Applications to MM

Transitive Closure

Given dir. graph $G=(V,E)$,
$\forall s,t \in V$, decide whether $\exists$ path $s \rightsquigarrow t$

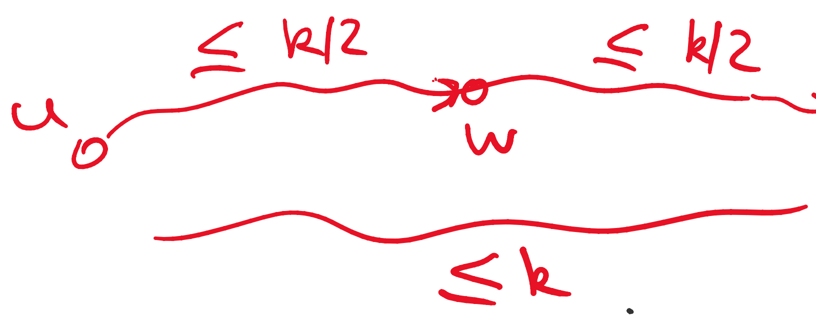
naive alg'ms: n DFSs/BFSs
$\Rightarrow O(\underline{mn}) \le O(n^3)$
by DP (Warshall)
$\Rightarrow O(n^3)$

Alg'm 1 repeated squaring

let $c_{uv}^{(\le k)} =$ true iff $\exists$ path $u \rightsquigarrow v$ with $\le k$ edges

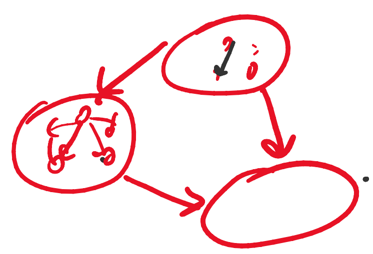
$$c_{uv}^{(\le k)} = \bigvee_{w \in V} \left(c_{uw}^{(k/2)} \wedge c_{wv}^{(k/2)} \right)$$

Boolean MM
i.e. $C^{(k/2)} \circ C^{(k/2)}$



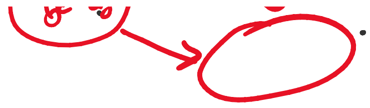
do $k = 1, 2, 4, 8, \ldots$
$\Rightarrow \boxed{O(n^{\omega} \underline{\log n})} \le O(n^{2.373})$

Alg'm 2: (Munro '71)

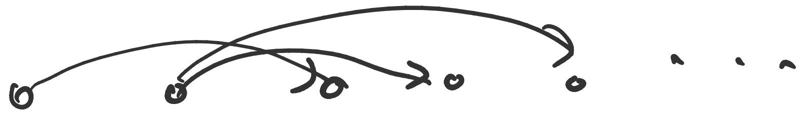


assume G is a DAG

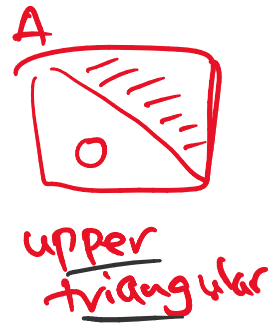
may assume G is a DAG
(by strongly conn. comps & meta-graph)



topologically sort vertices
i.e. all edges are of the form (i,j) with $i<j$



Idea - D & C where n is decreased

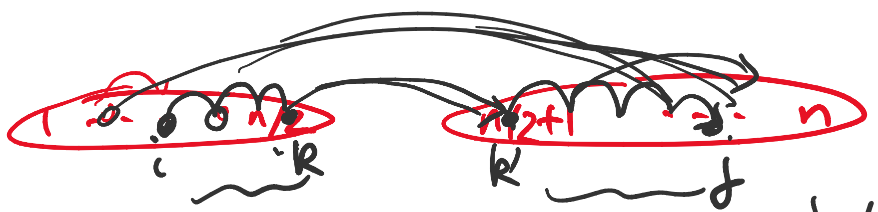


let $a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{else} \end{cases}$

to compute $a^*_{ij} = \begin{cases} 1 & \text{if } \exists \text{ path } i \rightsquigarrow j \\ 0 & \text{else} \end{cases}$

