Homomorphic Encryption

Lecture 18

And some applications

Homomorphic Encryption

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- Not covered today: Fully Homomorphic Encryption, which supports ring homomorphism (addition <u>and</u> multiplication of messages)

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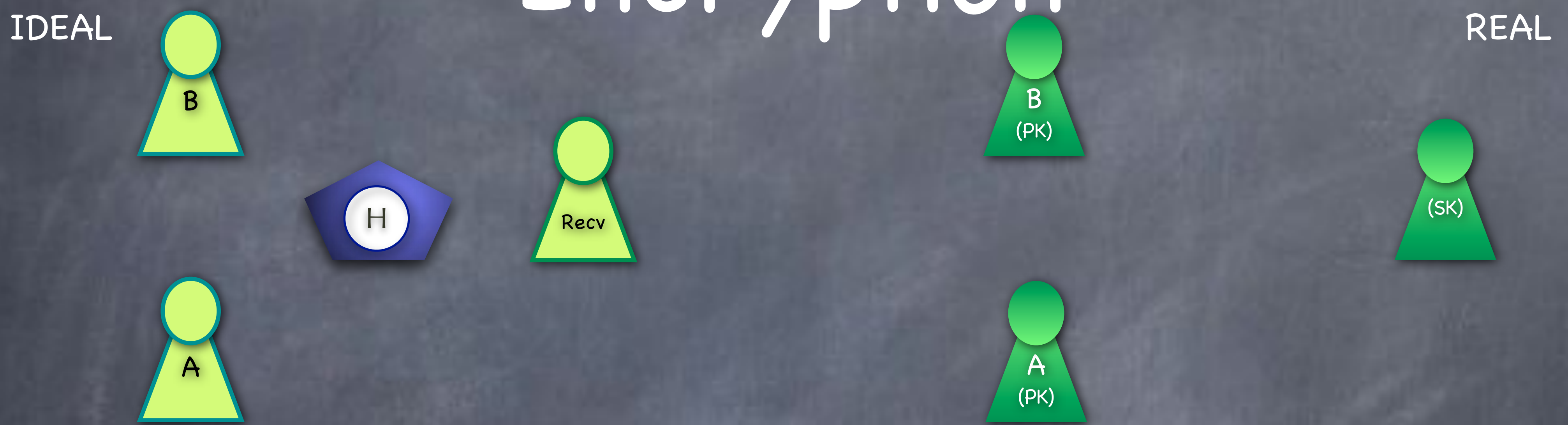
  - Rerandomization useful even without homomorphism

Unlinkable Homomorphic Encryption



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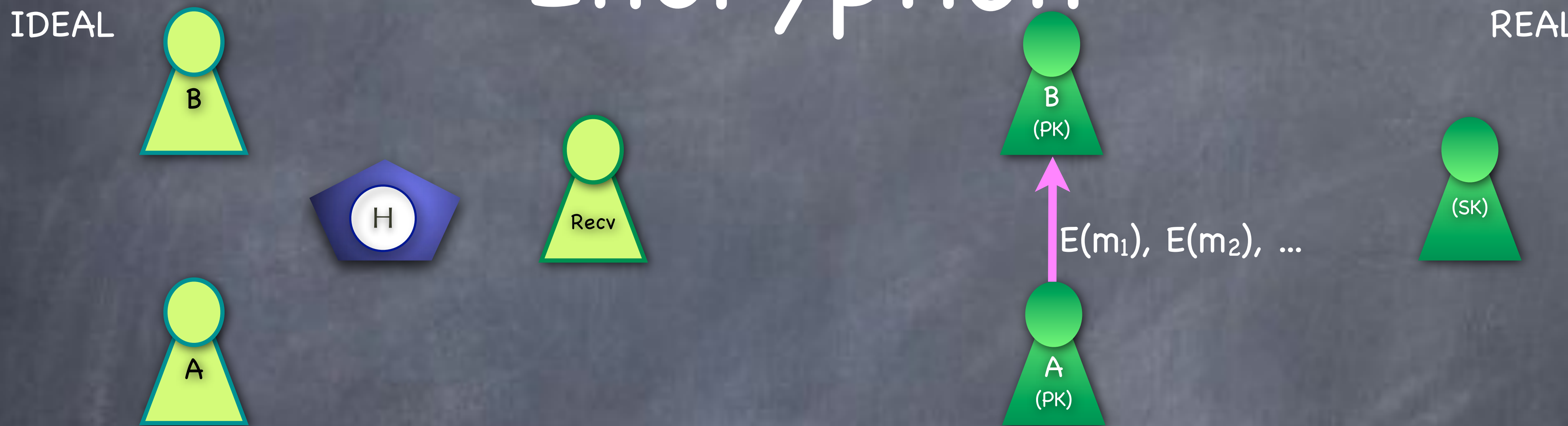

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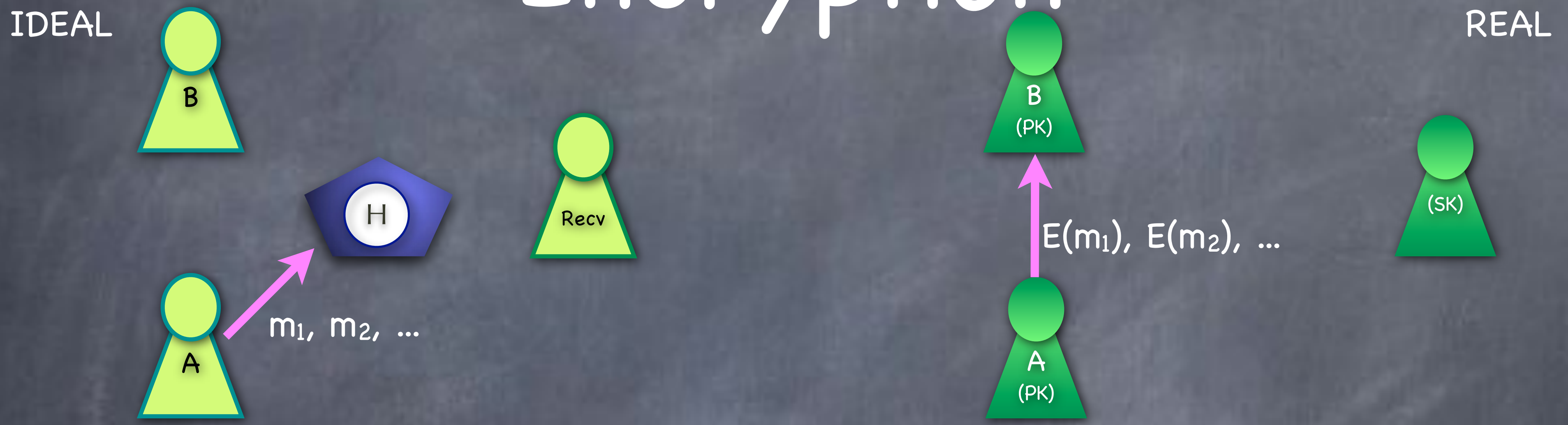

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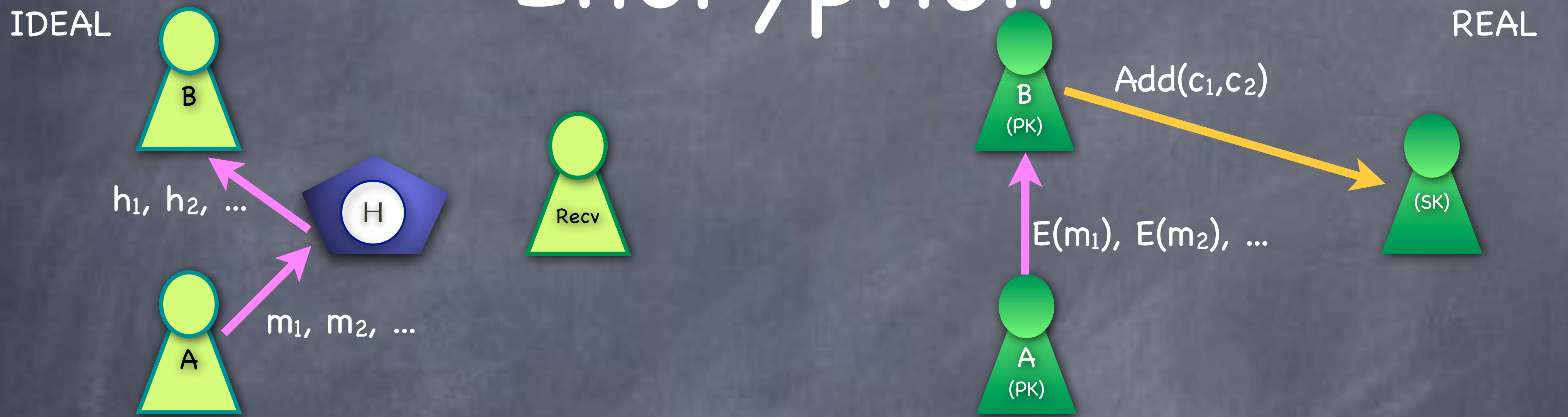

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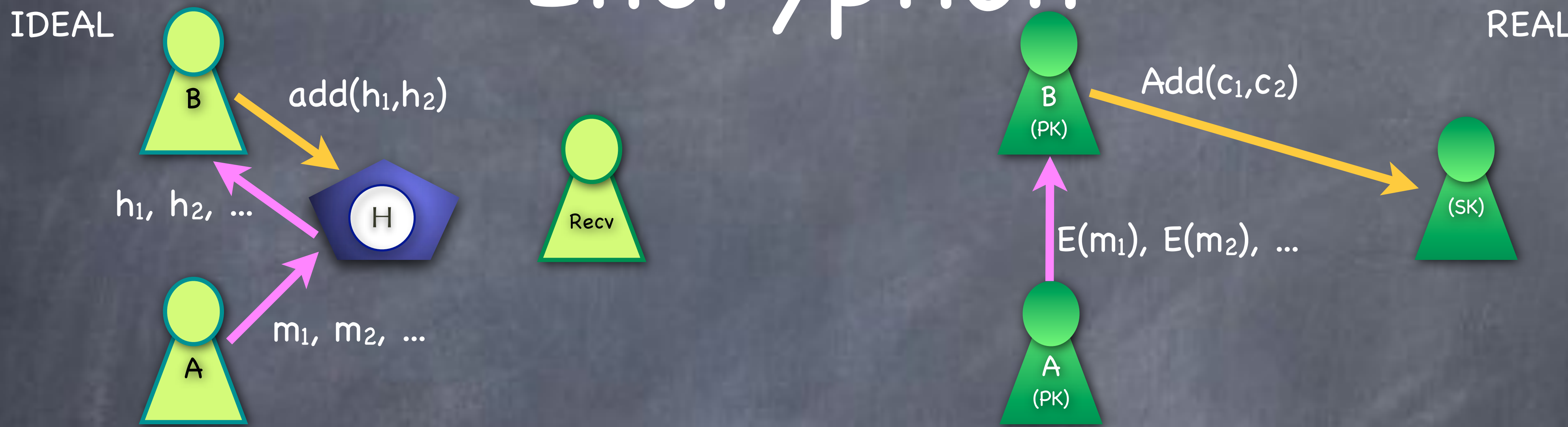

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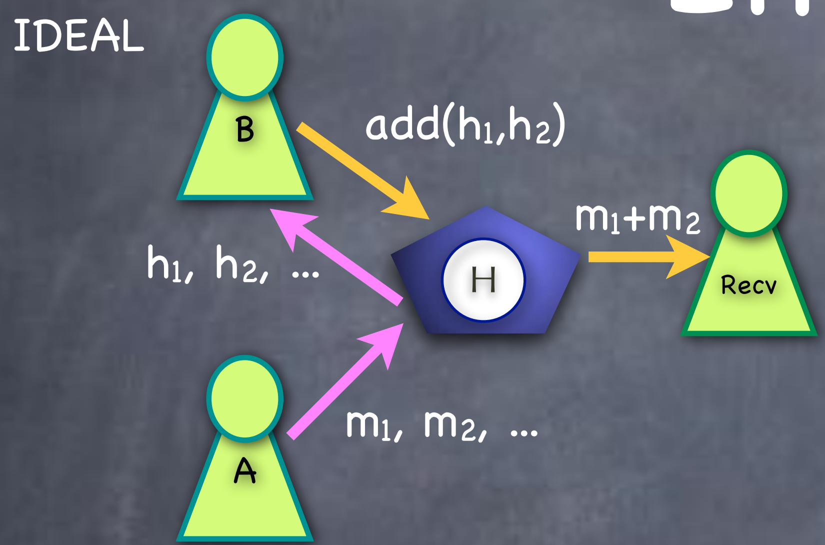

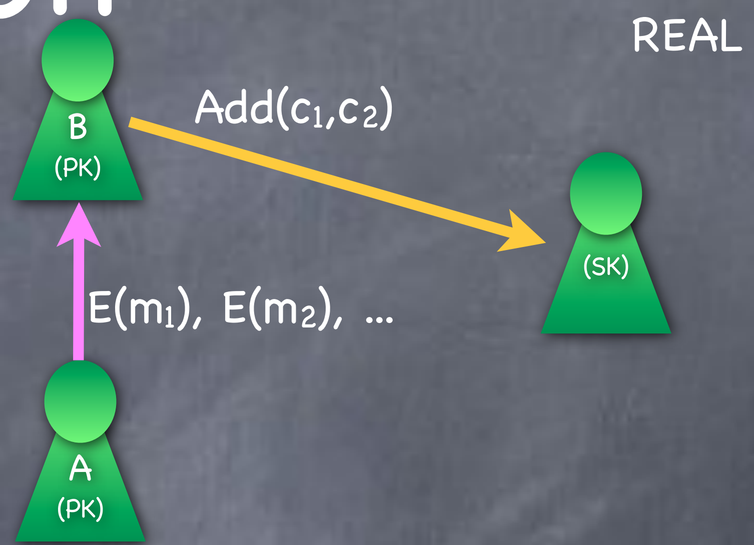

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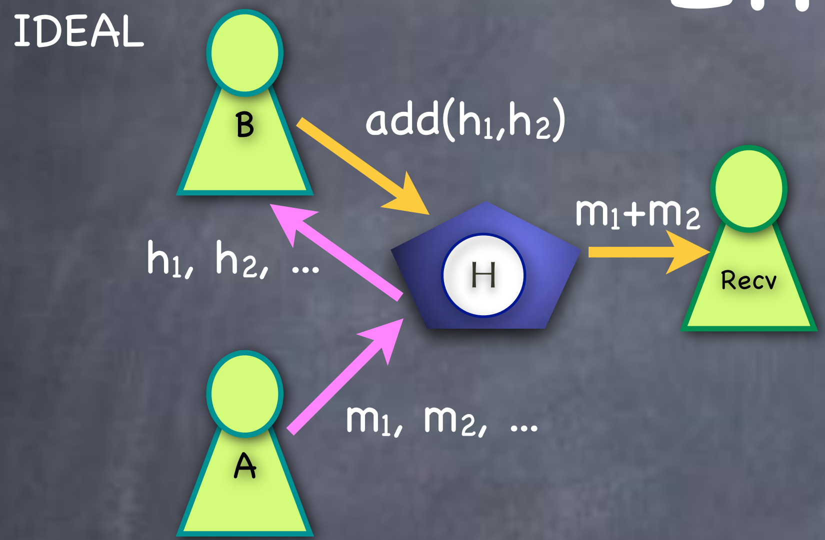

