LECTURE 22 (April 10^{th})

TODAY Area Laws

RECAP

Schmidt Decomposition Any state $|\psi\rangle$ can be written as

$$|\psi\rangle = \sum_{k=1}^{r} \sqrt{\lambda_k}\, \underbrace{|u_k\rangle_A \otimes |v_k\rangle_B}_{\text{orthonormal set}}$$

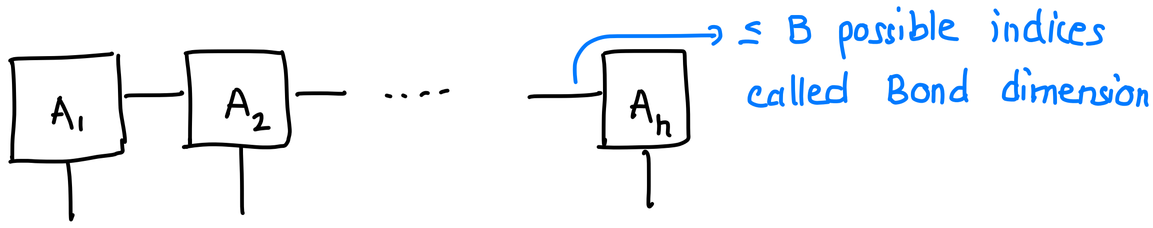
where Alice has A and Bob has B

Entanglement Entropy r is called the schmidt rank of $|\psi\rangle$ across cut (A,B)

$\sum_{k=1}^{r} \lambda_k \log \frac{1}{\lambda_k}$ is the entanglement entropy of $|\psi\rangle$ across (A,B)

Schmidt rank & entropy being small means that state has less entanglement across cut (A,B)

Matrix Product State

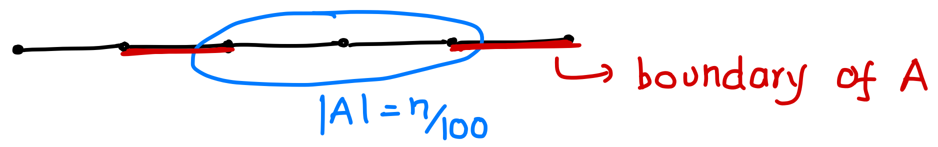


Area Laws

The area law conjecture says that any ground state $|\psi\rangle$ of a physically-relevant Local Hamiltonian has area law behavior, i.e.

> For any subset $A \subseteq [n]$ of qubits, the entanglement entropy is proportional to the size of the boundary of A (i.e. proportional to the area)

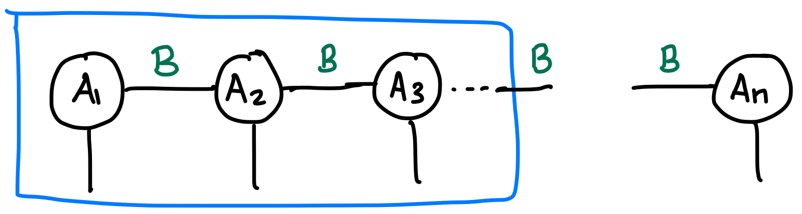
E.g. in 1-dimension :



Area law behavior : entanglement entropy $= O_d(1)$

In general, entanglement entropy could be as large as $\sim |A| \log d \sim n \log d$

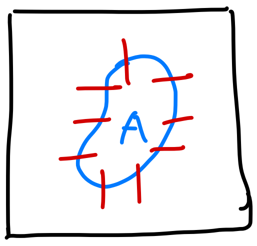
Also note that a matrix product state with constant bond dimension satisfies the area law across any cut



Entropy $\leq \log B$

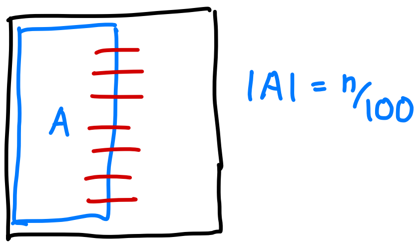
Schmidt-rank $\leq B$

In fact, one can also conjecture that ground states for 1-dimensional "physical" Hamiltonians must have a MPS description in 2-dimension:



