LECTURE 21 (March 3rd)

TODAY Tensor Networks & Area Laws

A quantum state on n qubits lives in a 2^n-dimensional space
This says that if we want to describe a physical system generically we will need to specify an exponential amount of information

But in practice we only care about a small corner of this exponential Hilbert space, e.g., states computed by a poly-size circuits, ground states of physically relevant Hamiltonians

Tensor Networks are a very powerful tool to describe such states with fewer parameters

Basics of Tensor Networks

A tensor is an array of numbers with a bunch of indices

e.g. A_{i_1, i_2, i_3, i_4}

We can view it as a vector $\sum A_{i_1 i_2 i_3 i_4} |i_1, i_2, i_3, i_4\rangle$

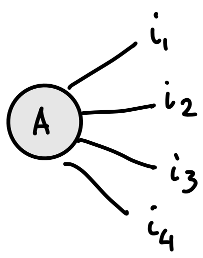
OR as a matrix/linear map $\sum A_{i_1, i_2, i_3, i_4} |i_1\rangle\langle i_2, i_3, i_4|$

OR

$$\sum A_{i_1 i_2 i_3 i_4} |i_1, i_2\rangle\langle i_3, i_4|$$

We are now going to represent tensors visually — a dot with a bunch of lines coming out of it

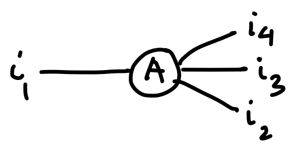
The tensor from before will be represented as



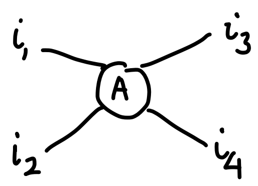
If we fix specific values, such as $i_1 = 1$, $i_2 = 2$, $i_3 = 3$, $i_4 = 2$
The tensor spits out the number A_{1232}