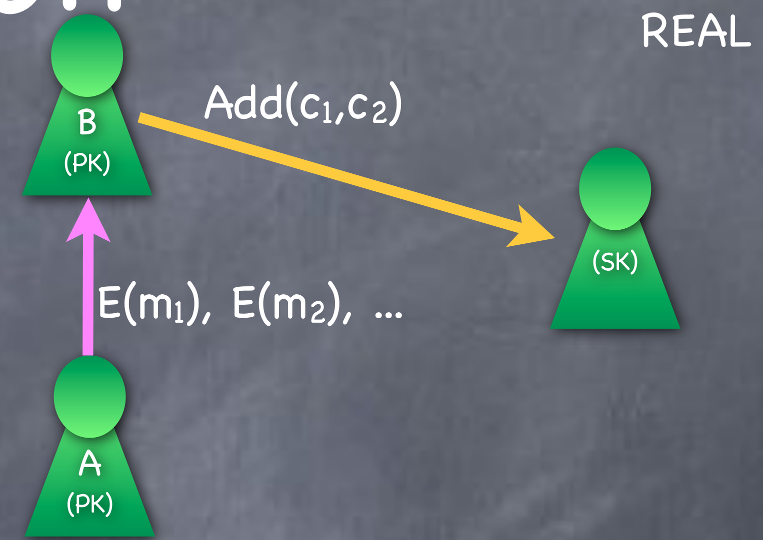

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- Functionality gives "handles" to messages posted; accepts requests for posting fresh messages, or derived messages
- Unlinkability: Above, receiver gets only the message m_1+m_2 in IDEAL; is not told if it is a fresh message or derived from other messages

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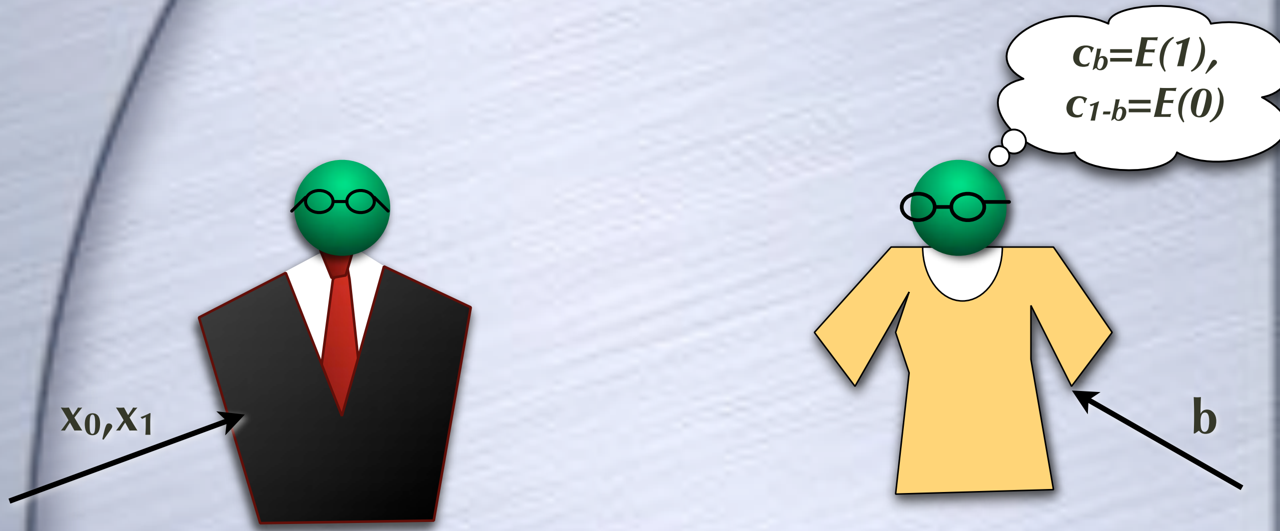

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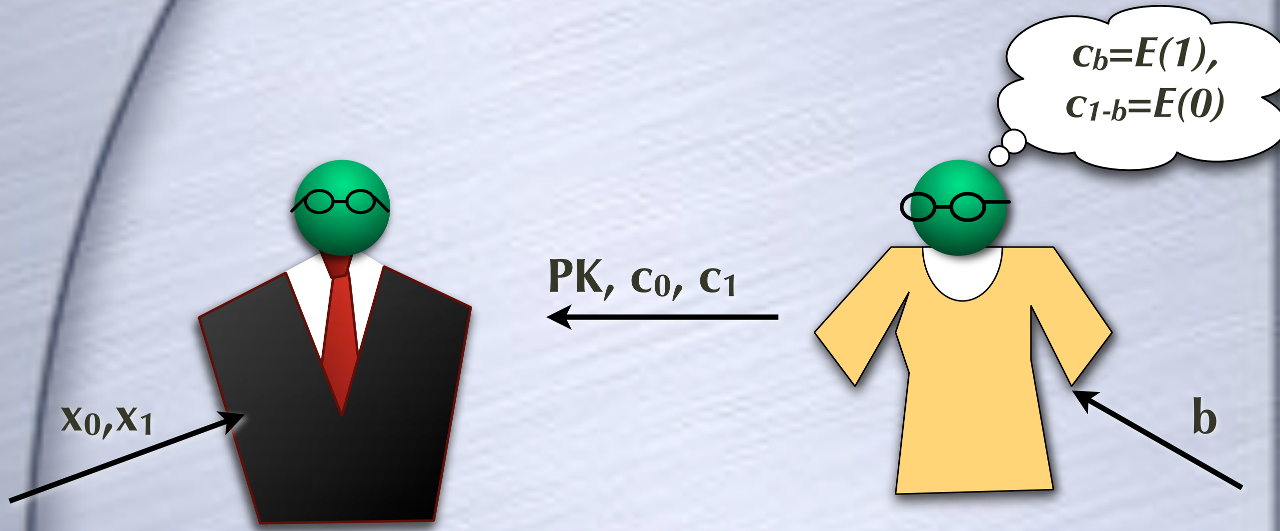

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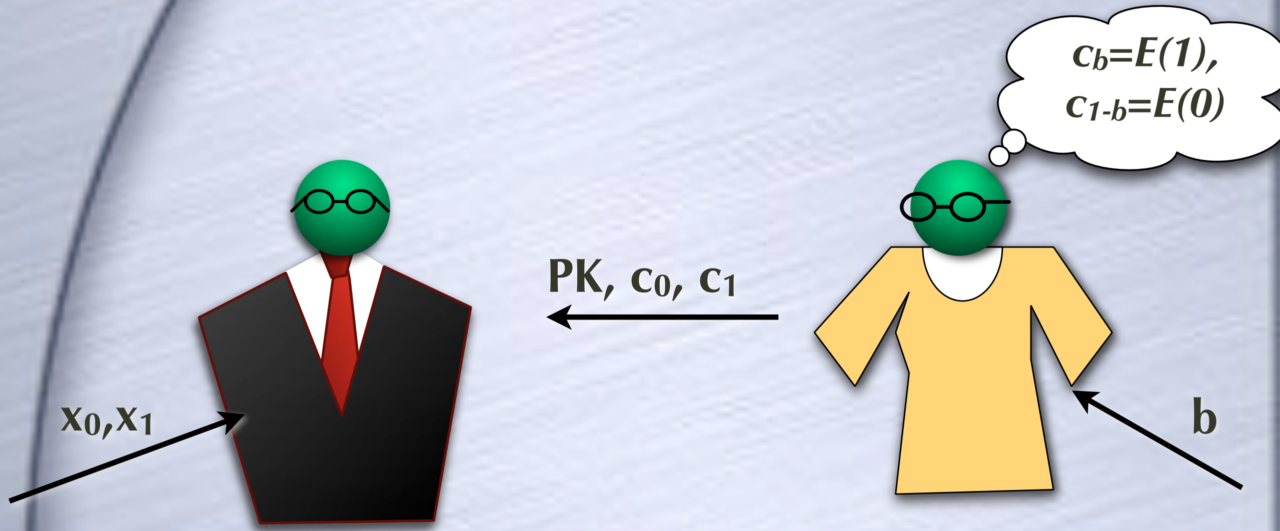

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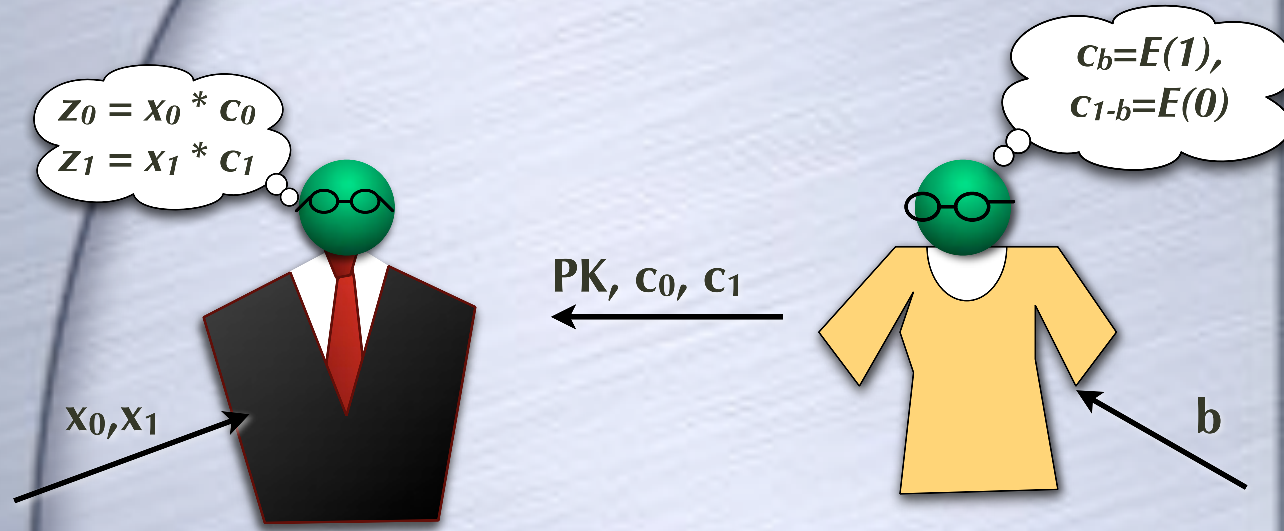

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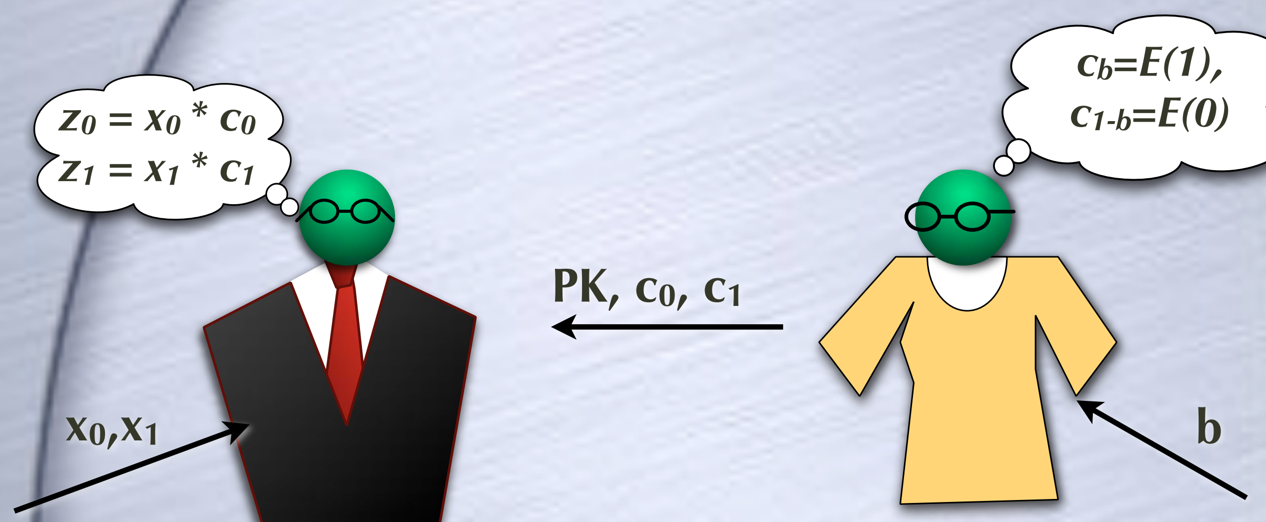

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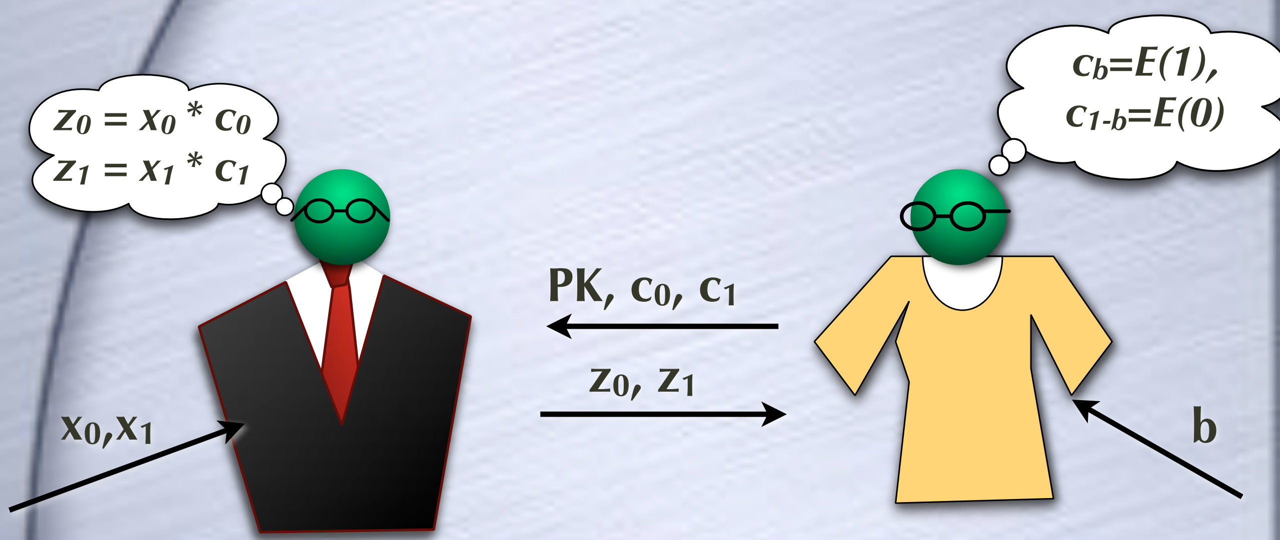

