CS 598RM: Algorithmic Game Theory, Fall 2020
HW 1 Solutions

Competitive Equilibrium: Warm-up

1. (3 points) Compute equilibrium prices and allocation for the following Fisher market. Show that the resulting allocation is Pareto optimal.

   Market with two agents $A = \{1, 2\}$, and two goods $G = \{1, 2\}$. Budgets of the agents are $B_1 = 5, B_2 = 2$, and their utility functions are $v_1(x_{11}, x_{12}) = 3x_{11} + 4x_{12}$ and $v_1(x_{11}, x_{12}) = x_{11} + 2x_{12}$.

2. (7 points) *(fairness properties)*

   Given a Fisher instance where budget of agent i is B_i (budget can also be thought of as weight/clout/importance for a fair-division task), show that a CE allocation satisfies the following:

   - It is weighted envy-free
   - It is weighted proportional
   - The weighted Nash welfare maximizing allocation gives a CE.

(a) We follow the steps of the algorithm discussed in lecture.

- We want to set initial prices such that $p_j \leq \min_i B_i/2 = 1$, and every item is in the MBB set of some agent. To satisfy the latter condition, we will set prices such that some agent has both goods in her MBB set. Without loss of generality, let this be agent 2, and set prices in the ratio of values of agent 2 for the items. Then to have the former condition, we will set prices to $[\frac{1}{2}, 1]$. It can be verified that the MBB sets at these prices are: Agent $1 : \{g_1\}$, Agent $2 : \{g_1, g_2\}$. At these prices, as both agents have budgets higher than the sum of prices of goods, there will be no tight-set. Hence, all goods are dynamic.
- We increase prices of all goods until some subset of goods becomes tight. At prices $[1, 2]$, we get the min-cut $\{s, g_2, a_2\}$. This is because a_1 can exhaust all her budget on g_2. Hence g_2 goes into the frozen set.
- Now increase the price of g_1 until either g_2 again becomes dynamic, or $\{g_1\}$ becomes a tight subset. In other words, either g_2 is an MBB good for a_1, or $a_1's$ budget is exhausted.
- At prices $[1.5, 2]$, g_2 becomes dynamic again. Hence, now we increase prices of both goods until we find a new tight subset.
- At prices $[3, 4]$, both goods become tight. Hence, both agents exhaust their budget, and every good is sold. The max flow gives the allocation g_1 to a_1 and half of g_2 to each of a_1 and a_2. As the market clears and every agent buys only MBB goods, this is a CE.

Pareto Optimality. Let $X^* = [X_1^*, X_2^*]$ be the equilibrium allocation. Suppose another allocation $Y = [Y_1, Y_2]$ Pareto dominates X^*. Then some agent, without loss of generality say agent 1, gets a bundle of value strictly more than her value for X_1^*. By definition of CE, agent 1 received the optimal valued bundle of price $B_1 = 5$. Hence the price of Y_1 has to be higher than 5. The total price of both items according to the equilibrium prices is 7, hence agent 2 receives a bundle of price less than 2. But as Y Pareto dominates X^*, her value for the bundle is at least equal to that of X_2^*. But then at the given prices, agent 2 could afford the bundle Y_2 and get higher value than she did in the CE, a contradiction.

(b) **1. Weighted envy-free.** Consider any pair of agents i, k. At the CE prices, agent i can afford to buy a fraction $x_k^* \frac{B_i}{B_k}$ of agent k's bundle. As i receives a bundle of highest possible value that she can afford (of price at most her budget), we have,

$$v_i(x_i^*) \geq v_i\left(x_k^* \frac{B_i}{B_k}\right) = \frac{B_i}{B_k} v_i(x_k^*),$$

where the last inequality follows as the valuation function v_i is additive.

Thus, we have weighted envy-freeness for all pairs of agents.

2. Weighted proportional. We know the CE allocation is weighted envy-free. Hence, for every agent i, we have, for all

$$\forall i' : \quad \frac{v_i(A_i)}{B_i}(B_{i'}) \geq v_i(A_{i'}).$$

Adding all these inequalities, we get,

$$\frac{v_i(A_i)}{B_i} \sum_{i'} B_{i'} \geq \sum_{i'} v_i(A_{i'}) \Rightarrow v_i(A_i) \geq B_i \frac{v_i(G)}{\sum_{i'} B_{i'}},$$

establishing weighted proportionality.

