CS 598TMC, Fall 2020
Timothy Chan

Homework 3 (due Nov 6 Friday 10:00am (CT))

Instructions: see previous homework.

1. [*30 pts*] Recall the triangulation problem from Homework 2: We are given a convex polygon P with vertices $v_1 v_2 \ldots v_n v_1$ in clockwise order. We are also given a positive integer weight matrix $w(v_i, v_j)$ (satisfying $w(v,v) = 0$ and $w(u,v) = w(v,u)$). We want to compute a triangulation that minimizes the sum of the weights of its edges.

   Assuming the APSP conjecture, prove a near-cubic lower bound for the triangulation problem (i.e., if the triangulation problem could be solved in $O(n^{3-\delta})$ time for some constant $\delta > 0$, then APSP for integer weights could be solved in $\widetilde{O}(n^{3-\delta'})$ time for some constant $\delta' > 0$).

2. [*30 pts*] Recall the min-area k-enclosing rectangle problem from class: given n points in two dimensions and $k \leq n$, find an axis-aligned rectangle with the minimum area that contains exactly k of the input points.

   Assuming the $(\min, +)$-convolution conjecture, we have proved a near quadratic lower bound as a function of n for the rectangle problem. Assuming the same conjecture, prove a near nk lower bound as a function of n and k for the rectangle problem (i.e., for any given function $\kappa(n)$, if the min-area $\kappa(n)$-enclosing rectangle for any set of n points could be found in $O((n\kappa(n))^{1-\delta})$ time for some constant $\delta > 0$, then the $(\min, +)$-convolution problem could be solved in $O(n^{2-\delta'})$ time for some constant $\delta' > 0$).

3. [*40 pts*] Recall the Hamming distance problem from class: Given a pattern string $p_1 \cdots p_m \in \Sigma^*$ and a text string $t_1 \cdots t_n \in \Sigma^*$ (with $m \leq n$), compute the Hamming distance D_i between $p_1 \cdots p_m$ and $t_{i+1} \cdots t_{i+m}$, for all $i = 0, \ldots, n-m$. In class, we described an $\widetilde{O}(n\sqrt{m})$-time algorithm by Abrahamson.

   Recall the "dominance string matching" problem from Homework 1: Given a pattern string $p_1 \cdots p_m \in \Sigma^*$ and a text string $t_1 \cdots t_n \in \Sigma^*$ with $\Sigma = [\sigma]$, decide whether there exists an index i such that $p_1 \leq t_{i+1}$, *and* $p_2 \leq t_{i+2}$, $\ldots$, *and* $p_m \leq t_{i+m}$. In Homework 1, you have given an $\widetilde{O}(n\sqrt{m})$-time algorithm for this problem, by modifying Abrahamson's algorithm.

   (a) Give a reduction of the dominance string matching problem to the Hamming distance problem that preserves run time up to polylogarithmic factors (i.e., if the Hamming distance problem could be solved in $T(n,m)$ time, then the dominance string matching problem could be solved in $\widetilde{O}(T(n,m))$ time). (Thus, this would imply an alternative solution to the Homework 1 question.)
   Hint: divide $[\sigma]$ into even vs. odd...

(b) Give a reduction of Boolean matrix multiplication to the Hamming distance problem, to prove an $\Omega(n^{\omega/2})$ lower bound in terms of n for the latter problem, where ω is the exponent for Boolean matrix multiplication. (Thus, if $\omega > 2$, this would rule out a near-linear algorithm for the Hamming distance problem.)

Hint: given two $s \times s$ matrices A and B, carefully show how the Boolean product AB can be computed by solving the Hamming distance problem on the following pattern and text strings over the alphabet $[s] \cup \{\#, \%, \$\}$:

$$
\begin{aligned}
p &= a'_{11}a'_{12}\cdots a'_{1s}a'_{21}a'_{22}\cdots a'_{2s} \qquad \cdots \qquad a'_{s1}a'_{s2}\cdots a'_{ss} \\
t &= \#^{s^2} b'_{11}b'_{21}\cdots b'_{s1}\#b'_{12}b'_{22}\cdots b'_{s2}\# \qquad \cdots \qquad \#b'_{1s}b'_{2s}\cdots b'_{ss}\#^{s^2},
\end{aligned}
$$

where

$$
\begin{aligned}
a'_{ik} &= \begin{cases} k & \text{if } a_{ik} \text{ is true} \\ \% & \text{else} \end{cases} \\
b'_{kj} &= \begin{cases} k & \text{if } a_{kj} \text{ is true} \\ \$ & \text{else.} \end{cases}
\end{aligned}
$$