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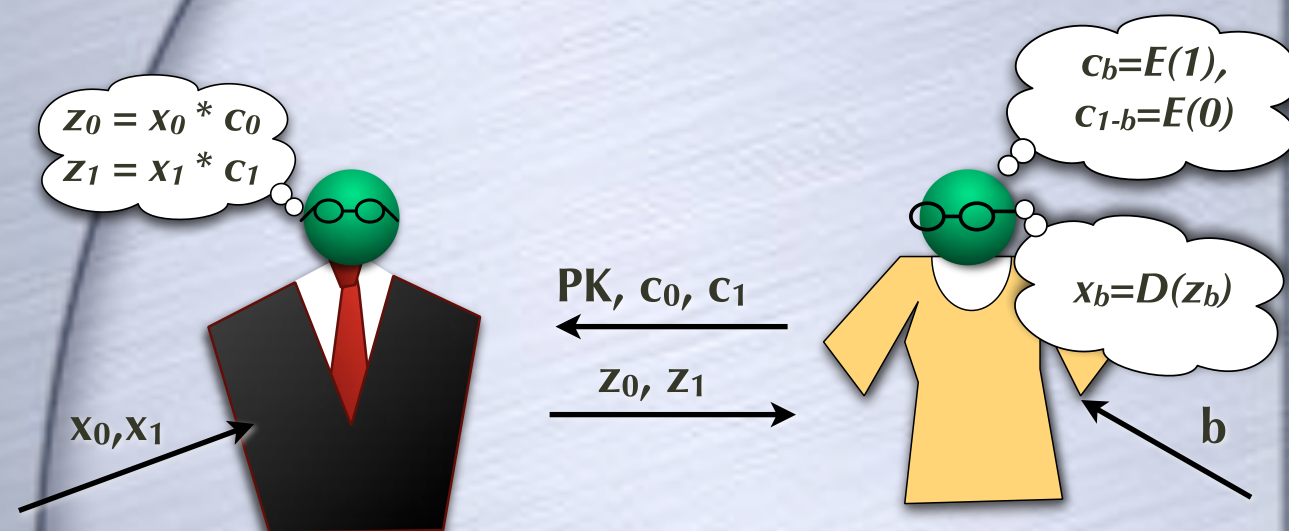

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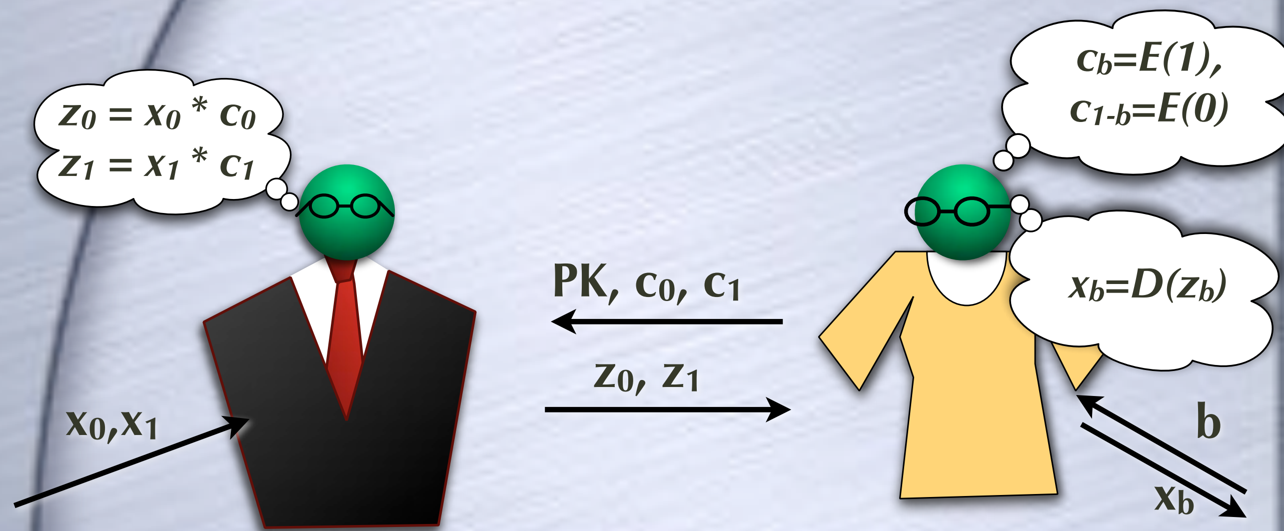

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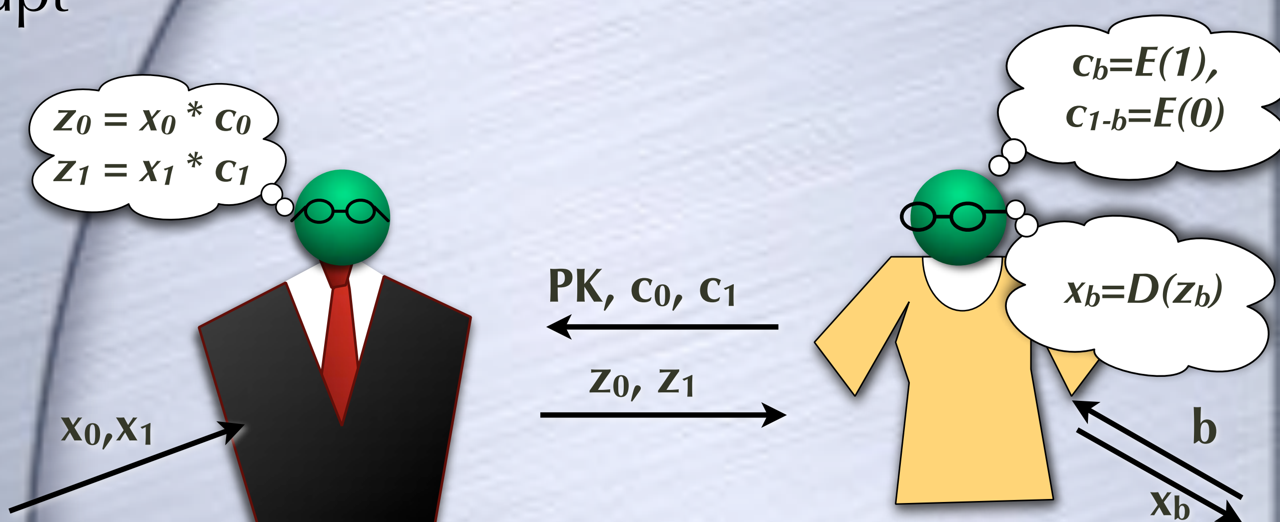

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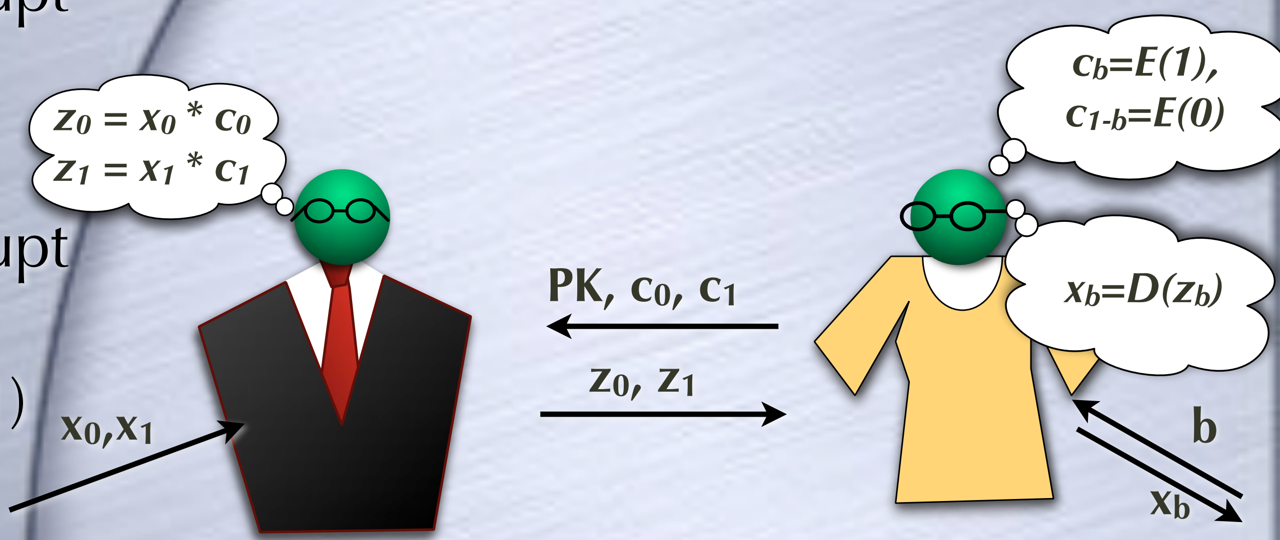

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  - When message space is $\mathbb{Z}_n$: additively homomorphic encryption

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- Unlinkability: ReRand(c) = c.Enc(0)

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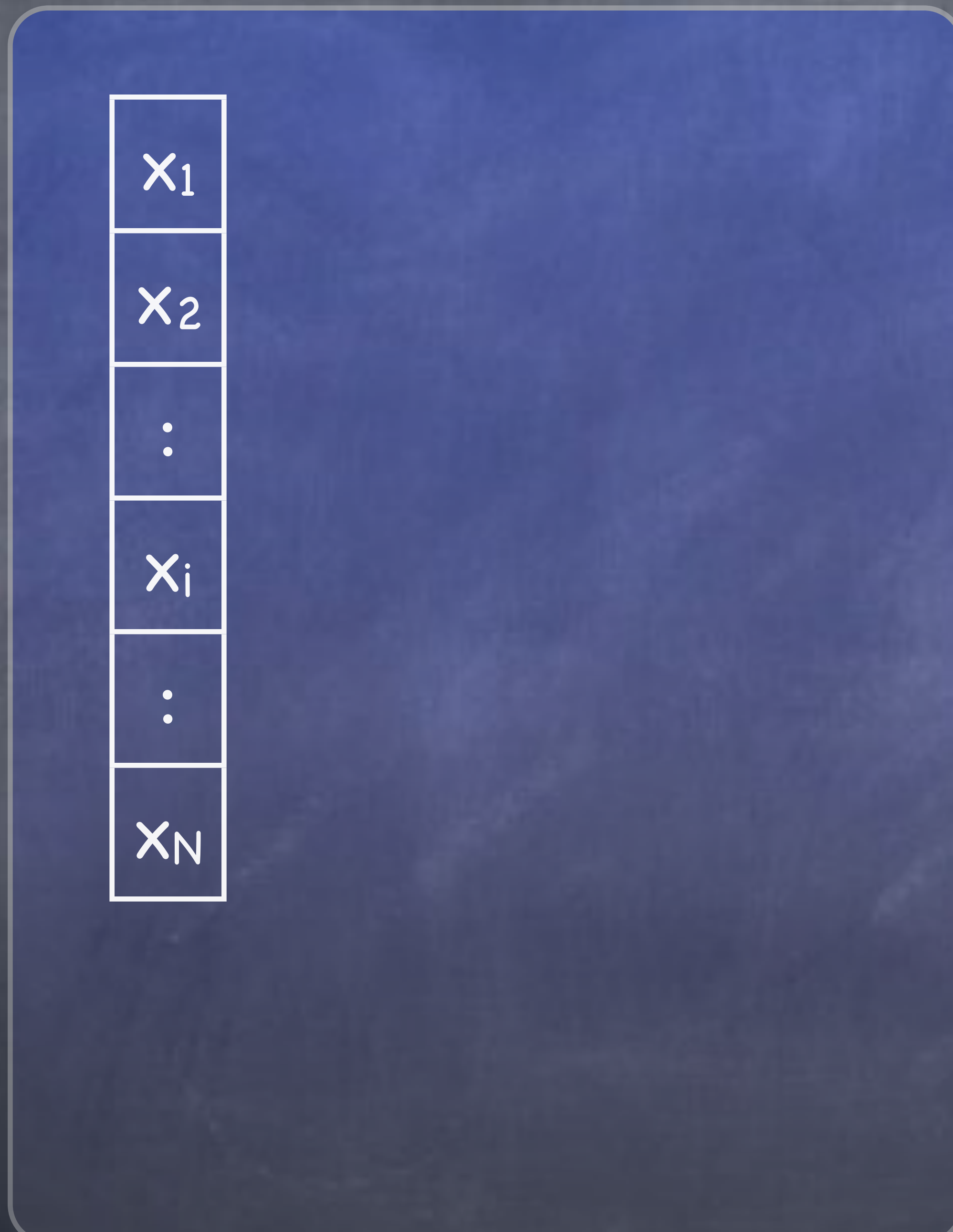
For Paillier, can use exponentiation

Private Information Retrieval

i



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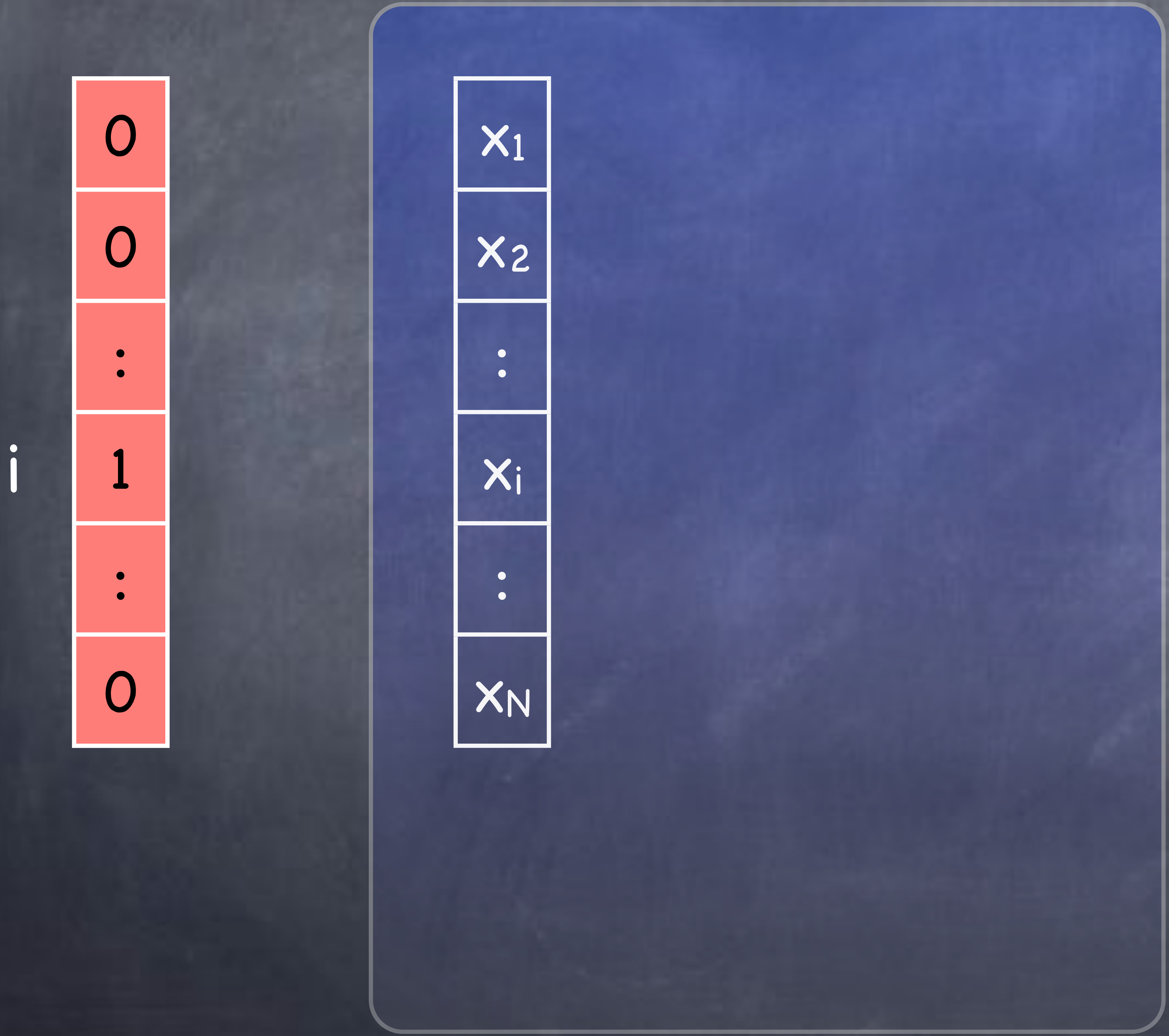