3. Weighted MNW allocation gives CE.

The optimization program for weighted MNW is as follows.

$$\begin{aligned} \max \sum_i B_i \log(\sum_j v_{ij} x_{ij}), \\ \sum_i x_{ij} \leq 1, \quad \forall j \in G, \\ x_{ij} \geq 0, \quad \forall i \in A, j \in G. \end{aligned}$$

An optimal solution to the program exists, as the feasible space is non-empty. Any integral allocation of goods is feasible. Hence, the KKT conditions are satisfied.

Using the KKT conditions, we will show the allocation and the dual variables of the first set of constraints together are a CE.

Let $p_j, j \in G$ be the dual variables corresponding to the first set of constraints, and $\lambda_{ij}, i \in A, j \in G$ those for the second set. Let $u_i := \sum_j v_{ij} x_{ij}$, the total value of agent i for allocation x. Then the duality conditions for x_{ij} ensure:

- $\frac{B_i}{u_i} v_{ij} - p_j - \lambda_{ij} = 0$
- $p_j \geq 0$
- $\lambda_{ij} \leq 0$

The first and third imply $\frac{u_i}{B_i} \geq \frac{v_{ij}}{p_j}$, for every j. This means that, at prices p_j, the bang-per-buck of the agent for her MNW allocation is at least that from any good. The complementary slackness conditions for the second set of constraints have $\lambda_{ij} x_{ij} = 0$, hence when $x_{ij} > 0$, that is, when a good is allocated in non-zero amount, $\frac{u_i}{B_i} = \frac{v_{ij}}{p_j}$, it gives the highest bang-per-buck value to the agent.

The complementary slackness conditions for the first constraints imply, $p_j(\sum_j x_{ij} - 1) = 0$. Hence, any good with price $p_j > 0$, has $\sum_j x_{ij} = 1$, that is, is fully allocated.

Finally, the total money spent by the agent is $\sum_j p_j x_{ij}$. We have,

$$\sum_j p_j x_{ij} = \sum_j \frac{B_i}{u_i} v_{ij} x_{ij} = \frac{B_i}{u_i} \sum_j v_{ij} x_{ij} = \frac{B_i}{u_i} u_i = B_i.$$

Hence, every agent spends all her money according to (x, p).

To summarize, at the MNW allocation x, according to the prices p corresponding to the dual variables, we have that every agent spends all her money, only on goods that are in her MBB set, and every good is fully allocated. Hence, the allocation is a CE.

Proportional response (PR) dynamics

Consider a proportional response function $f : D \to D$, where $D = \{\mathbf{b} \in \mathbb{R}_+^{mn} \mid \sum_{j\in G} b_{ij} = B_i\}$: for a $\mathbf{b} \in D$, if $\mathbf{b}' = f(\mathbf{b})$ then construct $\mathbf{b}'$ from $\mathbf{b}$ as follows:

Think of b_{ij} as the bid of agent i on good j. Price of good j is the total bids collected on it, i.e., $p_j = \sum_{i\in A} b_{ij}$. And the allocation is proportional to the bid, i.e., $x_{ij} = \frac{b_{ij}}{p_j}$. (In economics such a market implementation is known as *Trading-post*, introduced by Shapley and Shubik in 1977.)

Based on the utilities obtained, agents update their bids (proportional to the utility received from the previous bid):

$$b'_{ij} = B_i \frac{v_{ij} x_{ij}}{\sum_{k\in G} v_{ik} x_{ik}}$$

Show that, given a CE (p^*, X^*) the corresponding bids $b^*_{ij} = p^*_j x^*_{ij}$ for all (i, j) forms a fixed-point of f, i.e., $f(\mathbf{b}^*) = \mathbf{b}^*$.

Remark. In the proportional response dynamics agents update their bids as per function f every day. That is, starting with an arbitrary bid profile $b(0) \in D$, bids on day $t \geq 1$ is $b(t) = f(b(t-1))$. This is a well-studied dynamics that is known to converge to the fixed-point a.k.a. CE. The dynamics can be extended to more general utility functions like CES, gross-substitutes, and is known to converge for these too. See slides for the references.

We want to show that $f(\mathbf{b}^*) = \mathbf{b}^*$. That is, we must show that for all $i \in [n], j \in [m]$, the new bid corresponding to b^*_{ij} according to the update rule, is equal to b^*_{ij}.

