CS 598RM: Algorithmic Game Theory, Fall 2020
HW 2 (due on Wednesday, 28th Oct at 11:59pm CST)

Instructions:

1. We will grade this assignment out of a total of 60 points.
2. Feel free to discuss with fellow students, but write your own answers. If you do discuss a problem with someone then write their names at the starting of the answer for that problem.
3. Please type your solutions if possible in Latex or doc whatever is suitable. We will upload submission instructions on the course webpage and Piazza.
4. Even if you are not able to solve a problem completely, do submit whatever you have. Partial proofs, high-level ideas, examples, and so on.
5. Except where otherwise noted, you may refer to lecture slides/notes, and to the references provided. You cannot refer to textbooks, handouts, or research papers that have not been listed. If you do use any approved sources, make sure you cite them appropriately, and make sure to write in your own words.
6. No late assignments will be accepted.
7. By AGT book we mean the following book: Algorithmic Game Theory (edited) by Nisan, Roughgarden, Tardos and Vazirani. Its free online version is available at Prof. Vijay V. Vazirani's webpage.

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1. (a) (3 points) The following game has a unique Nash equilibrium. Find it, and prove that it is unique. (Hint: look for strict dominance.)

4, 0	1, 1	4, 0
1, 3	2, -1	3, 5
0, 1	3, 0	3, 0

   (b) (4 points). Construct a single 2×2 normal-form game that simultaneously has all four of the following properties:

      i. The game does not have a dominant strategy Nash equilibrium (at least one player does not have a dominant strategy).
      ii. The game is solvable by iterated weak dominance (so that one pure strategy per player remains).
      iii. In addition to the iterated weak dominance solution (which is a Nash equilibrium), there is a second pure-strategy Nash equilibrium.

iv. Both players strictly prefer the second equilibrium to the first. (Hints: the second pure-strategy equilibrium should not be strict; the pure strategy equilibria should be in opposite corners of the matrix.) If you cannot get all four properties, construct an example with as many of the properties as you can.

(c) (3 points). Consider the following game:

2, 5	1, 5
7, 1	0, 0

Find a correlated equilibrium that places positive probability on all entries of the matrix, except the lower-right hand entry. Try to maximize the probability in the upper-left hand entry.

(a) Let R denote the row player and C denote the column player. Furthermore, let r_1, r_2, r_3 denote R's strategies and c_1, c_2, c_3 denote C's strategies. Finally, let p_i (respectively q_i) denote the probability that R (respectively C) plays strategy r_i (respectively c_i), for $i = 1, 2, 3$.

First, notice that it is never in R's interest to play r_2. To see why this is the case, notice that r_2 is strictly dominated by the mixed strategy $s = (0.33, 0, 0.64)$ for R. This is because the expected payoff of r_2 is $q_1 + 2q_2 + 3q_3$, while the expected payoff for s is $4 \cdot 0.33q_1 + (0.33 + 3 \cdot 0.64)q_2 + (4 \cdot 0.33 + 3 \cdot 0.64)q_3$, which is strictly greater than $2q_1 + 2q_2 + 3q_3$, for any $q_1, q_2, q_3 \geq 0$ such that $q_1 + q_2 + q_3 = 1$.

Thus, in any NE, R will play r_2 with probability $p_2 = 0$. Knowing this, C will never play c_3, as c_3 yields higher payoff for C only if R plays r_2, i.e, $p_2 > 0$. In any other case where $p_2 = 0$, one of c_1 or c_2 is strictly better for C than c_3. Thus, we have reduced our 3×3 bimatrix game to a 2×2 bimatrix game

4, 0	1, 1
0, 1	3, 0

Note that there is no pure NE in this 2×2 game. So the only remaining choice is a mixed strategy where both the players randomize between the two remaining strategies. We have that if R plays r_1, they get a payoff of $4q_1 + q_2$, whereas if R plays r_3, they get a payoff of $3q_2$. Using the fact that $q_1 + q_2 = 1$ (as $q_3 = 0$), we get that R's payoff from strategy r_1 is $4 - 3q_2$. At NE, R's payoff from both strategies will be equal, and thus

$$4 - 3q_2 = 3q_2 \Leftrightarrow \begin{cases} q_2 = \frac{2}{3} \\ q_1 = \frac{1}{3} \end{cases}$$