Area law behavior

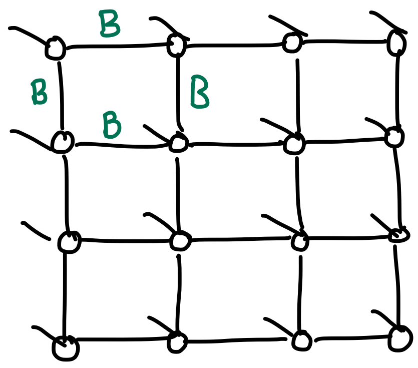
Entropy $\sim |\partial A|$



Area law behavior

Entropy $\sim \sqrt{n}$ as opposed to $n \cdot \log d$

One can also make a stronger conjecture about PEPS representations of such ground states



This conjecture is not true for a general local Hamiltonian but physicists believe it holds for most physically relevant Hamiltonians

What kind of extra conditions do we need (apart from grid Hamiltonians)?

① Spectral gap — There is a constant gap between the ground energy and second lowest energy

② Degeneracy of Ground space — If the ground state is unique it is conjectured that spectral gap is all that is needed

If the ground state is not unique, one can still expect the area law for instance for the maximally mixed state over the ground space but not all states in the ground space*

Of course, more possibilities are there for area laws

This conjecture was proven by Hastings & subsequent works in 1-dimension

In fact, such ground states also have an (approximate) MPS description

In 2- and higher dimension, the conjecture is still open, although there are some developments

A generalized area law on an arbitrary graph does not hold

Area Law in 1-dimensions

We will prove the area law for 1-dimensional Hamiltonians after making a few more assumptions

Let $H = \sum_{i=1}^{n-1} H_i$ where H_i acts on qudits i & $i+1$

We will assume that

1. ground state $|\psi\rangle$ of H is unique

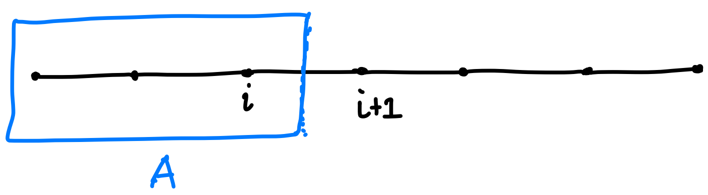
2. each H_i is a projector, i.e., $H_i^2 = H_i$

3. $\lambda_{min}(H) = 0$ → Since $0 \preceq H_i$, this can only happen if $|\psi\rangle$ is also the ground state of each local term H_i, i.e., $\langle\psi|H_i|\psi\rangle = 0$ as well

   Such a Hamiltonian is called frustration-free

4. spectral gap: the second lowest eigenvalue is $\Omega(1)$

$\boxed{\text{Theorem}}$ For any cut $(i, i+1)$, the entanglement entropy of $|\psi\rangle$ is $O(1)$.



The proof is a bit involved, so let us look at even simpler cases first

1. Diagonal Hamiltonians — In this case, the ground state is classical, i.e. a computational basis state $|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_n\rangle$, so an area law trivially holds

2. Commuting Hamiltonians — H_i & H_j commute $\forall i,j$ i.e. $H_i H_j = H_j H_i$

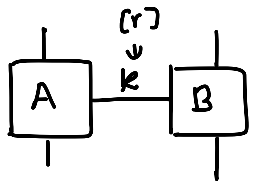
This case is non-trivial as the ground state could be entangled, but the proof is simple and illustrate the approach we will try for the general case

First, we need the notion of Schmidt Decomposition of an operator

We saw that for a cut (A,B), we can write a state

$$|\psi\rangle_{AB} = \sum_{k=1}^{r} \sigma_k |u_k\rangle_A \otimes |v_k\rangle_B$$

Similarly, we can also write an operator

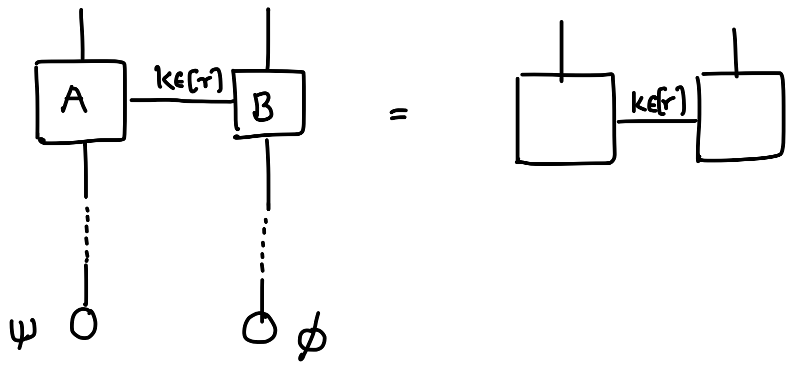


$= M = \sum_{k=1}^{r} A_k \otimes B_k$ where A_k only acts on A & B_k only acts on B