The updated bid, say b'_{ij}, is,

$$\begin{aligned}
b'_{ij} &= B_i \frac{v_{ij} x_{ij}}{\sum_{k\in G} v_{ik} x_{ik}} \quad \text{//apply given update rule} \\
&= B_i \frac{v_{ij} \frac{b^*_{ij}}{p_j}}{\sum_{k\in G} v_{ik} \frac{b^*_{ik}}{p_k}} \quad \text{//as allocation is proportional to the bid} \\
&= B_i \frac{\frac{v_{ij}}{p_j}(b^*_{ij})}{\frac{v_{ij}}{p_j} \sum_{k\in G} b^*_{ik}} \quad \text{//CE allocates MBB goods, thus } x_{ij} \neq 0 \Rightarrow \text{same bang-per-buck value} \\
&= B_i \frac{b^*_{ij}}{\sum_{k\in G} b^*_{ik}} \\
&= b^*_{ij}.
\end{aligned} \tag{1}$$

On the computation of CE

1. (7 points) Show that, for the case of Fisher model with binary valuations, *i.e.*, $v_{ij} \in \{0,1\}$ for all $(i,j) \in A \times G$, the algorithm we discussed terminates in $O(n)$ many iterations of the outer while loop (recall that $n = |A|$); take the starting prices to be $p_j = \epsilon$, $\forall j \in G$ where $0 < \epsilon < \min_{i \in A} B_i/m$.

   Remark. Observe that both the events of our algorithm can be computed in *strongly polynomial-time*. Therefore, the proof of the above statement shows that the algorithm runs in strongly polynomial time for binary instances.

2. (3 points) Consider a bi-valued HZ instance: for each agent i her value for good for good j is $v_{ij} \in \{a_i, b_i\}$ for all $j \in G$, where $0 \le a_i < b_i$. Reduce the computation of HZ equilibrium for this instance to finding HZ equilibrium in a binary valued instance where for all (i,j) pairs $v_{ij} \in \{0,1\}$.

(a) An iteration of the outer while loop terminates when either (a) a new tight-set is formed, that is, a subset of goods from the dynamic set freeze, or (b) goods from the frozen set become dynamic again, that is, an MBB edge is formed between some good in the frozen set and some dynamic agent. We will show that the latter condition never occurs. Then in every iteration, at least one good becomes frozen, hence the loop terminates in $O(n)$ iterations.

When valuations are binary and we start with the same price for all goods, the MBB value of every agent is $1/p_j$, as the only other bang-per-buck value is 0 for her zero valued goods, which is strictly smaller. Thus, all the goods valued at 1 by an agent are in her MBB. When a set of goods and agents is frozen, there is no MBB edge between the frozen goods and the dynamic agents. Hence, for all frozen goods, all agents who value the good at 1 are already frozen. When the prices of the dynamic goods are increased by a factor α, the MBB value of the dynamic agents change to $1/(\alpha p_j) > 0$, hence there will never be a new MBB edge from a good to an agent who has value 0 for the good. Particularly, there will be no MBB edge created between any dynamic agent and frozen good, hence no frozen good becomes dynamic again. Therefore, we only get Event 1 in every iteration, which reduces the size of the dynamic set by at least 1.

(b) The reduction is as follows. The set of agents and goods is the same. The valuations u_{ij} for the binary instance are defined based on the given valuations v_{ij} as,

$$u_{ij} = \begin{cases} 1 & \text{if } v_{ij} = b_i, \\ 0 & \text{otherwise.} \end{cases}$$

It suffices to show that for any set of prices, optimal bundles for the bi-valued instance and the reduced binary instance are the same. For unit bundles $\sum_j x_{ij} = 1$ and $\sum_j x'_{ij} = 1$,

$$x \text{ is optimal for the bi-valued instance} \equiv \sum_j v_{ij} x_{ij} \ge \sum_j v_{ij} x'_{ij}$$

$$\equiv \frac{1}{b_i - a_i} \sum_j v_{ij} x_{ij} - \frac{a_i}{b_i - a_i} \sum_j x_{ij} \ge \frac{1}{b_i - a_i} \sum_j v_{ij} x'_{ij} - \frac{a_i}{b_i - a_i} \sum_j x'_{ij}$$

$$\equiv \sum_j \frac{v_{ij} - a_i}{b_i - a_i} x_{ij} \ge \sum_j \frac{v_{ij} - a_i}{b_i - a_i} x'_{ij} \equiv \sum_j u_{ij} x_{ij} \ge \sum_j u_{ij} x'_{ij}$$

$$\equiv x \text{ is optimal for the binary instance.}$$

EFX

The following questions are regarding fair-division of a set of indivisible goods.

