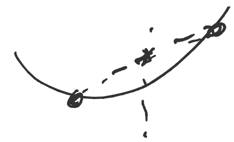
Known Properties:

$\omega(\cdot,\cdot,\cdot)$ is convex.



$$\underline{\omega(a,b,c)} = \omega(c,b,a) = \omega(b,a,c)$$ symmetry

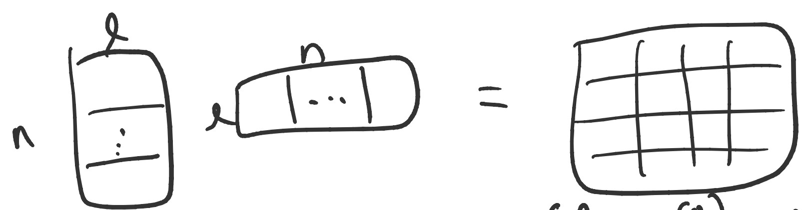
Quick Bds:

If $\ell \le n$,

$$M(n,\ell,n) = O\left(\left(\frac{n}{\ell}\right)^2 \cdot \ell^\omega\right) = O(\ell^{\omega-2} n^2). \checkmark$$

not tight

skinny



If $\ell > n$, $M(n,\ell,n) = O\left(\frac{\ell}{n} \cdot n^\omega\right) = O(\ell n^{\omega-1})$ $\checkmark$

not tight

fat

---

skinny case: $(\ell \le n)$

Coppersmith '82: $M(n, n^{0.172}, n) = \underline{O(n^2 \log^2 n)}$

Coppersmith '97: $M(n, n^{0.294}, n) = O(n^{2+\varepsilon})$

Le Gall '12: $n^{0.302}$

Le Gall, Urrutia '18: $M(n, \underline{n^{0.31389}}, n) = O(n^{2+\varepsilon})$

rectangle matrix mult. exponent & denoted α

$$1 \geq \alpha > \underline{0.31389}$$

fat case: $M(n, n^k, n) = O(n^{k+1+f(k)})$
$(k>1)$ with $f(k) \to 0$ as $k \to \infty$.

Appl'n to sparse case:

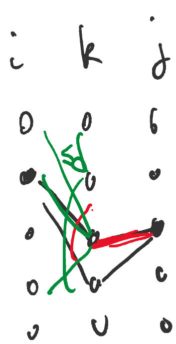
multiply 2 $n \times n$ matrices A, B with m non-zeros
$(m \le n^2)$

trivial bds: $\begin{cases} O(mn) \\ O(n^\omega) \end{cases}$

Yuster-Zwick '05: high-low trick

let $\text{deg}(k) = |\{i : a_{ik} \neq 0\}|$
$\left(\sum_k \text{deg}(k) \le m\right)$.

let $H = \{k : \text{deg}(k) > d\}$
$L = \{k : \text{deg}(k) \le d\}$

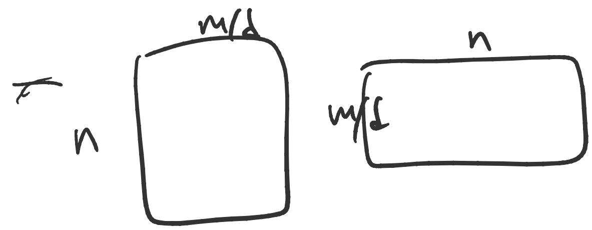


low case: compute $c_{ij}^L = \sum_{k \in L} a_{ik} b_{kj}$

$\Rightarrow O(dm)$ time (for each $b_{kj} \neq 0$, # candidates for $i = O(d)$)

high case: compute $c_{ij}^H = \sum_{k \in H} a_{ik} b_{kj}$

$|H| \le m/d$



$\Rightarrow M(n, \frac{m}{d}, n)$

$\Rightarrow \quad M(n, d, \ )$

total time $\boxed{O\left(\min_d \left(dm + M\left(n, \frac{m}{d}, n\right)\right)\right)}$

e.g. if $m \le n^{1+\frac{\alpha}{2}}$, set $d = n^{1-\frac{\alpha}{2}}$

1.156

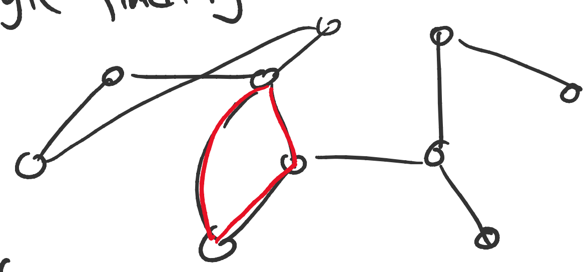
$$\Rightarrow O\left(n^2 + M(n, n^{\alpha}, n)\right)$$

$$= O(n^{2+\varepsilon}).$$

Appl 0: problems about matrices

e.g. inverse, $Ax=b$, det, ...