Viewing this tensor as a vector corresponds to the above view

One can also view this as a linear map

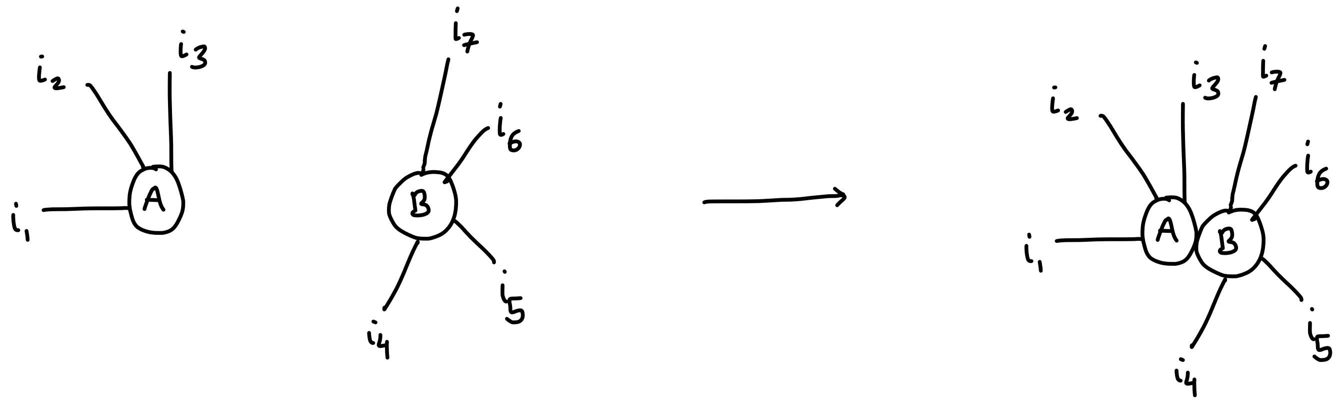


OR



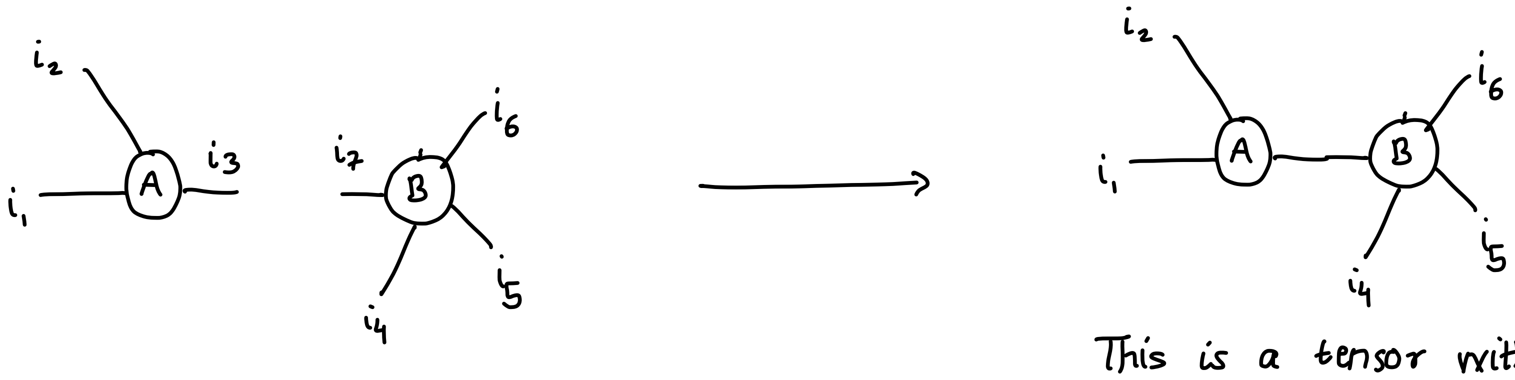
We can do two operations on tensors to create new tensors

<u>Tensor Product</u> This just puts the two tensors together (This is the same as the outer product)



The number that this tensor spits out is $A_{i_1 i_2 i_3} \cdot B_{i_4 i_5 i_6 i_7}$

<u>Contraction</u> This operation fuses two free legs

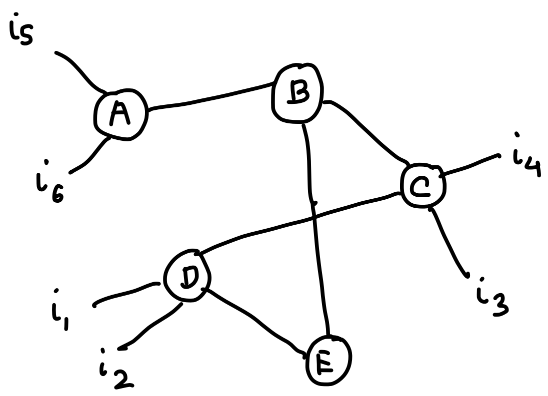


This is a tensor with 5 indices and on i_1, i_2, i_3, i_4, i_5 the value it spits out is

$$\sum_{i} A_{i_1 i_2 i} \; B_{i_4 i_5 i_6 i}$$

So we sum over all values taken by the index of the free legs

For a more complicated example



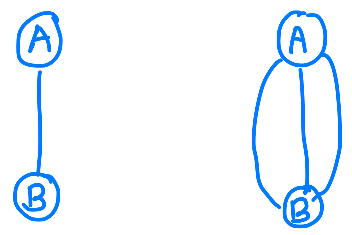
This is a tensor with 6 free indices and we sum over all possible internal labelings of the other edges

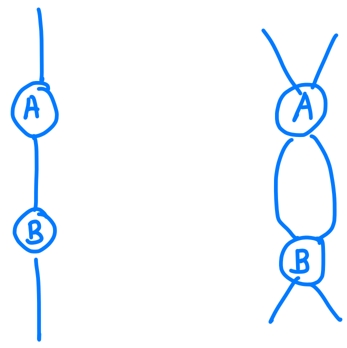
No matter how you put the initial tensors together you can check you get the same result

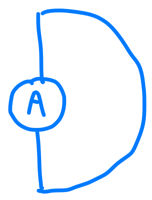
Such diagrams are called tensor networks

In-class Exercise

Interpret the following tensor networks

1. 

2. 

3. 