1. (2 points) Show an example with additive valuations for which the envy-cycle procedure does not give an EFX allocation.
2. (3 points) Show that an EFX allocation exist when agents have identical general monotone valuations.
3. (3 points) Design a polynomial-time algorithm to obtain an EFX allocation when agents have identical additive valuations.
4. (2 points) Design a polynomial-time algorithm to obtain an EFX allocation when there are two agents with additive valuations.

(a) The intuition for the counter example is as follows. We know the envy-cycle algorithm generates an EF1 allocation. Hence, we want EF1 to not be equal to EFX. Also, the source agent gets the last item, possibly creating new envy edges towards her. As these did not exist previously, removing the last item removes the envy. Hence, we can create an example where the last item added must have higher value than the previously added items.

Consider two agents and three items. The agent valuations are identical. Both agents have values $10, 20, 30$ for the items.

Suppose we run the envy cycle algorithm. The first two items go to distinct agents. The agent with the first item envies the other, hence gets the third item. The combined total value of the first and third item is higher than the second for both agents. Hence, the second agent envies the first. However, removing the lower valued item, of value 10, does not eliminate this envy, hence this is not an EFX allocation.

(b) Consider the MMS partition of any agent. As all agents are identical, this is an MMS allocation for all. We describe a procedure to obtain an EFX algorithm by modifying this allocation.

Order all agents in increasing order of values of bundles. If the allocation is not EFX then agent 1 must envy an agent $i > 1$ even after removing a good. Take the smallest such i. Then $v(A_1) < v(A_i \backslash \{g\})$, for some good $g \in A_i$. Take g from agent i and re-allocate to agent 1. We have that $v(A_1 \cup \{g\}) \geq v(A_1)$ and $v(A_i \backslash \{g\}) \leq v(A_i)$, as the valuations are monotone. Then as $v(A_i \backslash \{g\}) > v(A_1)$, the value of i's bundle after removing g is more than MMS. Hence, $v(A_1 \cup \{g\}) = v(A_1)$, otherwise the new allocation has higher MMS value. Thus, agent 1 still has the lowest valued bundle after obtaining g.

Repeat this process until the EFX property holds for all i with 1. As the allocation is EFX according to 1, it is EFX according to every agent. The process will converge in at most m steps, as every step adds one item to the fixed smallest bundle (of agent 1) and there are at most m items. Hence, an EFX allocation always exists.

Note: One can also start with any allocation instead of the MMS allocation. Then while the identity of agent 1 may change, the value of the smallest bundle will always increase. As this value cannot be arbitrarily high, this process also terminates.

(c) The algorithm is as follows. Order all items in decreasing order of valuations. Assign items via the envy cycle procedure.

We claim that this algorithm gives an EFX allocation after allocating the first j goods, for every $j \in [m]$, and prove this by induction on j. For the base case, when $j = 1$, the allocation is EFX by definition. Suppose the allocation is EFX after assigning j goods, for all j less than some t. Then the $(t+1)^{st}$ good is assigned after eliminating all envy-cycles, to a source agent, say s. For every agent $i \neq s$, $v_i(A_i) \geq v_i(A_s)$, for the bundle A_s assigned before allocating the last good. Every new envy edge created after assigning this good and eliminating cycles will be towards the agent receiving $A_s \cup \{t+1\}$. For the agent who envies this bundle, we have $v_i(A_i) < v_i(A_s \cup \{t+1\})$, and $v_i(A_i) \geq v_i(A_s)$ as s was a source agent. As $t+1$ is the last item allocated, it is the lowest valued item in $A_s \cup \{t+1\}$. Hence, all new envy edges satisfy the EFX property. All other envy edges satisfy this property by inductive hypothesis, completing the proof.

Hence, the allocation after assigning all goods, returned by the algorithm, is EFX.

(d) The algorithm is as follows: Create an EFX allocation to one agent's valuation function. Assign to the other agent the higher valued bundle according to their valuation function.

This allocation is EFX according to the first agent's valuation function, and the second agent does not envy the first agent's bundle, hence EFX for the second agent too. Thus, the algorithm gives an EFX allocation.

MMS and Prop1

Suppose there are m indivisible goods, and n agents with additive valuations.

1. (2 points) Show that MMS allocations exist when $n = 2$.
2. (3 points) EF1 implies 1/n-MMS.
3. (2 points) Show an example where an MMS allocation is not EF1.
4. (3 points) Show that envy-freeness up to one item (EF1) implies proportionality up to one item (Prop1), but Prop1 does not imply EF1.