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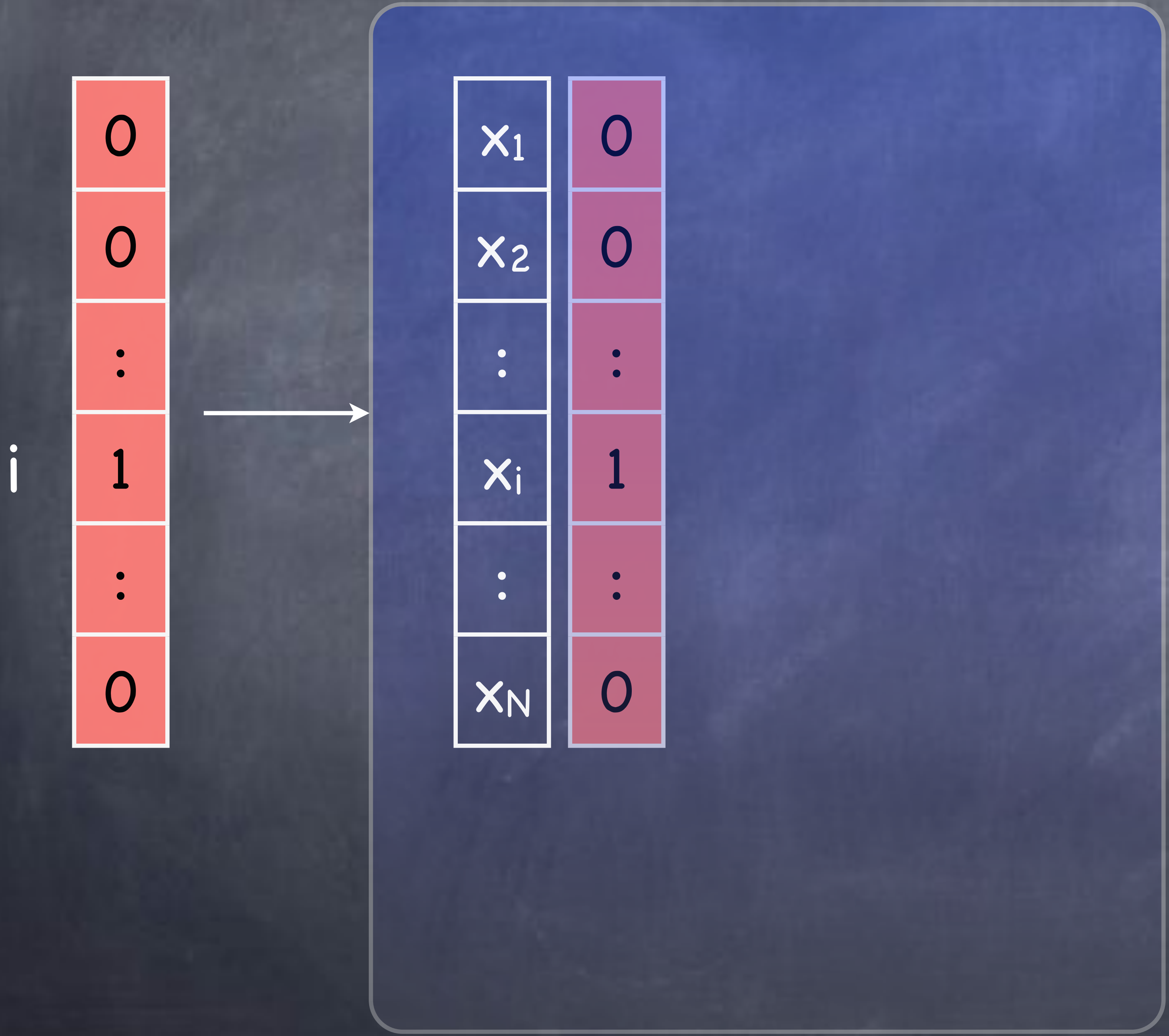

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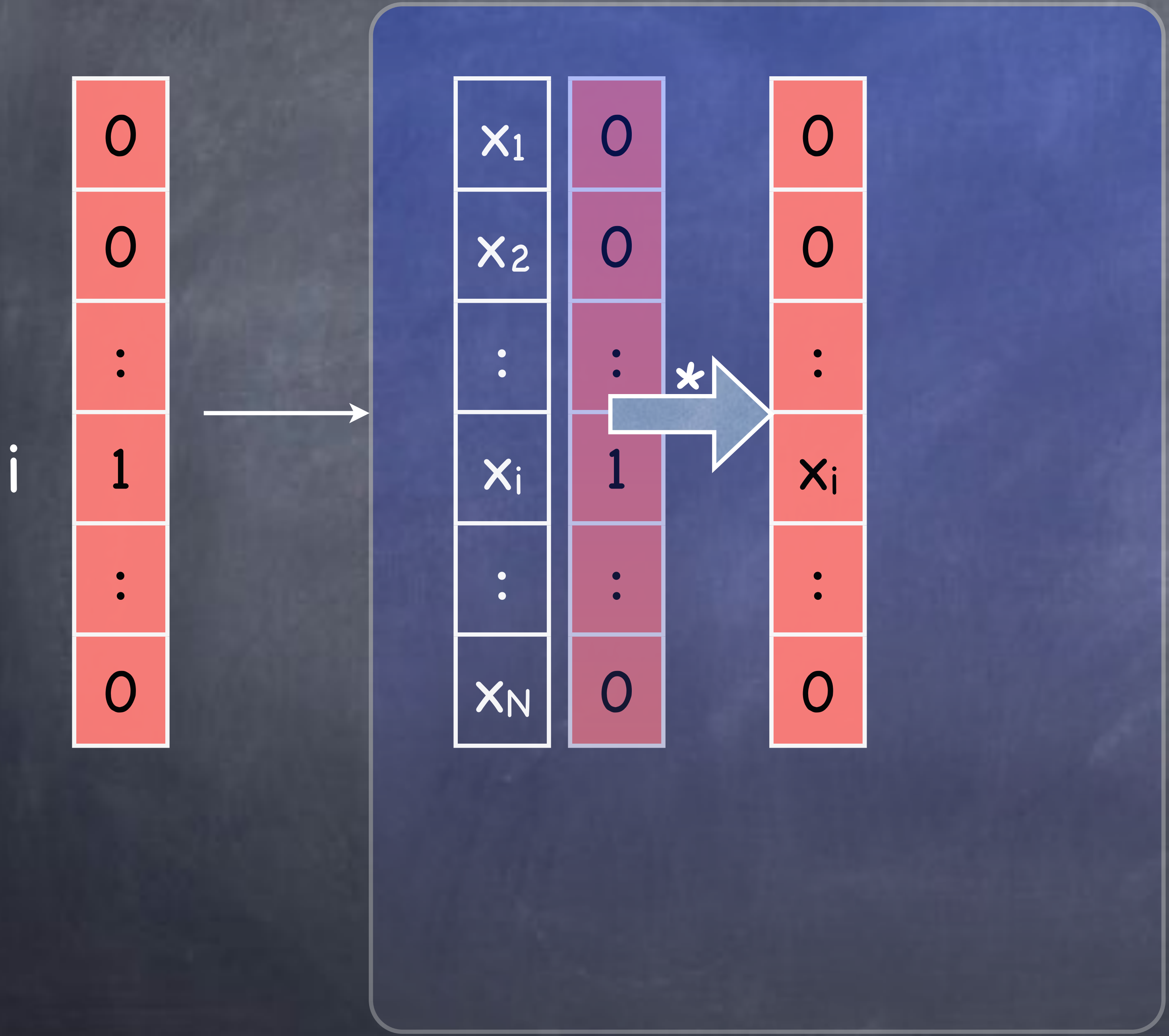

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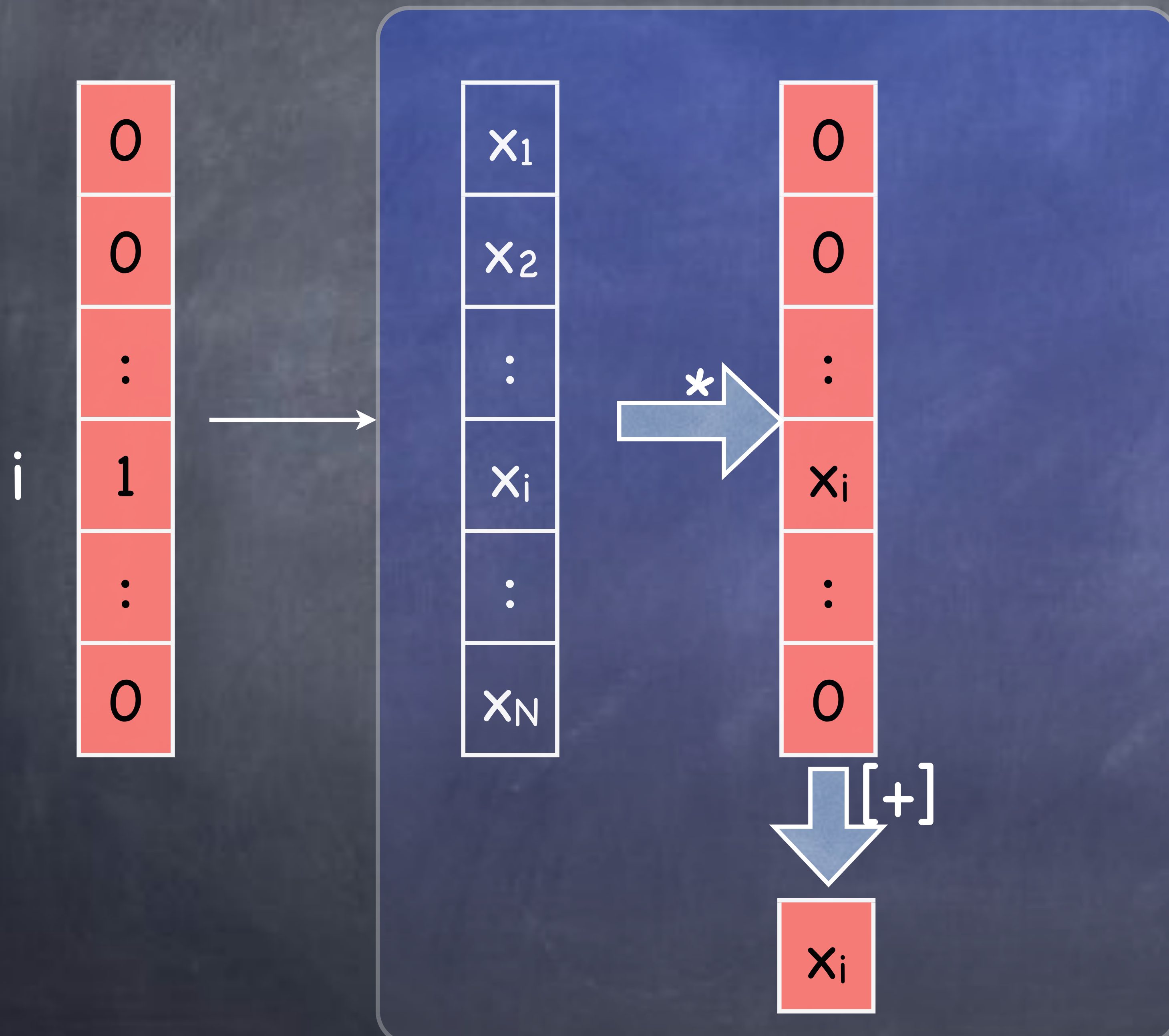

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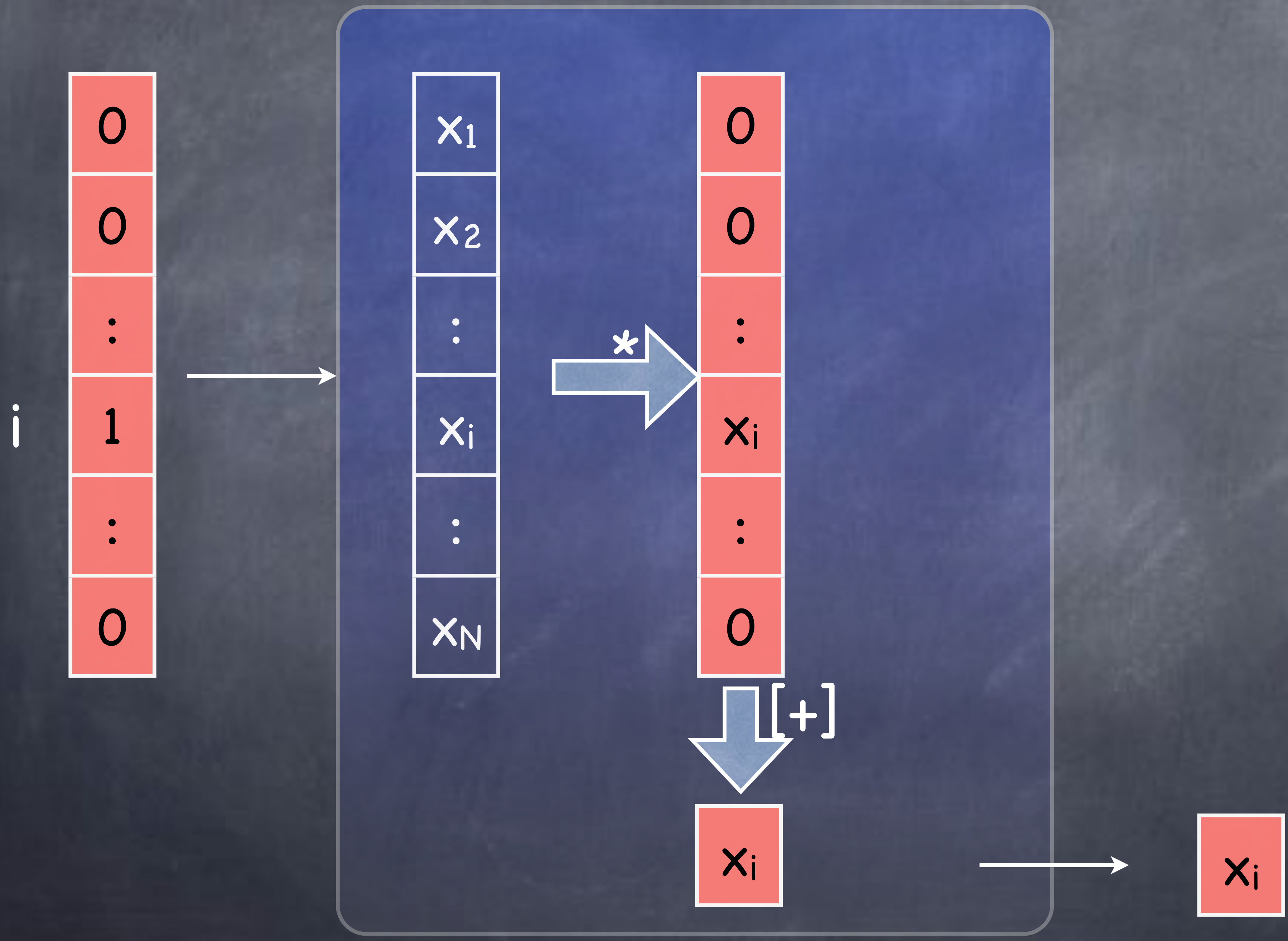

