v) \le d(u,w) + d(w,v) \quad (\Delta \text{ ineq}).$$

$$\Rightarrow \quad |d(u,v) - d(u,w)| \le d(w,v)$$
("Δ diff. ineq.")

Modified Lemma Given $n \times n$ matrices A, B where all entries of B are in $[c] \cup \{\infty\}$ but all entries of A are arbitrary ints s.t. $\forall i,j,k,k'$, if $b_{kj}, b_{k'j} \ne \infty$, $|a_{ik} - a_{ik'}| \le 2c$.

& follows from Δ diff ineq.

Then can compute $(\min,+)$-MM in $\tilde{O}(cn^{\omega})$ time.



$c_{ij} = \min_k (a_{ik} + b_{kj})$

$|a_{ik} - a_{ik'}|$
$\le b_{kk'} \le b_{kj} + b_{k'j}$
$\le 2c.$