Similarly, if C plays c_1, they get a payoff of p_3, whereas if C plays c_2, they get a payoff of p_1. Using the fact that $p_1 + p_3 = 1$ (as $p_2 = 0$), we get that C's payoff from strategy c_1 is $1 - p_1$. At NE, C's payoff from both strategies will be equal, and thus

$$1 - p_1 = p_1 \Leftrightarrow \begin{cases} p_1 = \frac{1}{2} \\ p_3 = \frac{1}{2} \end{cases}$$

Since the NE equations had a unique solution, the one we gave above, we conclude that this is the only NE of this game.

(b) Consider the following game

4, 2	3, 3
4, 4	0, 1

Let R denote the row player and C denote the column player. Furthermore, let r_1, r_2 denote R's strategies and c_1, c_2 denote C's strategies. This game satisfies all four requested properties:

- Notice that C does not have a dominant strategy, as if R plays r_1, then C's best-response strategy is c_2 and if R plays r_2, C's best-response strategy is c_1.
- Note that r_2 is weakly dominated by r_1. Once we remove r_2, c_2 will strictly dominate c_1. Thus, we are left with (r_1, c_2) which is a NE.
- Notice that the bottom-left entry of the matrix represents another NE. Indeed, if R plays r_2 and C plays c_1, no player can gain higher payoff by unilaterally switching strategies.
- Clearly, both players prefer the second pure strategy NE, as both get higher payoffs in the bottom-left entry $(4, 4)$ than the top-right entry of the matrix $(3, 3)$.

(c) Once again, let R denote the row player and C denote the column player. Furthermore, let r_1, r_2 denote R's strategies and c_1, c_2 denote C's strategies. Suppose the mediator selects (r_1, c_1) with probability p_1, (r_1, c_2) with probability p_2 and (r_2, c_1) with probability p_3 (the mediator selects (r_2, c_2) with probability 0 as instructed by the problem statement). Thus, $p_1 + p_2 + p_3 = 1$, and for the probability distribution to be a CE, we have

$$\begin{cases} 2p_1 + p_2 \geq 7p_1 \\ 7p_3 \geq 2p_3 \\ 5p_1 + p_3 \geq 5p_1 \\ 5p_2 \geq 5p_2 \end{cases}$$

where the first inequality corresponds to R not wanting to deviate if the mediator suggests r_1 to R, the second inequality corresponds to R not wanting to deviate if the mediator suggests r_2 to R, the third inequality corresponds to C not wanting to deviate if the mediator suggests c_1 to C and the fourth inequality corresponds to C not wanting to deviate if the mediator suggests c_2 to C.

Clearly, all inequalities except the first are trivially satisfied. Thus, our system is

$$\begin{cases} 2p_1 + p_2 \geq 7p_1 \\ p_1 + p_2 + p_3 = 1 \end{cases} \iff \begin{cases} p_2 \geq 5p_1 \\ p_1 + p_2 + p_3 = 1 \end{cases}$$

In order to maximize p_1, the first inequality must be tight, i.e. $p_2 = 5p_1$. As p_3 must be positive, let it be some small probability δ. Thus our CE is

$$\begin{cases} p_1 = \frac{1}{6}(1 - \delta) \\ p_2 = \frac{5}{6}(1 - \delta) \\ p_3 = \delta \end{cases}$$

2. Consider a symmetric 2 person game between Alice and Bob, with the same strategy set S for both players. Let $A(i,j)$ and $B(i,j)$ denote the payoff of Alice and Bob respectively, when Alice plays i and Bob plays j. We say that the game is symmetric if we have that $A(i,j) = B(j,i)$ for all $i, j \in S$, *i.e.*, $B = A^T$.

   (a) (2 points) Can a symmetric game have a pure Nash equilibria? (even if all values $A(i,j)$ are different?)

   (b) (2 points) Do all symmetric games have pure Nash equilibria?

   (c) (6 points) J. Nash showed that every symmetric game has a symmetric equilibrium, i.e., a probability distribution $x \in \Delta(S)$ such that (x,x) is an Nash equilibrium. Using this fact derive a quadratic program to compute a symmetric equilibrium of game (A, A^T).