4. What is the identity operator tensor interpreted as a vector?

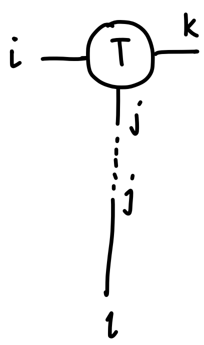
5. Give a picture of M^* given M

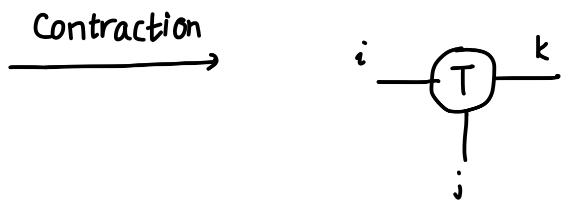
6. Prove that trace is cyclic using tensor networks

Some conventions

One can follow some conventions that are probably not standard but useful

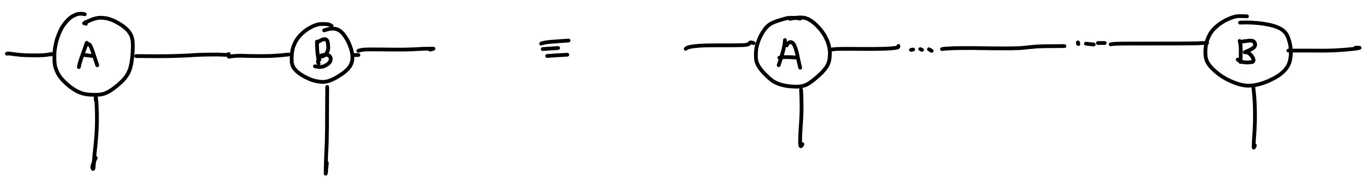
- Identity Matrix is just drawn as a line since





Contraction with the identity matrix (or unnormalized EPR pair) does not do anything

One can also go in the reverse direction and split a tensor

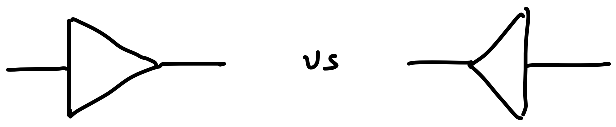


- Symmetric vs Not symmetric (OR Hermitian vs Non-Hermitian)

- Matrix vs Transpose (OR Conjugate Transpose)

So, symmetric matrices have the same transpose

- Projections vs Isometries

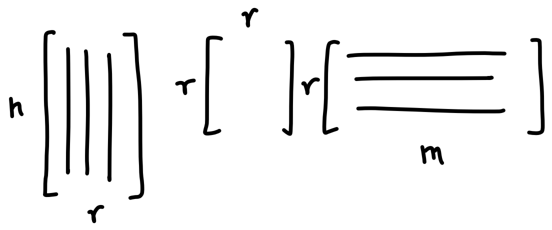


Schmidt Decomposition and Entanglement Entropy

For us the most relevant tensor network representation is the Schmidt decomposition

To state what it is let us first recall singular value decomposition of matrices

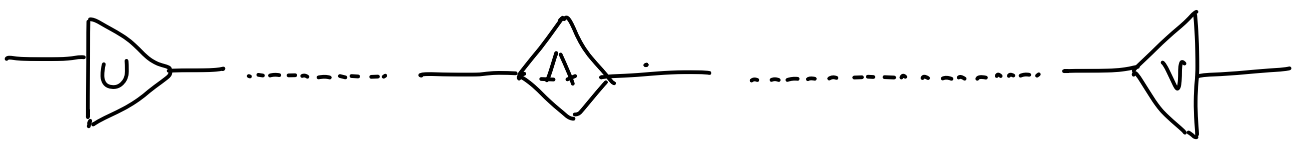
Any matrix $\underset{n\times m}{A} = \underset{n\times r}{U}\,\underset{r\times r}{\Lambda}\,\underset{r\times m}{V^{+}}$ =

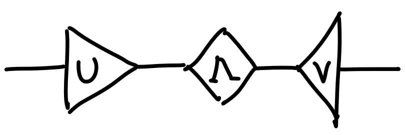


where U and V have orthonormal columns
and Λ is a diagonal matrix of singular values σ_k

non-zero singular values = rk(A)

Writing $U = \sum_{k=1}^{r} |u_k\rangle\langle u_k|$ & $\Lambda = \sum_{k=1}^{r} \sigma_k |u_k\rangle\langle v_k|$ & $V = \sum_{k=1}^{r} |v_k\rangle\langle v_k|$



We get A =  $= \sum_{k=1}^{r} \sigma_k |u_k\rangle\langle v_k|$

One can also view this matrix as a vector $|A\rangle$ in which case we can write it as

$$|A\rangle = \sum_{k=1}^{r} \sigma_k |u_k\rangle \otimes |v_k\rangle$$

This is called the Schmidt Decomposition across this cut

non-zero terms is called the $\underbrace{\text{Schmidt rank}}_{SR(|A\rangle)}$ and $\underbrace{\sum_{k=1}^{r} \sigma_k^2 \log \frac{1}{\sigma_k^2}}_{S(|A\rangle)}$ is called entanglement entropy