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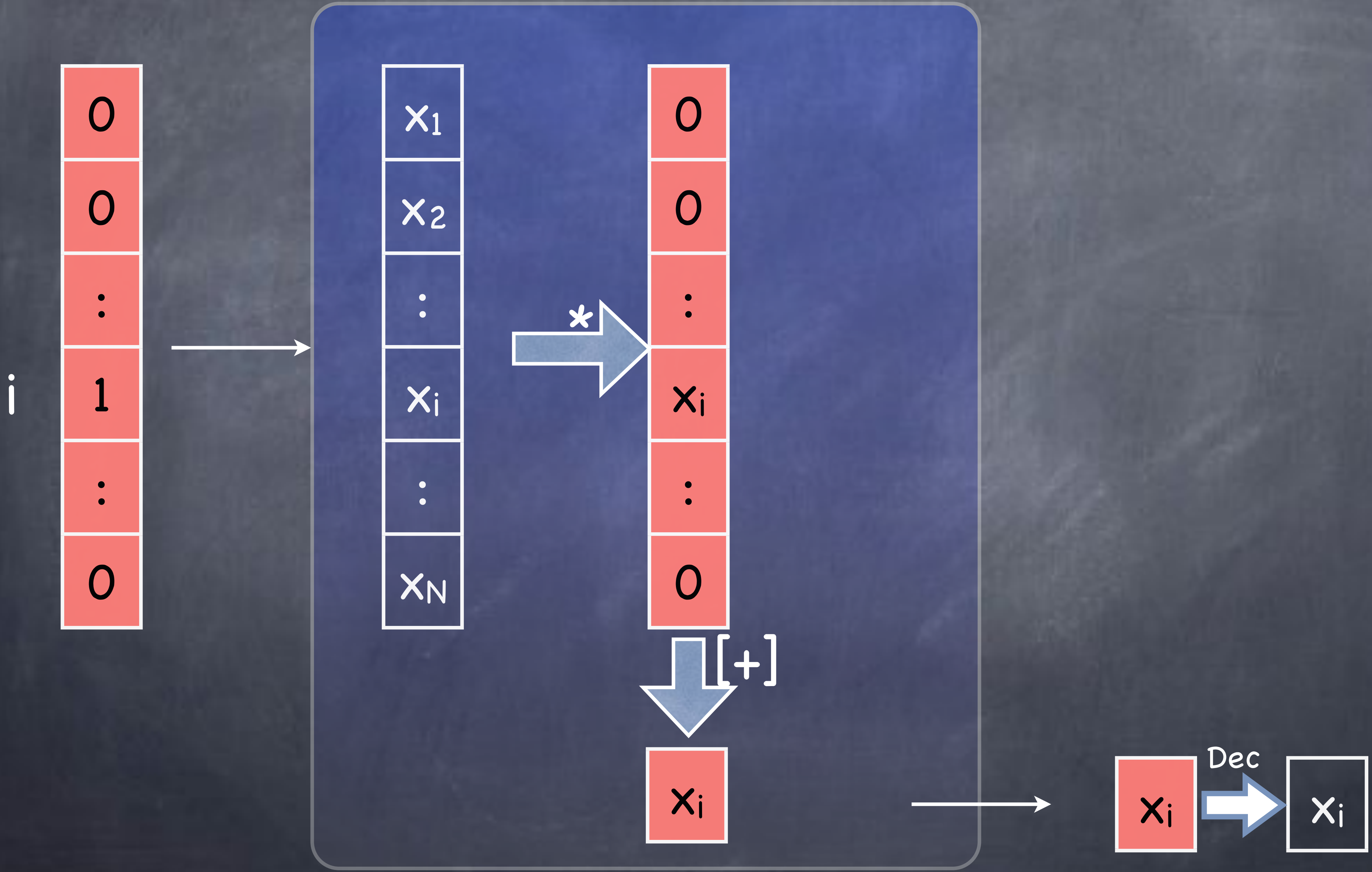

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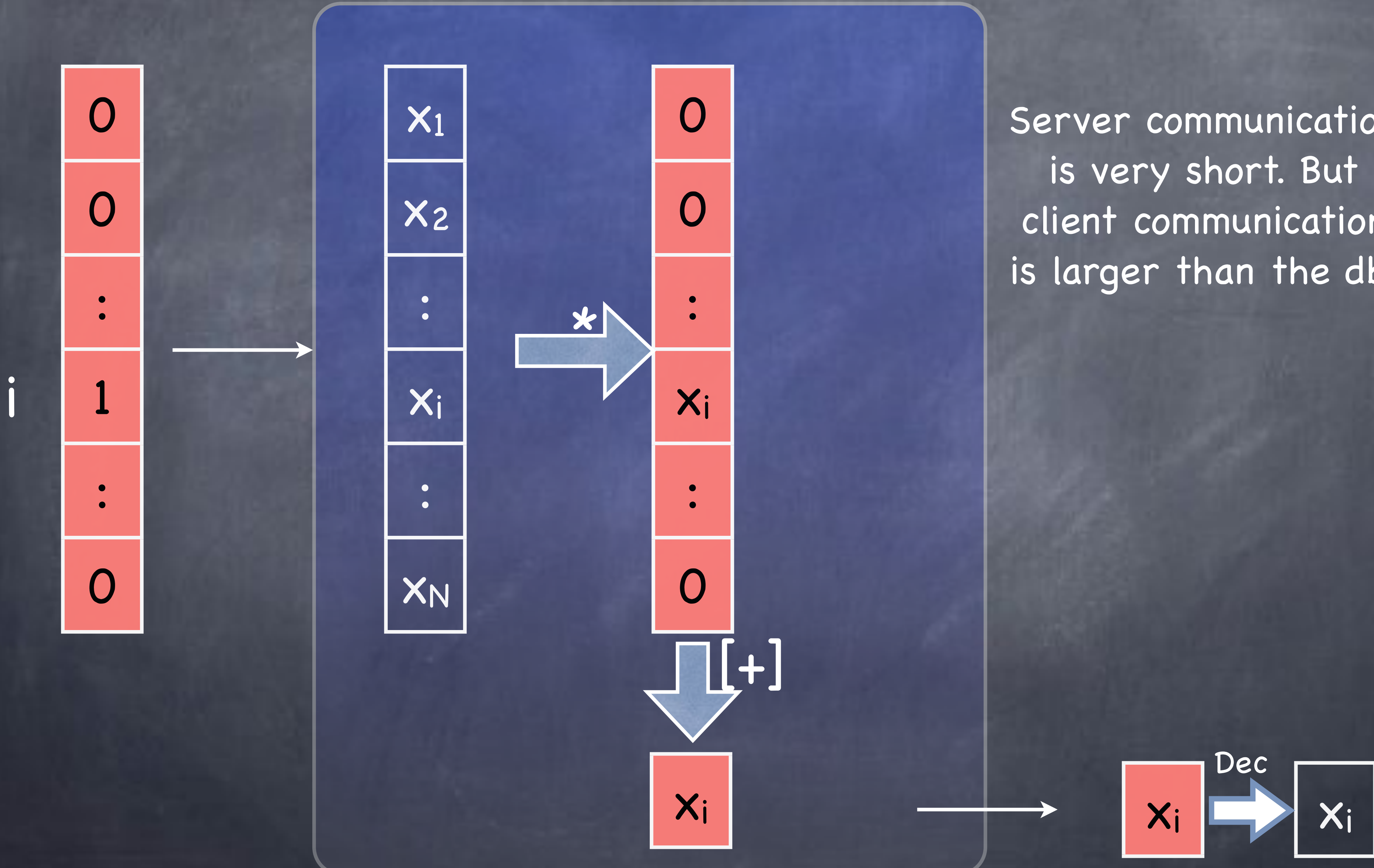

Server communication is very short. But client communication is larger than the db!

Private Information Retrieval

x_{11}					x_{1N}
x_{21}					x_{2N}
:					:
x_{i1}			x_{ij}		x_{iN}
:					:
x_{N}					x_{NN}

Private Information Retrieval

0
0
:
1
:
0

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:					:
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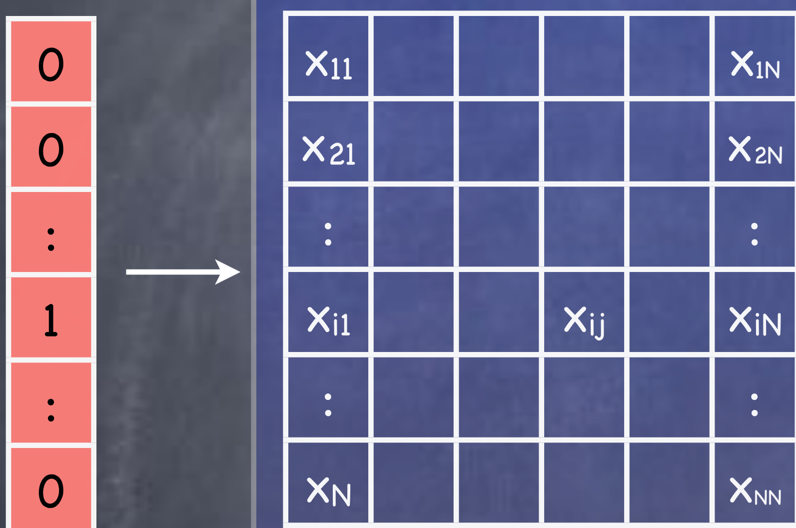

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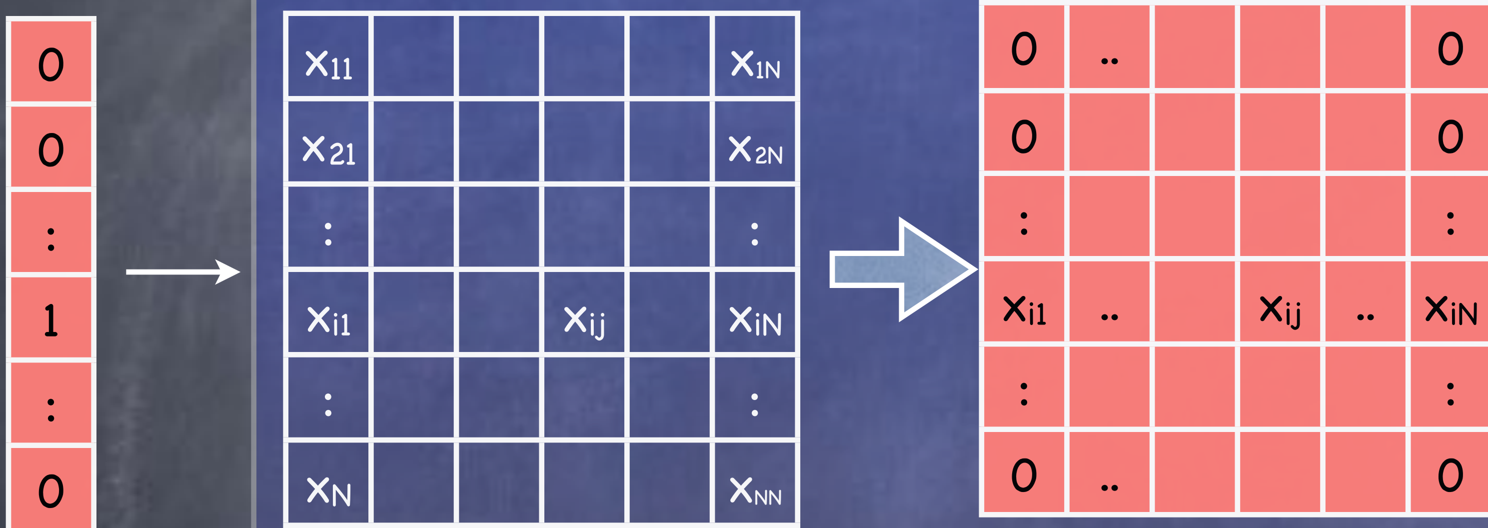

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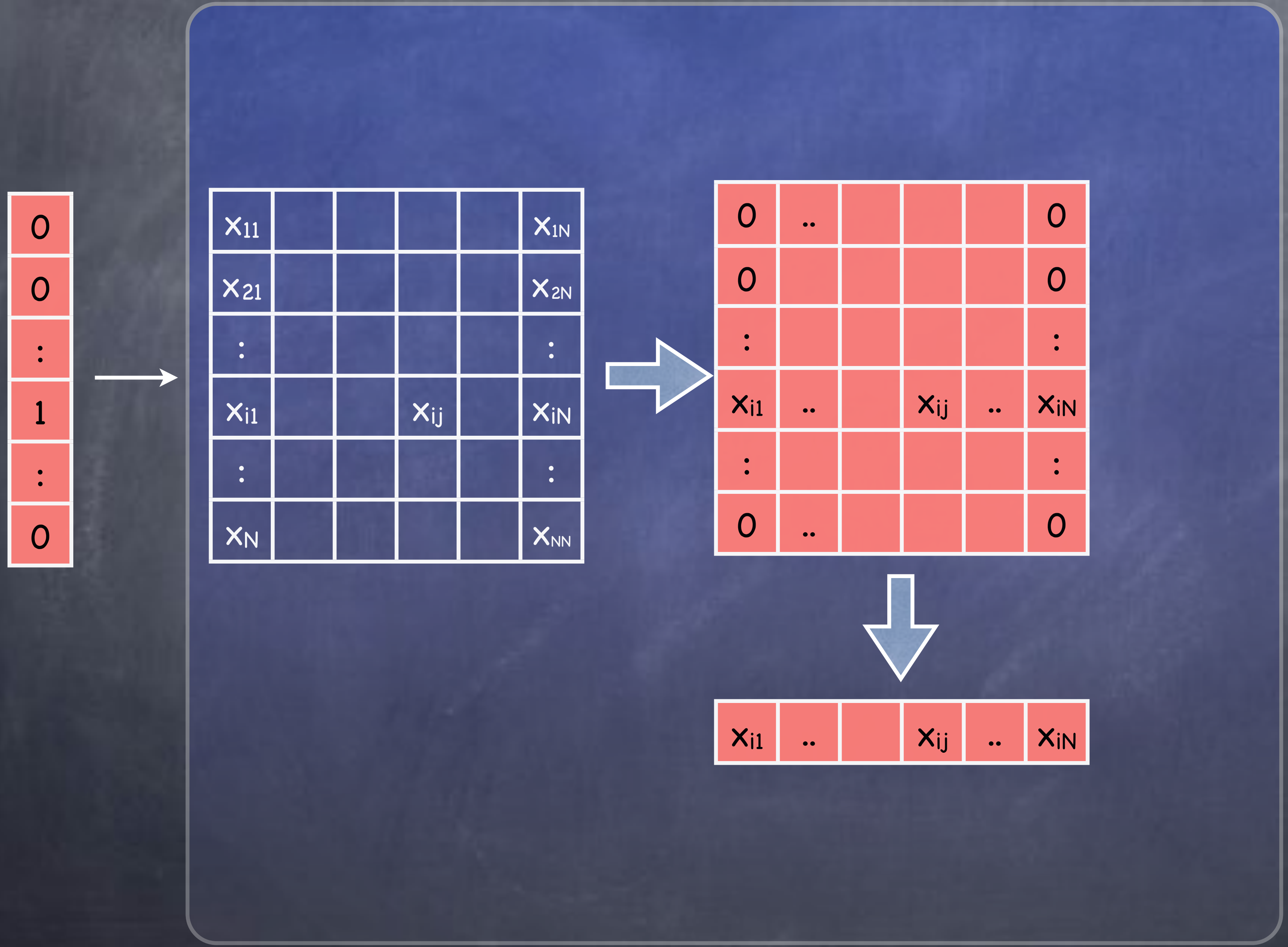