(a) Yes. Consider the following game

1, 1	2, 3
3, 2	4, 4

This game is symmetric, as $B = A^T$, and has a pure Nash equilibrium, the bottom-right entry of the matrix where both players get a payoff of 4.

(b) No. Consider the rock, paper, scissors game

0, 0	-1, 1	1, -1
1, -1	0, 0	-1, 1
-1, 1	1, -1	0, 0

This game is symmetric but has no pure Nash equilibrium. Specifically, the only NE of the game is the mixed NE where both players play $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

(c) We take the quadratic program to find a Nash equilibrium (x,y) studied in class, and substitute $B = A^T$, $x = y$ and $\pi_A = \pi_B$ to get one for a symmetric game. The program is as follows.

$$
\begin{aligned}
\max \quad & x^T(A + A^T)x - 2\pi_A \\
s.t. \quad & (Ax)_i \le \pi_A \qquad \forall i \\
& (x^T A^T)_i \le \pi_A \qquad \forall i \\
& x \in \Delta_n
\end{aligned}
$$

Note that as Ax is a 1-dimensional vector, and $(Ax)^T = x^T A^T$, $(Ax)_i = (x^T A^T)_i$. Hence the second set of constraints is redundant. Also, the objective function is maximized when the payoff of both players is π_A. Hence, we can further reduce the program by eliminating π_A as find an x that satisfies,

$$
\begin{aligned}
& (Ax)_i \le x^T A x \qquad \forall i \\
& x \in \Delta_n
\end{aligned}
$$

3. The 1-dimensional Sperner's problem is defined on a 1-dimensional grid from $[0, 2^n - 1]$, with each integer being a grid point. There are two colors, red and blue, represented by 0 and 1 respectively. There is a Boolean circuit named *Color,* which outputs the color (0/1 bit) of a grid point given its bit representation, such that, Color(0) = red, Color($2^n - 1$) = blue, and the remaining grid points get any color.

   (a) (4 points) Show that there exists an integer $0 \le k \le 2^n - 1$ such that Color(k)=red and Color($k+1$)=blue. Furthermore, we can compute it in $O(n)$ calls to the Boolean circuit "Color".

   (b) (6 points) Show that checking if there are more than one such ks is NP-complete (hint: reduce from 3-SAT).

(a) To the contrary suppose no such k exists. Then, for each $k < 2^n - 1$ if Color(k)=red then Color($k+1$)=red. Since Color(0) is red, by induction it follows that Color($2^n - 1$) = red, and we arrive at a contradiction.

The following binary search algorithm will find such a k.

```
If Color(1)=blue then output k = 0. If Color(2^n − 2)=red then output k = 2^n − 2.
Initialize: l = 0 and h = 2^n − 1.    While l < h − 1 do
  k = ⌊(l + h)/2⌋.
  If Color(k)=red and Color(k + 1)=blue then output k and Break.
  If Color(k)=red then l = k else h = k
EndWhile
Output k = l.
```

Correctness. The algorithm maintains the following invariant: $l < h$, Color(l)=red and Color(h)=blue.

Since we know that there is always a $k \in [l, h]$ with Color(k)=red and Color($k+1$)=blue, either such a k will be found at the current pivot during the while loop, or at the end of the while loop when $l = (h - 1)$.

 # calls to the Color circuit. Initially $h - l < 2^n$ and in every step this difference reduces by half, therefore the while loop can execute at most $O(n)$ times each making $O(1)$ calls to Color. The remaining calls to Color is $O(1)$.

(b) Reduction from 3-SAT: Given a 3-SAT instance $\phi(x)$ with n variables $(x_1, \ldots, x_n)$ and m clauses, we will construct a Sperner instance on $[0, 2^{(n+1)} - 1]$ grid.

In what follows, by $\phi(k)$ for an integer k we mean $\phi(x)$ where $x = (x_1, \ldots, x_n)$ is the binary representation of k with x_1 being highest significant bit (hsb) and x_n being the lowest significant bit (lsb). By $int(x)$ we mean the integer represented by binary string x considering x_1 as hsb and x_n as lsb.