For example, $|u_k\rangle \otimes |v_k\rangle$ has Schmidt rank 1 and Entanglement entropy 0
$\frac{1}{\sqrt{D}} \sum_{k=1}^{D} |k\rangle \otimes |k\rangle$ has Schmidt rank D and Entanglement entropy $\log(D)$

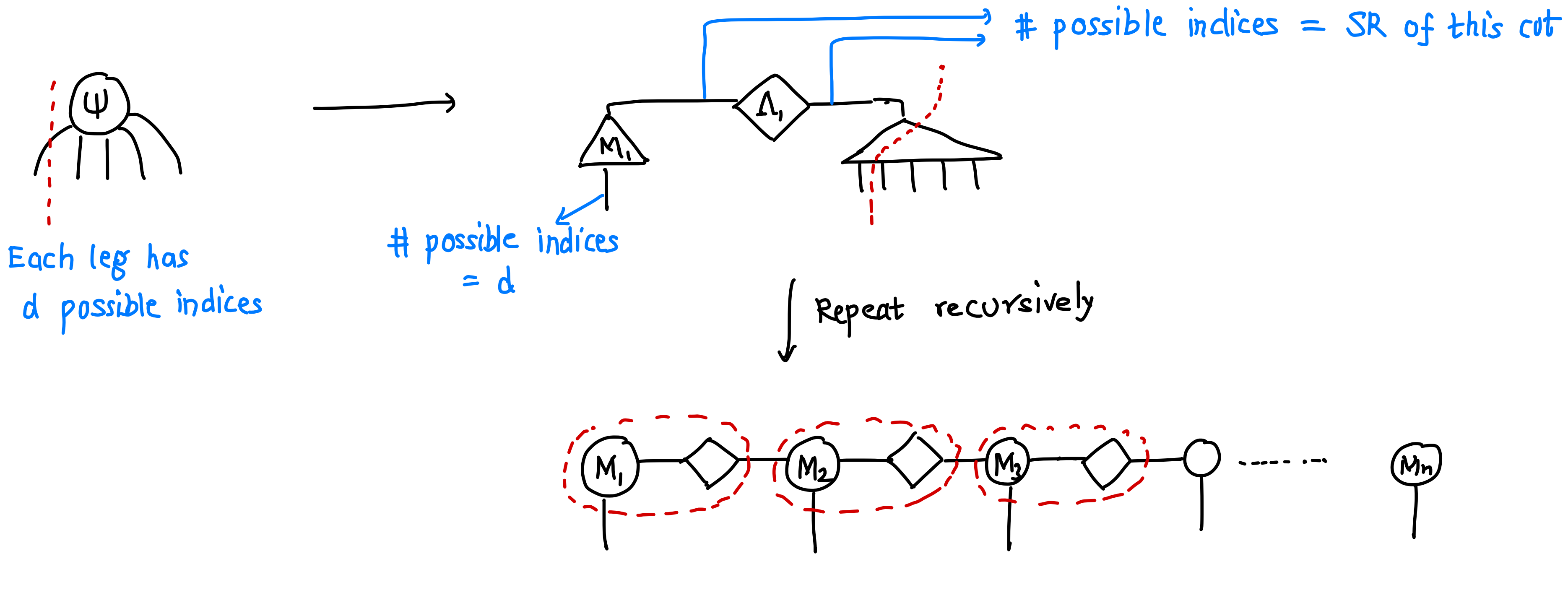
In general, $0 \le S(|A\rangle) \le \log SR(|A\rangle)$, so if SR or S(A) is small, it means the state doesn't have a lot of entanglement