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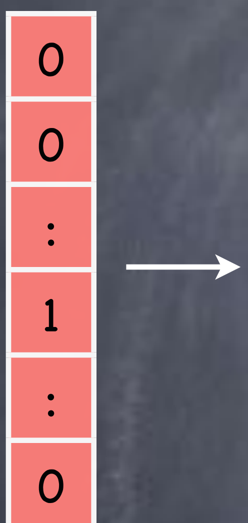
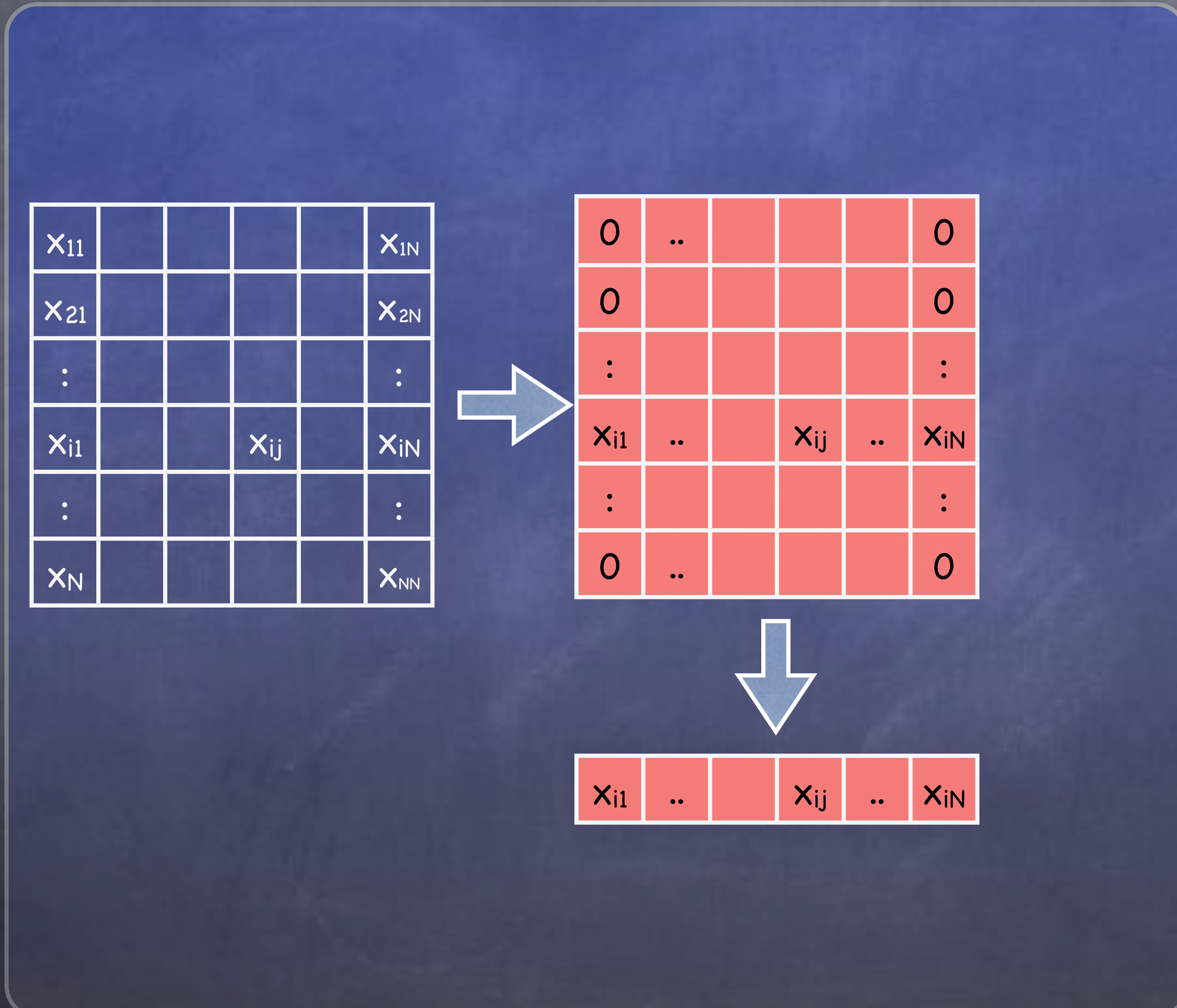
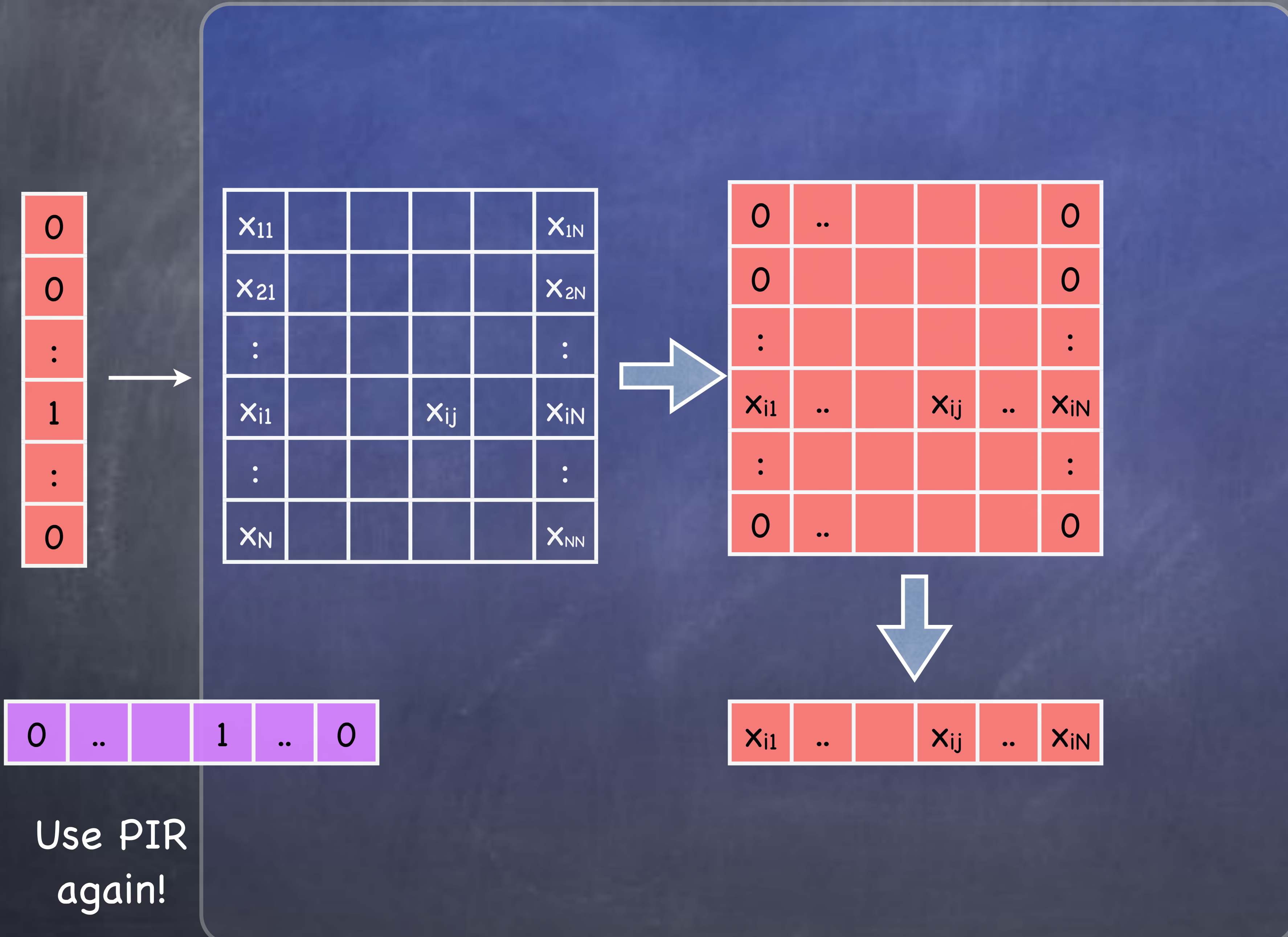




Use PIR again!

Private Information Retrieval



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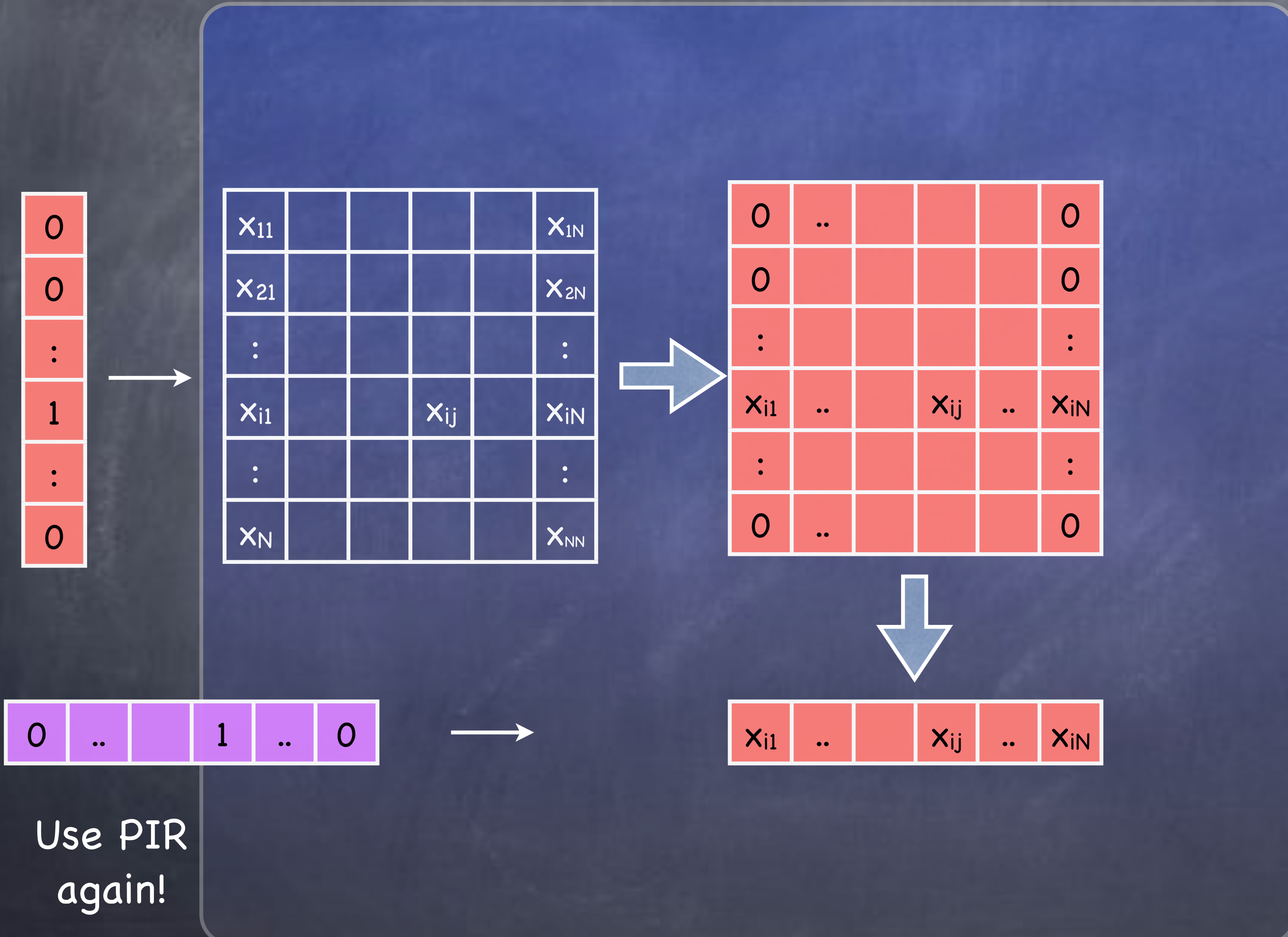