If $\phi(0) = 1$, then return ϕ is satisfiable. If $\phi(1) = 1$, then return ϕ is satisfiable.

Otherwise, construct a Sperner instance where Color function is implemented as follows: Let Color(0)=red and Color(1)=blue. For $k \in [1, 2^n - 1]$, Color(k)=red if $\phi(k) = 1$, otherwise Color(k)=blue. For $k \in [2^n, 2^{n+1} - 1]$ let Color(k)=blue.

Correctness. Clearly, if ϕ is satisfiable at binary representation of 0 or 1, we correctly return that it is satisfiable. If not, then we create a Sperner instance with the above implementation of Color. It creates a dummy solution at $k = 0$.

If there is another solution k' then surely $1 < k' < 2^n$ and $\phi(k') = 1$. On the other hand, if ϕ is satisfiable, then let $k' = \max\{int(x) \mid \phi(x) = 1, x \in 0, 1^n\}$. Clearly, $1 < k' < 2^n$, Color(k')=red and Color($k' + 1$)=blue. This implies that the Sperner has at least two solutions, one at $k = 0$ and another at k'.

4. (10 points) In the town of Gamica, there is exactly one police patrol, two police stations (s and t), and one robber. The robber plans to target a house on one of the streets at night, say on street $i \in [n]$. If street i is patrolled by the police then they get r_i reward while the robber gets ζ_i cost, otherwise police's cost is c_i and the robber's reward is ρ_i.

   Every night, the police can patrol exactly one route from station s to station t. At night all the streets are one way, and the street network forms a DAG. Design a polynomial time algorithm to compute Stackelberg strategy of the police.

   (Hint: What is the corresponding DBR problem? Can that be solved in polynomial time?)

Consider the DAG corresponding to the street layout of Gamica, i.e., edges of the DAG correspond to the streets, and vertices are ends of streets. Let E be the set of all $s-t$ paths in Gamica, and let each path in E be denoted by $e = [e_1 \cdots e_n]$, where $e_i = 1$ if street i is on the path, and $e_i = 0$ otherwise.

Suppose the strategy of the police is $p = [p_e]_{e \in E}$, i.e., she patrols path e with probability p_e, and suppose attacking street i is the robber's best response to p. We will compute the p that maximizes the police's utility for which i is the robber's best response. The Stackelberg strategy of the police is the p that maximizes her utility over all the strategies corresponding to each $i \in [n]$.

The following program computes the p for a given i. For this, let the probability that the police patrols street i be denoted by x_i. x_i is the sum of all probabilities that she patrols some path containing i, i.e., $\sum_{e:i \in e} p_e$.

$$
\begin{aligned}
\max \quad & x_i \cdot r_i + (1 - x_i) \cdot c_i \\
s.t. \quad & x_i \cdot \zeta_i + (1 - x_i)\rho_i \geq x_k \cdot \zeta_k + (1 - x_k)\rho_k, \quad \forall k \in [n] \backslash \{i\} \\
& x_i = \sum_{e:i \in e} p_e, \quad \forall i \in [n] \\
& \sum_e p_e = 1, p_e \geq 0, \quad \forall e \in E
\end{aligned}
$$

But this program has an exponential number of variables ($|E|$). Consider the dual. Let y_k, a_i, b be the variables corresponding respectively to the first, second and third set of constraints.

$$
\begin{aligned}
\min \quad & \sum_{k \neq i} (\rho_k - \rho_i) y_k - b \\
s.t. \quad & b \geq e_i((r_i - c_i) - \sum_{k \neq i} y_k(\rho_i - \zeta_i)) + \cdot \sum_k e_k \cdot y_k(\rho_k - \zeta_k), \qquad \forall e \in E \\
& y_k \geq 0
\end{aligned}
$$

Define $w_i = (r_i - c_i) - \sum_{k \neq i} y_k(\rho_i - \zeta_i)$, and $w_k = y_k(\rho_k - \zeta_k)$, for all $k \neq i$. As the first set of constraints must be satisfied for all paths, they must be satisfied for the max weight $s-t$ path as well. As the graph is a DAG, finding the weight of the max weight $s-t$ path can be done in polynomial time. Hence, checking if a given b, y_is are feasible can be done in polynomial time, by checking that b is greater than the weight of the max weight $s-t$

path according to the weights w_i defined above. In other words, there is a polynomial time separation oracle for the above LP.