Write $A = \left(\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right)$

Then $A^* = \left(\begin{array}{c|c} A^*_{11} & A^*_{11} \cdot A_{12} \cdot A^*_{22} \\ \hline 0 & A^*_{22} \end{array}\right)$



$A^*_{11} \cdot A_{12} \cdot A^*_{22}$ $\quad \bigvee_{k,k'} a^*_{ik} \wedge a_{kk'} \wedge a^*_{k'j}$

2 recursive calls + 2 MMs

$\Rightarrow \quad T(n) = 2T\left(\frac{n}{2}\right) + O(n^\omega)$

$\Rightarrow \quad \boxed{O(n^\omega)}$ without extra logs

Rmk. without top. sort, 4 recursive calls...

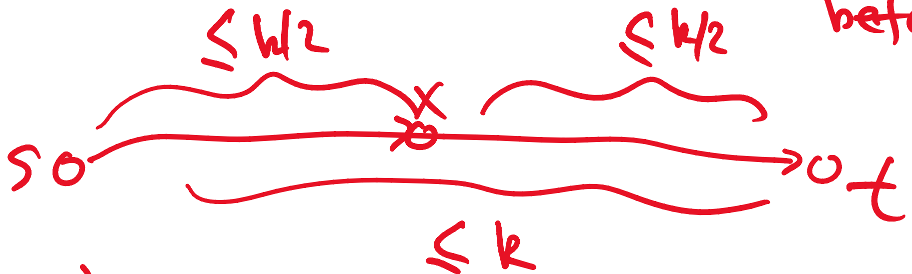
All-Pairs Shortest Paths (APSP)

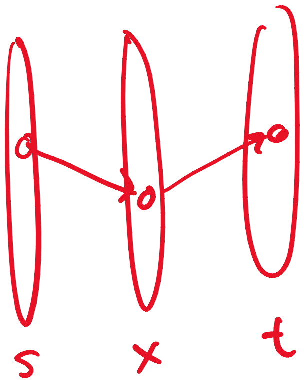
Given $G=(V,E)$, with edge wts, $\forall s,t \in V$, compute $d(s,t)$ = shortest path dist from s to t.

repeated squaring generalizes...
Munro generalizes (w/o top sort)...

$$d^{(\le k)}(s,t) = \min_{x \in V} \left(d^{(\le k/2)}(s,x) + d^{(\le k/2)}(x,t) \right)$$

computed before





(min, +)-MM: also called distance product

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will consider unweighted case
or small integer weight case
i.e. all wts are in $[c] \cup \{\infty\}$

$[c]$ = $\{1,\dots,c\}$

Zwick's Alg'm ('02)

Lemma Given $n \times n$ matrices A, B where all entries are in $[c] \cup \{\infty\}$.

can compute (min, +)-MM

$$c_{ij} = \min_k (a_{ik} + b_{kj})$$

in $\boxed{\tilde{O}(c\, n^{\omega})}$ time.

in $\boxed{\tilde{O}(c\, n^{\omega})}$ time.

Pf: Let $a'_{ik} = M^{a_{ik}}$, $b'_{kj} = M^{b_{kj}}$

Compute standard MM for A', B'

$$c'_{ij} = \sum_{k=1}^{n} a'_{ik} \cdot b'_{kj}$$

$$= \sum_{k=1}^{n} M^{a_{ik}+b_{kj}}$$

$$= \underline{\text{something}} \times M^{\max_k (a_{ik}+b_{kj})} + \cdots$$

lower-order terms

$\Rightarrow$ get $(\max,+)$-MM.

$\Rightarrow$ get $(\min,+)$-MM $\qquad c-a_{ik}$

$\qquad c-b_{kj}$

Set $M = n+1$.

$\Rightarrow$ need to work with numbers in $[(n+1)^c]$

$\underline{O(c \log n)}$ - bit

each arithmetic op takes $\tilde{O}(c)$ time.

by $\underline{\text{FFT}}$. $\square$