Matrix Product States

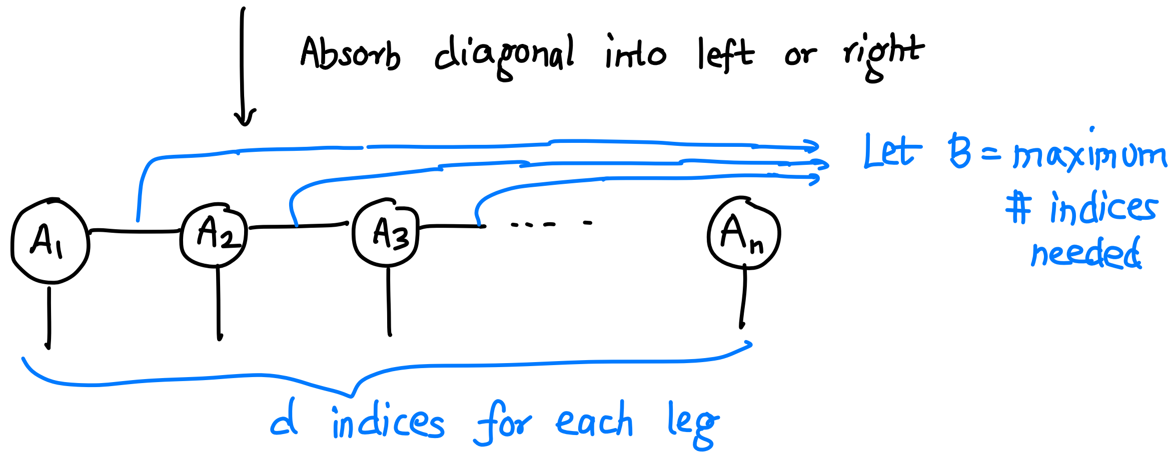
Consider a quantum state $|\psi\rangle = \sum_{i_1,\dots,i_n \in [d]} \psi_{i_1 i_2 \dots i_n} |i_1 \dots i_n\rangle$ on N qudits of d-dimension

To describe a generic state we need d^n numbers

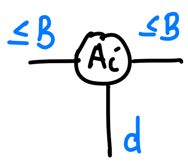
Suppose we do a Schmidt Decomposition by splitting it into first qudit & the rest



$\sum_{i_1 \dots i_n \in [d]} \mathrm{Tr}[A_{i_1} A_{i_2} \cdots A_{i_n}] |i_1, \dots, i_n\rangle =$



This is called a matrix product state & B is called the bond dimension

Total # parameters needed to describe each tensor  $\leq dB^2$

For the entire MPS, we need $O(ndB^2)$ parameters

In general B is exponential in n, but if B is small, these quantum states have low entanglement & small description

One can also compute energy of such states in poly(n,d,B) time classically by repeated matrix multiplication (exercise)

Characterizing which systems have such states is of great importance
For instance, ground states of QMA-hard hamiltonians cannot be MPS (assuming QMA ≠ NP)

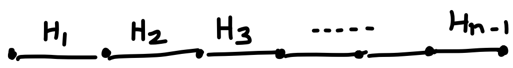
There are also higher dimensional generalizations (not on a line) called PEPS (projected entangled pair states) which we won't introduce

Area Laws

Recall our motivating question: what kind of local Hamiltonians have simple ground states (e.g. matrix product states) ?

Let us look at Local Hamiltonians on a grid :

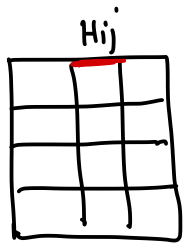
In 1-dimensions, there are n qudits arranged on a line and local Hamiltonian term acts on neighboring qudits



$$H = \sum_i H_i$$

where each $0 \preceq H_i \preceq I$ and H_i acts non-trivially on qudit i & $i+1$

In 2-dimensions, qudits are on a grid and H_{ij} acts on two neighboring qubits i,j in the grid

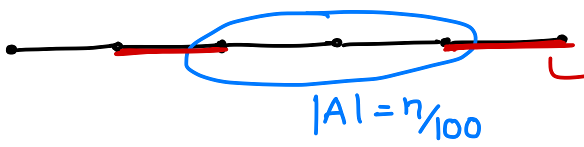


$$H = \sum_{ij \sim edge} H_{ij}$$

The area law conjecture says that any ground state $|\psi\rangle$ of a physically-relevant Local Hamiltonian has area law behavior, i.e.

> For any subset $A \subseteq [n]$ of qubits, the entanglement entropy is proportional to the size of the boundary of A (i.e. proportional to the area)

E.g. in 1-dimension :



Area law behavior : entanglement entropy $= O_d(1)$

In general, entanglement entropy could be as large as $\sim |A| \log d \sim n \log d$

One can make even stronger conjecture that the ground state has a MPS description

NEXT TIME More on this and a proof