Using the separation oracle, we solve the dual and find the optimal objective value, the expected optimal utility of the police, for each pure strategy of the robber. We return the strategy corresponding to the highest utility among these.

5. Consider the extensive-form game shown in the figure.

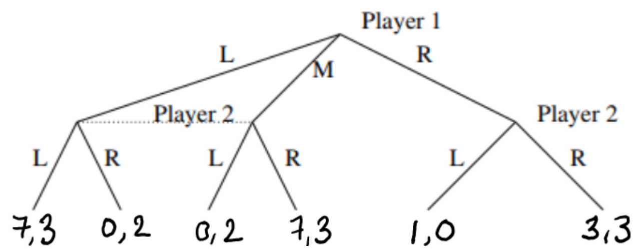

Figure 1: An extensive-form game with imperfect information.

(a) (3 points). Give the normal-form representation of this game.

(b) (3 points). Give a Nash equilibrium where player 1 sometimes plays left. (Remember that you must specify each player's strategy at *every* information set.)

(c) (4 points). Characterize the subgame perfect equilibria of the game. (Remember that you must specify each player's strategy at *every* information set.)

(a) Player 1 has 3 strategies available, while Player 2 has 4 strategies available, as Player 2 cannot distinguish between Player 1 playing strategy L or M. Let Player 1 be the row player, and Player 2 be the column player. Then the normal-form representation of the game is

	LL	LR	RL	RR
L	7, 3	7, 3	0, 2	0, 2
M	0, 2	0, 2	7, 3	7, 3
R	1, 0	3, 3	1, 0	3, 3

(b) Consider the strategy $x = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ for Player 1 and the strategy $y = \left(0, \frac{1}{2}, 0, \frac{1}{2}\right)$ for Player 2. We claim that (x, y) is a Nash equilibrium.

To see why, notice that, given x, if Player 2 plays y, they get an expected payoff of

$$\frac{1}{2}\left(\frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 2\right) + \frac{1}{2}\left(\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 3\right) = \frac{5}{2}$$

and they cannot get strictly higher payoff when Player 1 plays x no matter how they mix their strategies. Similarly, given y, the pure strategy R for Player 1 is dominated by the x. This is because, if Player 1 plays x, they get an expected payoff of

$$\frac{1}{2}\left(\frac{1}{2}\cdot 7 + \frac{1}{2}\cdot 0\right) + \frac{1}{2}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 7\right) = \frac{7}{2}$$

which is strictly higher than R's payoff for Player 1 which is equal to 1 (given y). Thus, (x, y) is a Nash equilibrium.

(c) There are 2 subgames for this game, only one of which is a proper subgame. Since any subgame perfect Nash equilibrium (SPNE) of the game has to constitute a NE in every subgame, we start from the proper subgame to eliminate as many NE of the whole game as possible.

The proper subgame is where Player 1 plays the pure strategy R. In this subgame, R strictly dominates L for Player 2. Thus, the normal form representation of the reduced game is the following.

	LR	RR
L	7, 3	0, 2
M	0, 2	7, 3
R	3, 3	3, 3

The strategy set $(1/2, 1/2, 0)$ for Player 1 for the strategies L, M, R, gives her an expected payoff of

$$\frac{1}{2}(p \cdot 7 + (1-p) \cdot 0) + \frac{1}{2}(p \cdot 0 + (1-p) \cdot 7) = \frac{7}{2},$$

when Player 2 plays LR with probability p. Hence it dominates over the strategy R that gives Player 1 a payoff of 3. Thus player 1 will never play R in any SPNE, giving the reduced game,

	LR	RR
L	7, 3	0, 2
M	0, 2	7, 3

It can be verified that the pure strategy sets (L, LR) and (M, RR) are Nash equilibria for the subgame, as well as the entire game.