Appl 1: triangle finding



Given dir/undir graph $G=(V,E)$, $|V|=n$, $|E|=m$

decide $\exists$ triangle, i.e. $u,v,w \in V$ s.t. $(u,v), (v,w), (w,u) \in E$.

naive alg'm: $O(n^3)$ time

better?

let $a_{uv} = \begin{cases} 1 & \text{if } (u,v) \in E \\ 0 & \text{else} \end{cases}$ (adj matrix) for each u, v

$$c_{vu} = \boxed{\bigvee_{w \in V} (a_{vw} \wedge a_{wu})} \leftarrow A \cdot A$$

return true if c_{vu} true & $(u,v) \in E$.

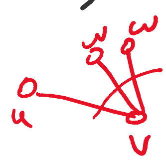
Boolean MM!

reduces to standard MM

$\sum a_{ik} b_{kj}$

$$\sum_{w \in V} a_{vw} a_{wu} \leftarrow$$

$$\Rightarrow \boxed{O(n^{\omega})} \text{ time } = O(n^{2.373})$$

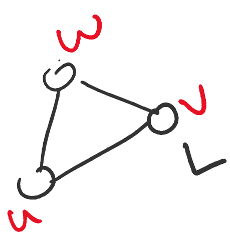


Sparse case? trivial $O(mn)$

Alon, Yuster, Zwick '97: high-low trick again

let $H = \{ v \in V : \deg(v) > d \}$ $\quad \sum \deg(v) = O(m)$

$L = \{ v \in V : \deg(v) \leq d \}$



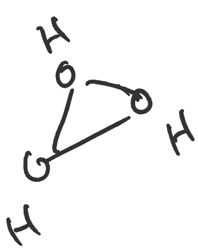
Case 1. at least one of vertices in triangle is low

$\Rightarrow$ for each $(u,v) \in E$ check neighbors w of v

$\Rightarrow O(dm)$ time.

Case 2. all 3 vertices in triangle is high.

$|H| \leq O(\frac{m}{d})$



$$O\left(\left(\frac{m}{d}\right)^{\omega} \right)$$

$\Rightarrow$ total time $O\left(dm + \left(\frac{m}{d}\right)^{\omega}\right)$ $\quad d = \frac{m^{\omega}}{d^{\omega}}$

Total time $O\left(dm + \left(\frac{m}{d}\right)^{\omega} \right)$

$dm = \frac{m^{\omega}}{d^{\omega}}$

$d^{\omega+1} = m^{\omega-1}$

Set $d = m^{\frac{\omega-1}{\omega+1}}$

$\Rightarrow \boxed{O\left(m^{\frac{2\omega}{\omega+1}} \right)} = O(m^{1.41})$

(without MM, $\omega = 3 \Rightarrow O(m^{3/2})$)

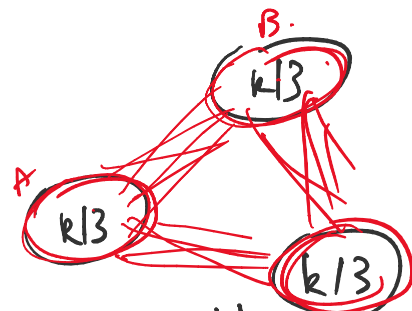
Rmk - extensions: k-cycle for const k

Alon, Yuster, Zwick: $O(n^{\omega})$ time

"color-coding"

k-clique for const k:

if k is div by 3,



define $G' = (V', E')$

$V' = (k/3)$-clique $\quad (O(n^{k/3})$ vertices)

$AB \in E'$ iff $A \cup B$ is $(2k/3)$-cliq.

$\Rightarrow$ reduce to triangle finding in G'

$\Rightarrow O\left((n^{k/3})^{\omega} \right) = \boxed{O\left(n^{k\omega/3} \right)}$

$= O(n^{0.792k})$

better than brute-force n^k.

Rmk - what about weighted graphs?

min-wt triangle

for each u, v,

compute $c_{vu} = \underbrace{\min_{w \in V} (a_{vw} + a_{wu})}_{\text{"(min,+)-MM"}}$

e.g. Strassen not work!

is subcubic possible?