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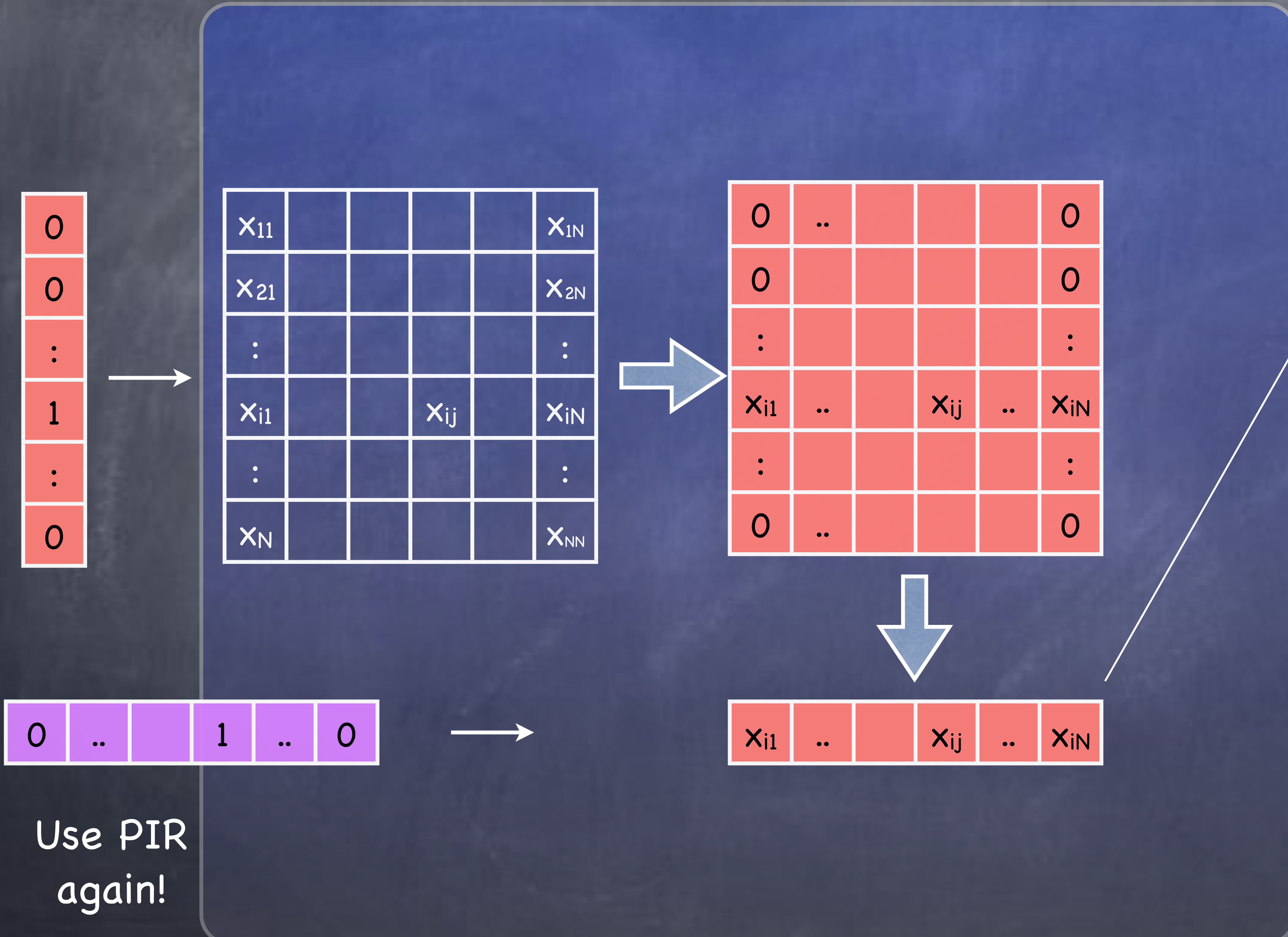

Considering ciphertext as plaintext for the sub-PIR

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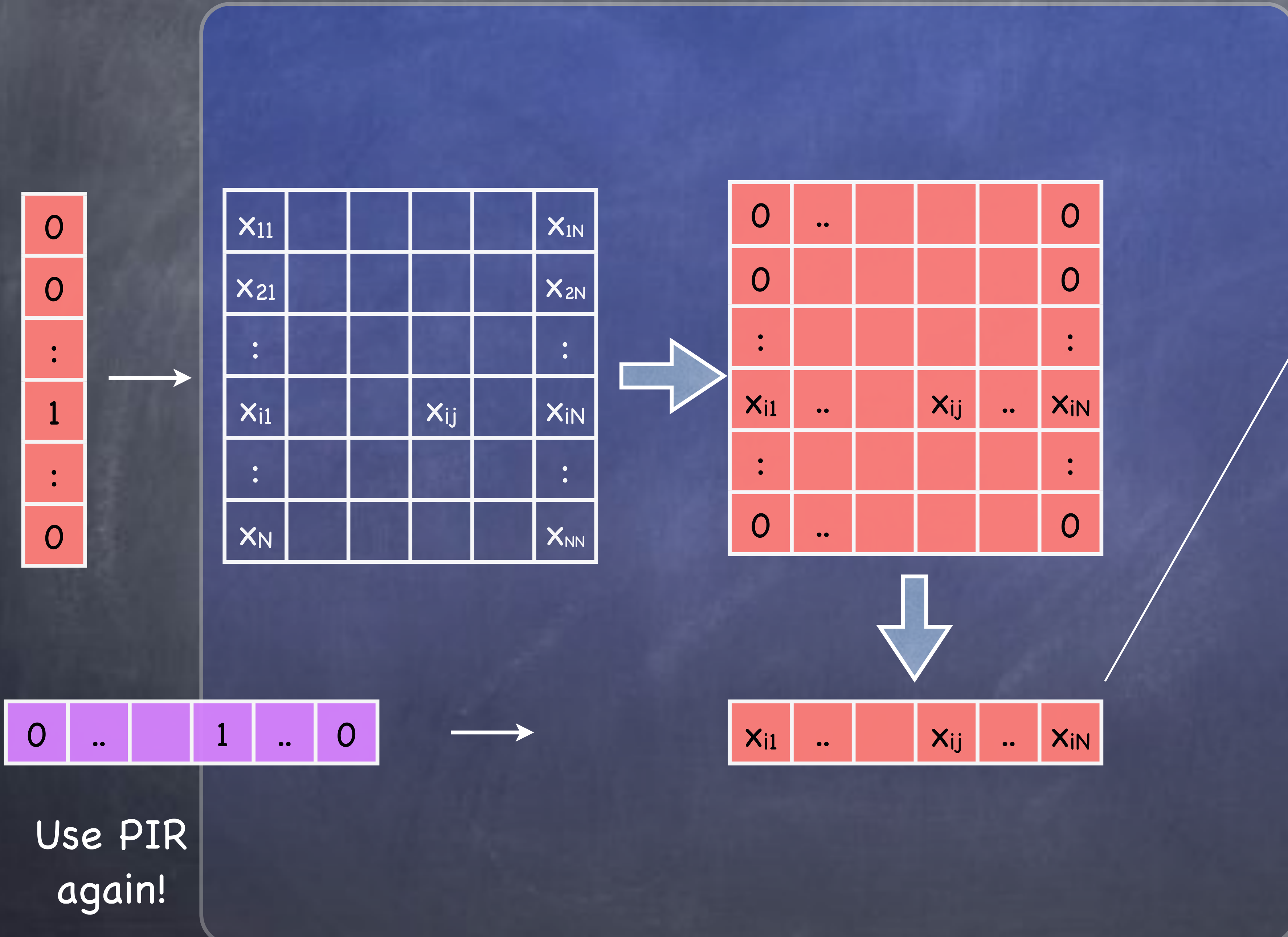

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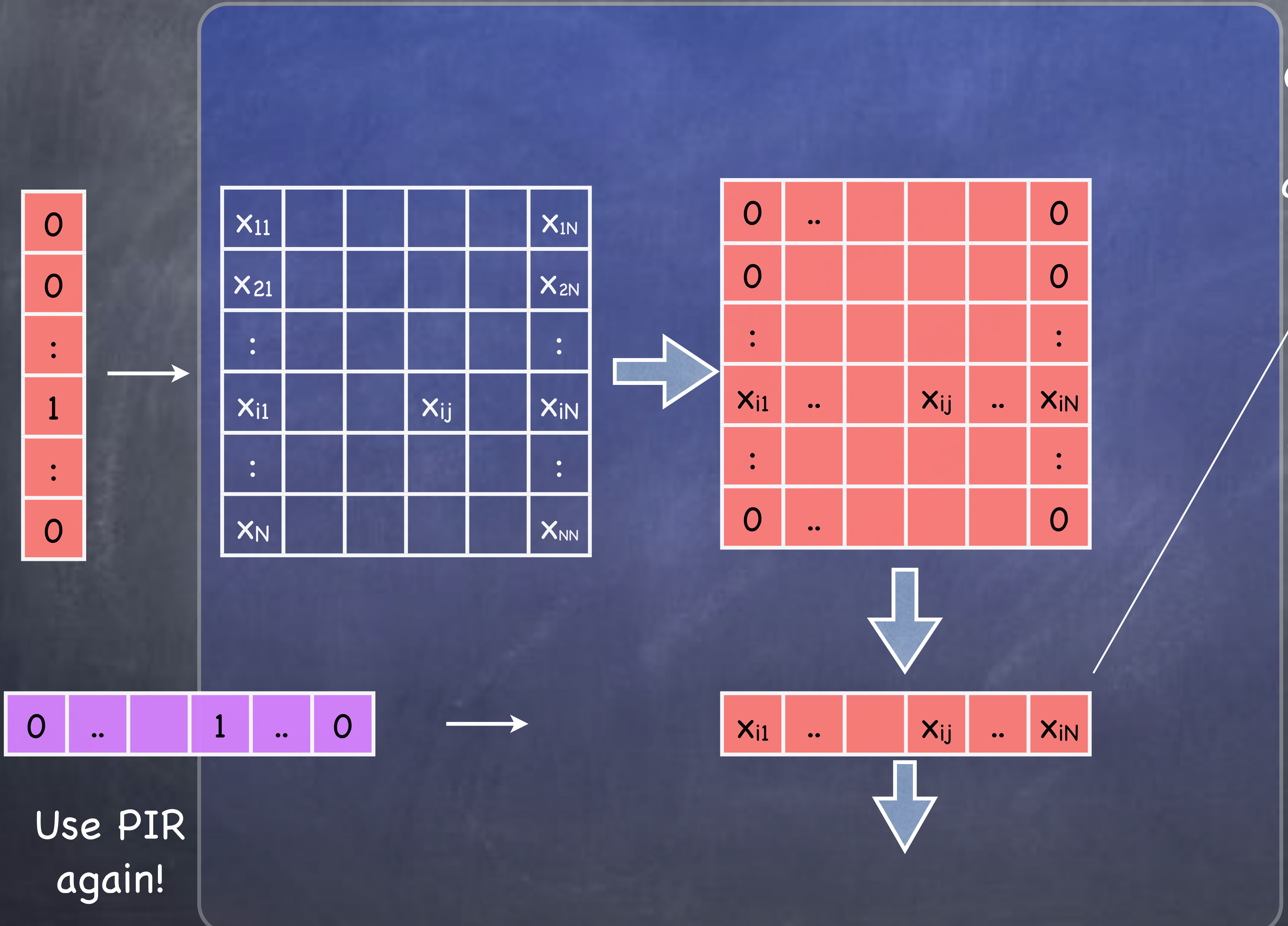

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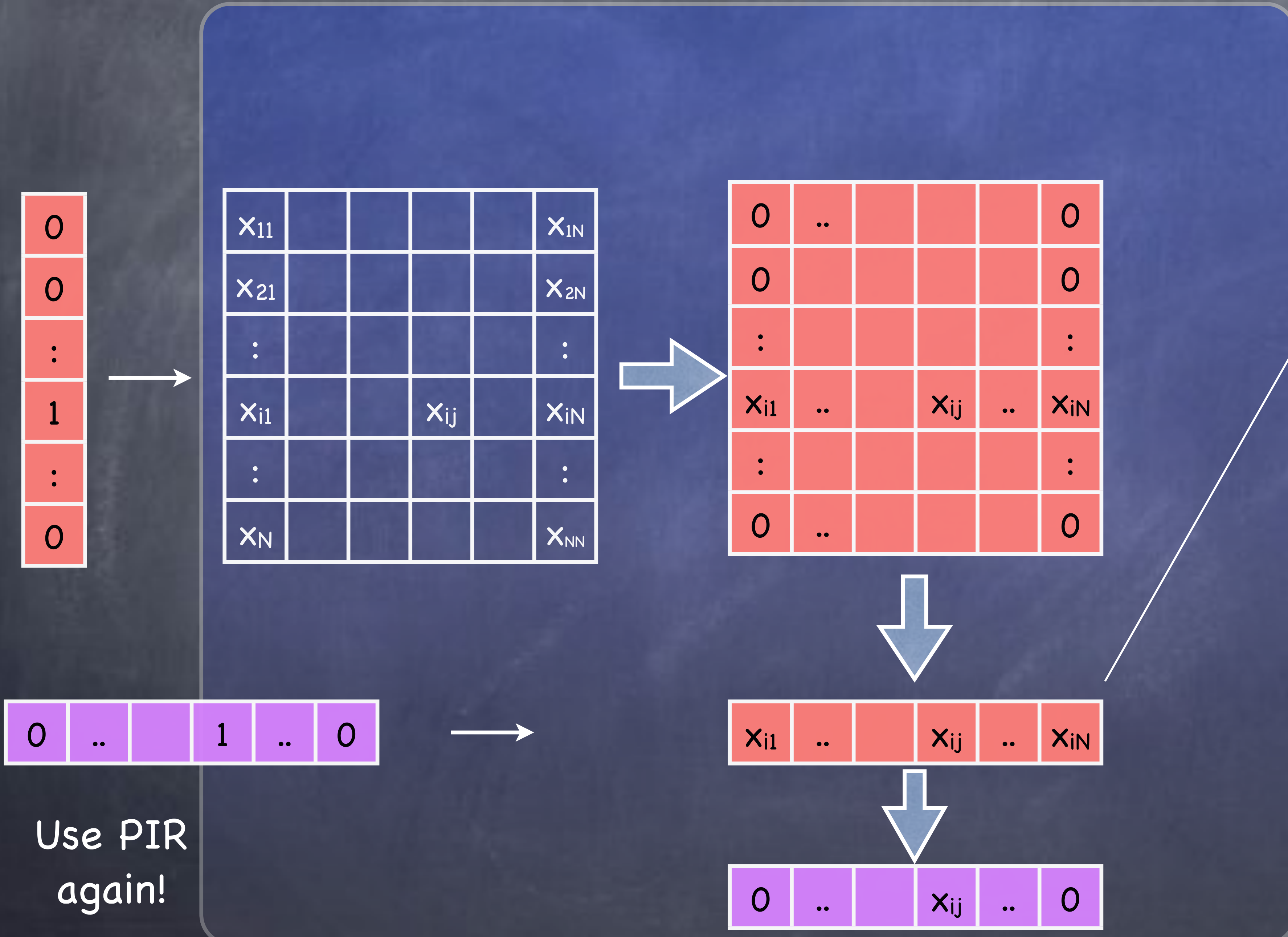