We use the following in solving (a) and (b). Denote agent i's value for all items by V_i. First, by definition of MMS, the MMS value of agent i is the value of the smallest bundle in her MMS partition. By pigeonhole principle, this can be at most the average value of all the items, i.e., MMS $\leq V_i/m$.

(a) The cut and choose protocol gives an MMS allocation when $n = 2$. The agent who cuts will divide all items according to her MMS partition. The other agent chooses the bigger part according to her, hence gets a bundle of value at least half of all goods, thus at least her MMS. The first agent has at least MMS value for both parts, hence also gets an MMS valued bundle. Thus, MMS allocations always exist when $n = 2$.

(b) By definition of EF1, for every agent i, we have

$$\forall k \in [n],\ \exists g_k \in [m]: \quad v_i(A_i) \geq v_i(A_k \backslash \{g\}) = v_i(A_k) - v_i(g_k)$$

Adding these inequalities for all $k \neq i$, we get,

$$(n-1) * v_i(A_i) \geq \sum_{k \neq i, k \in [n]} v_i(A_k) - \sum_{k \neq i} v_i(g_k).$$

$$\Rightarrow n * v_i(A_i) \geq \sum_{k \in [n]} v_i(A_k) - \sum_{k \neq i} v_i(g_k). \tag{2}$$

Now consider the MMS partition of agent i. There is at least one part which does not have any item from the g_ks above. The MMS value of this agent is at most the value of this part. This part can have at most all other items except the g_ks, thus have value at most $V_i - \sum_{k \neq i} v_i(g_k)$. Hence, $MMS \leq v_i(A_k) - \sum_{k \neq i} v_i(g_k)$. Combined with equation (2), we get,

$$n * v_i(A_i) \geq MMS \Rightarrow v_i(A_i) \geq \frac{1}{n} MMS.$$

(c) Intuitively, we want to create an instance where the MMS value of agents are low. Even after giving all except some (say one) agents an MMS valued bundle, the last few agents should get bundles of value far bigger than MMS, and also with a lot of goods. These agents' allocations will violate the EF1 property.

Consider 3 agents, denoted as $1, 2$ and 3, and 5 goods $g_j, j \in [5]$. The valuations of the agents are as follows. Agent i values good g_i at 1, and the remaining goods from the first 3, i.e. $g_j, j \in [3]$, at 0. All agents value the last two goods at 50 each. It can be verified by enumerating all allocations that every agent's MMS partition can allocate goods g_1, g_2, g_3 to one agent, g_4 to another agent, and g_5 to the third agent. Every agent's MMS value is thus 1.

Consider the allocation that gives agents 1 and 2 the goods g_1, g_2 respectively, and gives the remaining goods to agent 3. This is an MMS allocation. However, after removing any one good from 3's allocation, the value of her bundle for both 1 and 2 is higher than 1, hence it is not EF1.

(d) EF1⇒Prop1. Let v_i^* be the good of highest value to agent i. Then from equation (2), we have,

$$n * v_i(A_i) \geq \sum_{k \in [n]} v_i(A_k) - (n-1)v_i^*,$$

$$\Rightarrow v_i(A_i) + v_i^* \geq \frac{1}{n} \sum_{k \in [n]} v_i(A_k), \tag{3}$$

hence the given allocation is Prop1.

Prop1$\not\Rightarrow$ EF1. Equation (3) is satisfied for every agent. The intuition to design a counter example is that the agent's allocation must have negligible value, and most of the value that satisfies equation (3) must come from v_i^*. Some other agents gets most of the goods, which violates EF1.

Consider two agents and two goods. Both agents have value 1 for each good. Consider the allocation where one agent gets no good and the other gets both. This allocation is Prop1 for the first agent, as adding the value of any one good gives her total value 1, which is equal to the average value of all items. The second agent values her own allocation more than the average, hence this is a Prop1 allocation for her as well.

But removing any good from the second agent's bundle still keeps the remaining bundle of higher value than agent 1's allocation, hence this is not an EF1 allocation.

Max Nash Welfare w/ Indivisible Goods

Consider a fair-division instance with set M of m indivisible goods, and n agents with additive valuations. Show that an allocation that maximizes the Nash welfare (MNW) over the set $\Pi(M)$ of feasible integral allocations, i.e., $\arg\max_{(A_1,\ldots,A_n)\in\Pi(M)} \Pi_{i=1}^n v_i(A_i)$,

1. (4 points) is EF1 + PO.
2. (2 points) may not be EFX.
3. (4 points) is EFX when agents have identical valuations.