Exercise: Give a tensor network proof of this

If Schmidt-rank of operator M denoted $SR(M) \leq r$, then applying it to a product state (which has schmidt rank 1) can increase the schmidt rank by at most r

$$M|\psi\rangle_A \otimes |\phi\rangle_B = \sum_{k=1}^{r} A_k|\psi\rangle \otimes B_k|\phi\rangle$$



For a general state $|\psi\rangle_{AB}$, the Schmidt rank increases by a factor of r if we apply M

Let's go back to the area law proof: $H = \sum H_i \otimes \mathbb{I}$ where H_i's are projectors that commute

Moreover, ground state $\langle\psi| H_i |\psi\rangle = 0 \;\; \forall i$

This means that $|\psi\rangle \in$ orthogonal complement of the space on which H_i projects

projector on this space is $\mathbb{1} - H_i$

Now, consider the operator $(\mathbb{1}-H_1)(\mathbb{1}-H_2) \cdots (\mathbb{1}-H_{n-1})$

Claim This is a projector with schmidt rank at most d^2 and it projects on to the ground state $|\psi\rangle$, i.e.

$$P = (\mathbb{1}-H_1) \cdots (\mathbb{1}-H_{n-1}) = |\psi\rangle\langle\psi|$$

Before proving this claim, let us see why this would imply an area law

Take any product state $|\phi\rangle$, say a computational basis state, that has non-zero overlap with $|\psi\rangle$. We have that $SR(|\phi\rangle) = 1$.

Then, consider the state $\frac{P|\phi\rangle}{\|P|\phi\rangle\|} = |\psi\rangle$ since $P = |\psi\rangle\langle\psi|$

Moreover, since $SR(P) \leq d$, $SR(|\psi\rangle) \leq d \cdot SR(|\phi\rangle) = d$

↳ The normalization does not affect the schmidt rank

Thus, Entanglement entropy across any cut is $\log d^2 = O(1)$

This proves the area law in the commuting case assuming the claim

<u>Proof of claim</u> $P^2 = P$ since each term commutes and $\mathbb{1} - H_i$ is a projector

Moreover, $\langle\psi|P|\psi\rangle = 1$ so $|\psi\rangle \in \text{span}(P)$

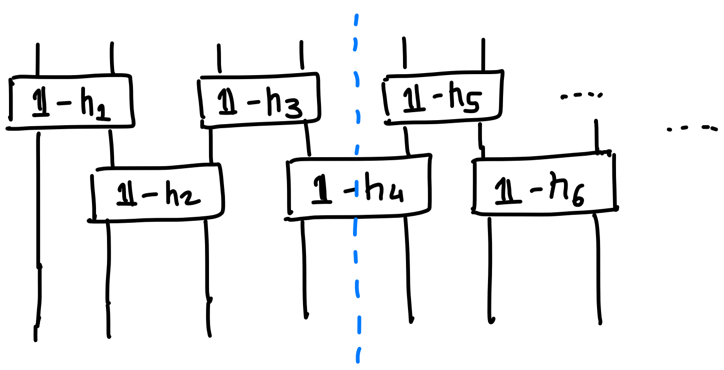
Last, any state $|\sigma\rangle \in \text{span}(P)$ must have ground energy 0 and since the ground state is unique by assumption

$$P = |\psi\rangle$$

Why is the Schmidt rank of P small ?

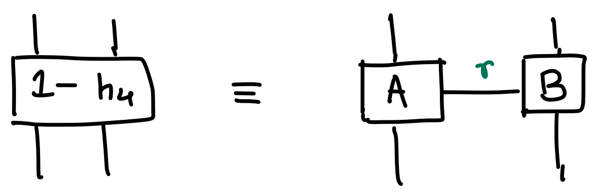
Note each operator $\underbrace{\mathbb{1} - H_i}_{= \underbrace{(\mathbb{1} - h_i)}_{d^2 \times d^2 \text{ matrix}} \otimes \mathbb{I}}$ is acts non-trivially on qubits i & $i+1$

P =



→ suppose this is the cut

Let us write



Schmidt rank $r \leq d^2$

It follows that $SR(P) \leq d^2$

NEXT TIME Non-commuting case and complexity of quantum states