Let us compute all mixed equilibria now. In every SPNE Player 1 will play $x = (q, 1-q, 0)$ and Player 2 will play $y = (0, p, 0, 1-p)$ for some $0 < p, q \leq 1$. In order for a strategy profile to be a SPNE though, it has to be a NE for the whole game as well, as any game is a subgame of itself. Thus, the set of SPNE is the set of strategies (x, y) that are NE of the whole game.

In order to guarantee that such a strategy profile is a NE of the whole game we have to guarantee that both Players do not have incentive to deviate. Player 1's expected payoff is

- $7pq$, if they play strategy L.
- $(1-q)(7(1-p))$, if they play strategy M.

In order for Player 1 to have no incentive to deviate, we have to have

$$7pq = 7(1-p)(1-q) \Leftrightarrow p + q = 1$$

Similarly, Player 2's expected payoff is,

- $3pq + 2p(1-q) = 3p(1-p) + 2p^2$, if they play strategy LR.
- $2(1-p)q + 3(1-p)(1-q) = 2(1-p)^2 + 3p(1-p)$, if they play strategy RR.

In order for Player 2 to have no incentive to deviate, we have to have

$$3p(1-p) + 2p^2 = 3p(1-p) + 2(1-p)^2 \Leftrightarrow p = 1/2 \Rightarrow q = 1/2$$

Thus, the set of SPNE of the game is the set of strategies (x, y) where (a) $x = (1/2, 1/2, 0),\ y = (1/2, 0, 1/2, 0)$, (b) $x = (1, 0, 0),\ y = (0, 1, 0, 0)$ and (c) $x = (0, 1, 0),\ y = (0, 0, 0, 1)$.

6. (10 points) Consider an atomic selfish routing game in which all players have the same source vertex and sink vertex (and each controls one unit of flow). Assume that edge cost functions are non-decreasing, but do not assume that they are affine. Prove that a pure-strategy Nash equilibrium can be computed in polynomial time. Be sure to discuss the issue of fractional vs. integral flows, and explain how (or if) you use the hypothesis that edge cost functions are non-decreasing.

   [Hint: Recall the Rosenthal's potential function. You can assume without proof that the minimum-cost flow can be solved in polynomial time. If you haven't seen the min-cost flow problem before, you can read about it in any book on "combinatorial optimization".]

Given the graph $G(V,E)$ of the routing game, consider the new multi-graph $G'(V',E')$, where $V' = V$, and for every edge $e \in E$, there are n edges $(e,i), i \in [n]$. We design a flow network over G' as follows. Let the capacity of each edge be 1, and the cost of edge (e,i) be $c_e(i)$.

Every selection of pure strategies of all players in the routing game with potential ϕ can be mapped to a flow through G' of cost ϕ as follows. If there are x units of traffic through edge $e \in G$, then send a unit flow through each edge $(e,i), i \in [x]$ in G'. If $x_{(e,i)}$ is the indicator variable for whether the flow through edge (e,i) is 1 or 0, the cost of the flow is $\sum_{(e,i)} c_e(i) \cdot x_{(e,i)}$. This can be re-written as $\sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$, which is the Rosenthal potential of the game, hence equal to ϕ.

It then follows that if there is a selection of strategies of all players corresponding to a min-cost flow through G', then it minimizes the Rosenthal potential function, hence is a Nash equilibrium of the game. We now show there is such a min-cost flow, by showing two properties.

First, we know that there is an integral min-cost flow from s-t in G', as the edge capacities are all integers. We will consider an integral min-cost flow.

Next, if there is non-zero (hence, as the flow is integral, 1 unit) flow through edge (e,i) in the min-cost flow, we can assume there is a unit flow through all (e,i') with $i' < i$, by re-directing the flow through the edges (e,i') if this is not so. As the cost functions are non-decreasing, no such re-direction can increase the cost of the resulting flow.

Thus, consider the integral min-cost flow that for every edge $e \in G$, has a unit flow through all edges (e,i_e) for some $i_e \in [n]$ and zero flow through all edges (e,i_e^+), for all $i_e^+ > i_e$. Such a flow can be mapped to a pure strategy set of the players, by directing i_e units of traffic through edge e for all e. As this strategy set corresponds to a min-cost flow, it minimizes the Rosenthal potential, hence is a Nash equilibrium.