(a) PO: Proof by contradiction. Suppose the MNW allocation was not PO. Then there is another Pareto dominating allocation, say P. As all agents receive in P at least the same valued bundles as in the MNW allocation, and at least one agent gets a higher valued bundle, the Nash welfare product of P will be higher than the MNW allocation, a contradiction.

EF1: Proof by contradiction. Suppose the MNW allocation was not EF1. Then there is a pair of agents, say 1 and 2, such that $v_1(A_1) < v_1(A_2\backslash\{g\})$, for every $g \in A_2$. We will show that there is some good g in A_2 such that removing it from A_2 and giving it to A_1 increases the Nash welfare product, a contradiction.

We want,

$$\frac{(v_1(A_1)+v_1(g))(v_2(A_2)-v_2(g))}{v_1(A_1)v_2(A_2)} > 1 \Leftrightarrow \left(1+\frac{v_1(g)}{v_1(A_1)}\right)\left(1-\frac{v_2(g)}{v_2(A_2)}\right) > 1$$

$$\Leftrightarrow 1+\frac{v_1(g)}{v_1(A_1)}-\frac{v_2(g)}{v_2(A_2)}-\frac{v_1(g)v_2(g)}{v_1(A_1)v_2(A_2)} > 1 \Leftrightarrow \frac{v_1(g)}{v_1(A_1)} > \frac{v_2(g)}{v_2(A_2)}\left(1+\frac{v_1(g)}{v_1(A_1)}\right)$$

$$\Leftrightarrow \frac{v_1(g)}{v_2(g)} > \frac{v_1(A_1)+v_1(g)}{v_2(A_2)}.$$

If we show some good for which the last inequality is true, then so is the first in this series of implications, completing the proof. We proceed to prove this.

Consider the good from A_2 with the highest value of the ratio $\frac{v_1(g)}{v_2(g)}$, and for which $v_2(g) \neq 0$. As 1 envies 2, there is at least one good positively valued by 1 in A_2; if 2 did not value this good positively, we could re-allocate it to 1 and increase the Nash welfare product. Hence, such a good g is well-defined.

Simple algebra shows for any two fractions $\frac{a}{b} > \frac{c}{d}$, we have $\frac{a}{b} > \frac{a+c}{b+d}$.

Hence,

$$\frac{v_1(g)}{v_2(g)} \geq \frac{\sum_{j\in A_2:v_2(j)>0} v_1(j)}{\sum_{j\in A_2:v_2(j)>0} v_2(j)} = \frac{\sum_{j\in A_2:v_2(j)>0} v_1(j)}{v_2(A_2)} \geq \frac{\sum_{j\in A_2} v_1(j)}{v_2(A_2)} > \frac{v_1(A_1)+v_1(g)}{v_2(A_2)}.$$

Here the second-last inequality follows for the case when there is no good valued positively by 1 and zero by 2. This is without loss of generality, as otherwise we could re-allocate this good to 1 and increase the Nash welfare.

The last inequality follows as 1 envies 2 even upon removing any good, hence even g, from $2's$ allocation.

(b) From part (a), we know that the good with the highest ratio of $v_1(g)/v_2(g)$ and non-zero value can be re-allocated to reduce envy and increase NSW. The intuition is that this good should not be the lowest valued good for agent 2, hence showing the allocation is not EFX.

Consider two agents and three goods. The values of the agent 1 are $1, 0, 100$, and of 2 are $0, 1, 10$. Then the MNW allocation gives goods 1 and 3 to agent 1, and 2 to agent 2. 2 envies $1's$ allocation even after removing good 1 from A_1, hence this allocation is not EFX.

(c) We will assume there is no good valued zero in all except the smallest valued bundle, as all such goods can be re-allocated to the smallest bundle without changing the Nash welfare.

For contradiction, assume the MNW allocation is not EFX. Then there is a pair of agents, say 1 and 2, such that $v(A_1) < v(A_2 \backslash \{g\})$, where g is the smallest valued good from A_2. Form a new allocation by re-allocating g to 1. Let $u := v(\{g\}), v_1 := v(A_1)$ and $v_2 := v(A_2)$. The ratio of the Nash welfare of the new and old allocations is,

$$\frac{(v_1 + u)(v_2 - u)}{v_1 v_2} = \left(1 + \frac{u}{v_1}\right)\left(1 - \frac{u}{v_2}\right) = 1 + \frac{u(v_2 - v_1 - u)}{v_1 v_2} > 1.$$

Here the last inequality follows as $v_1 < v_2 - u$ from the assumption that the allocation is not EFX.