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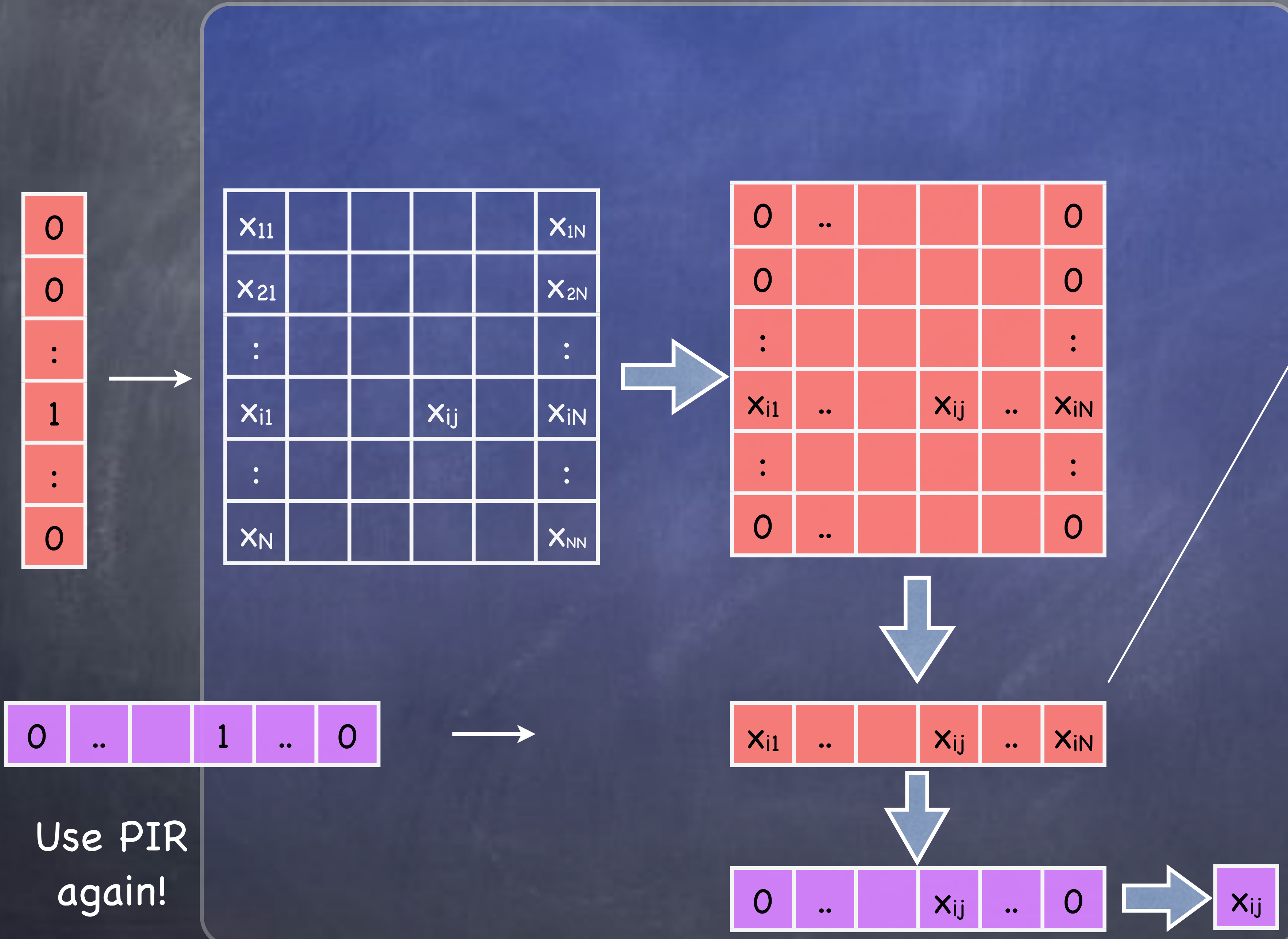

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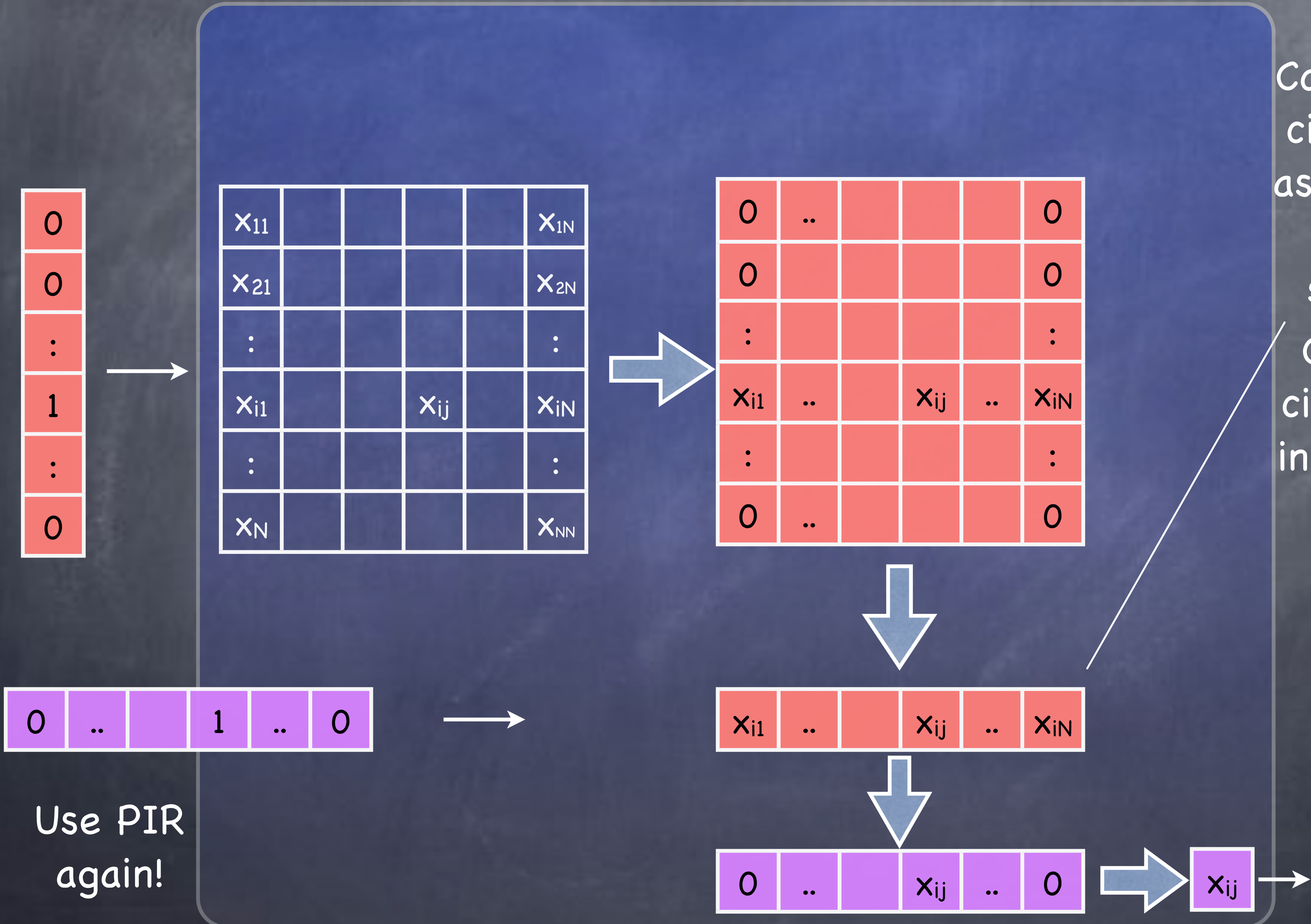

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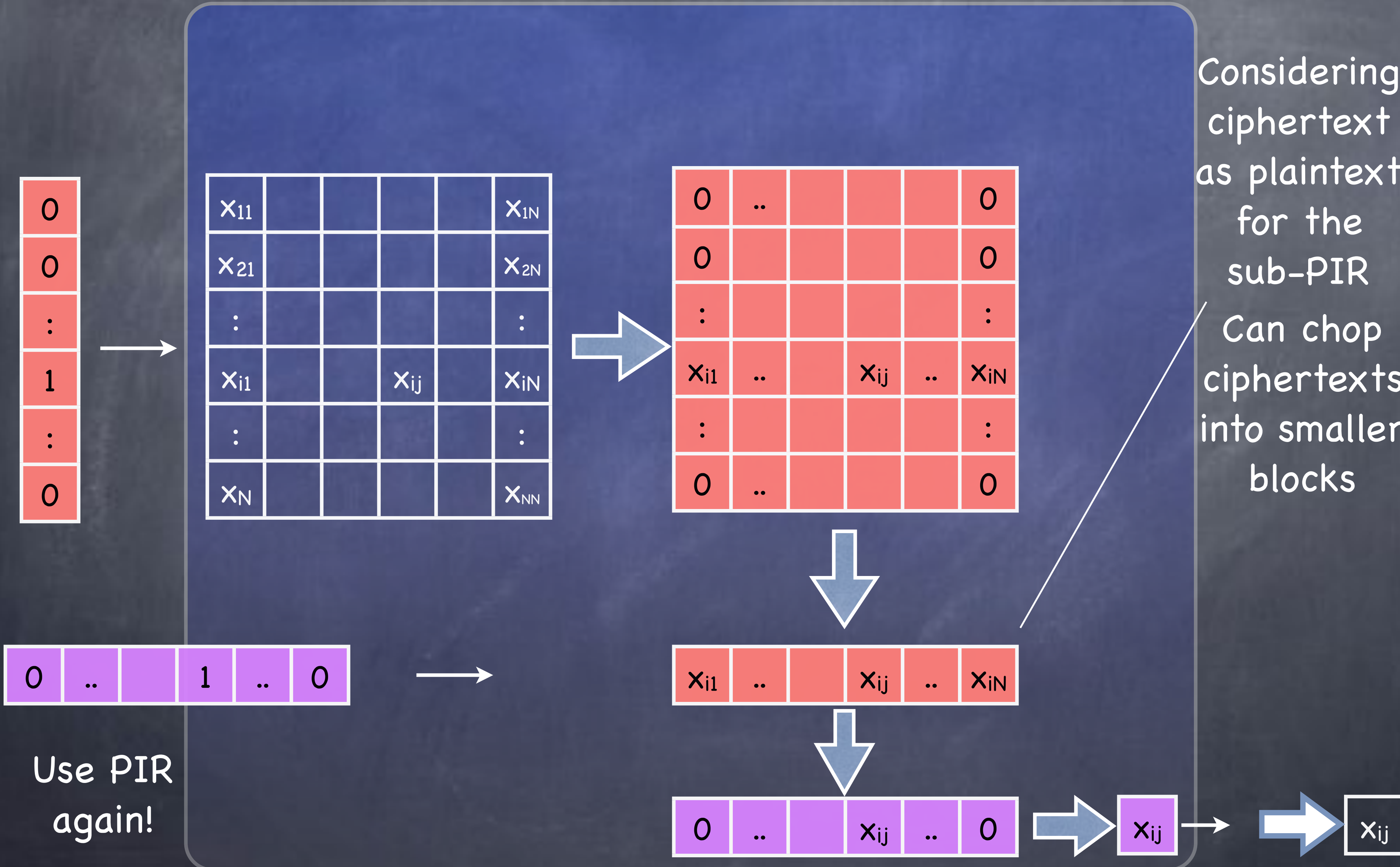

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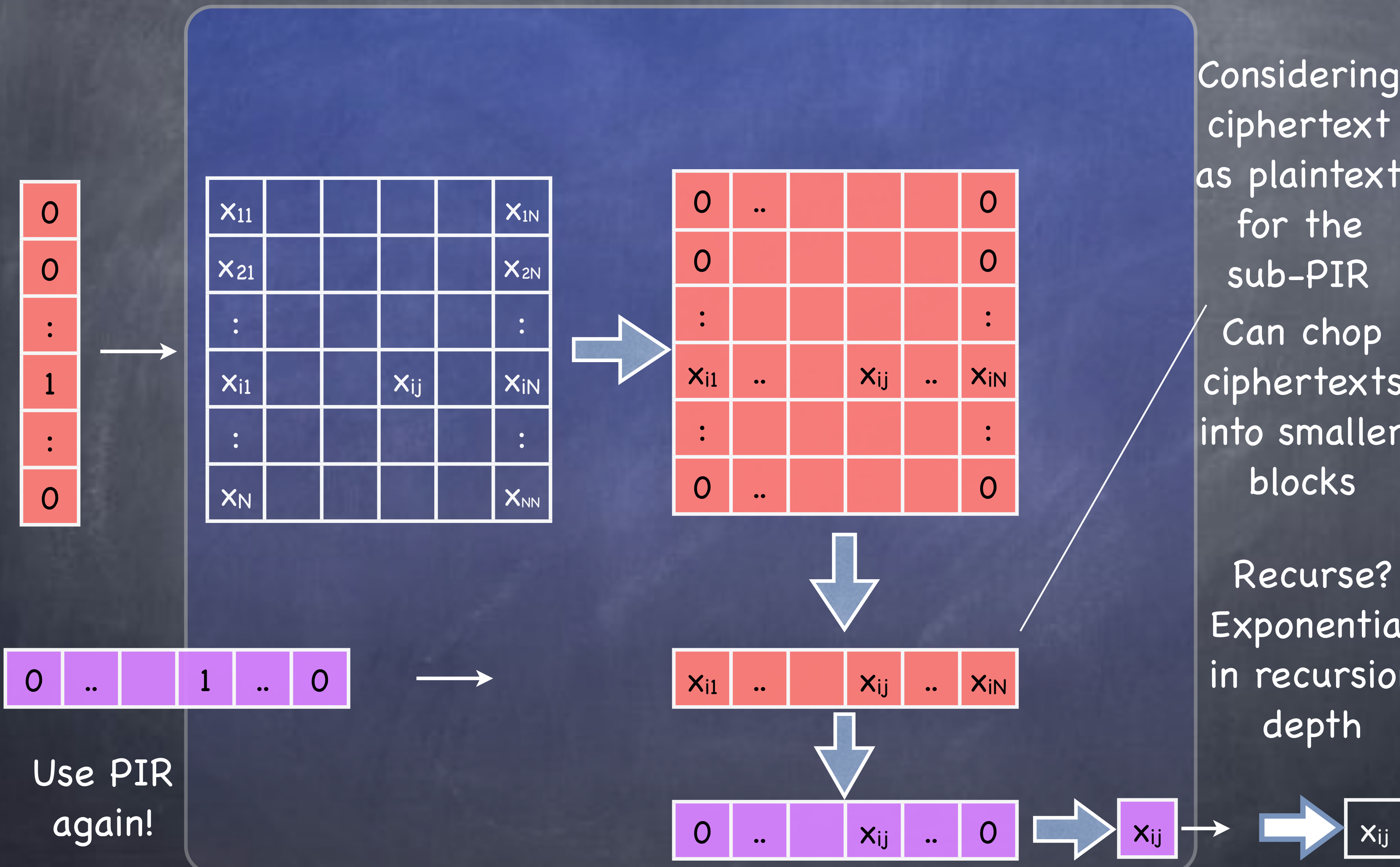

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- Does such a family of encryption schemes exist?

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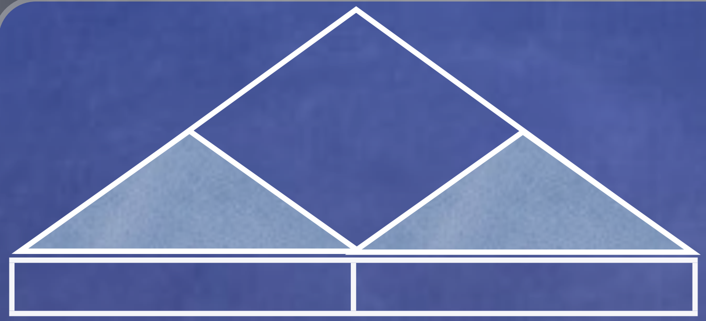
- Unlinkability: ReRand(c) = c.Enc(0) (using same s in Enc as for c)

Final PIR protocol



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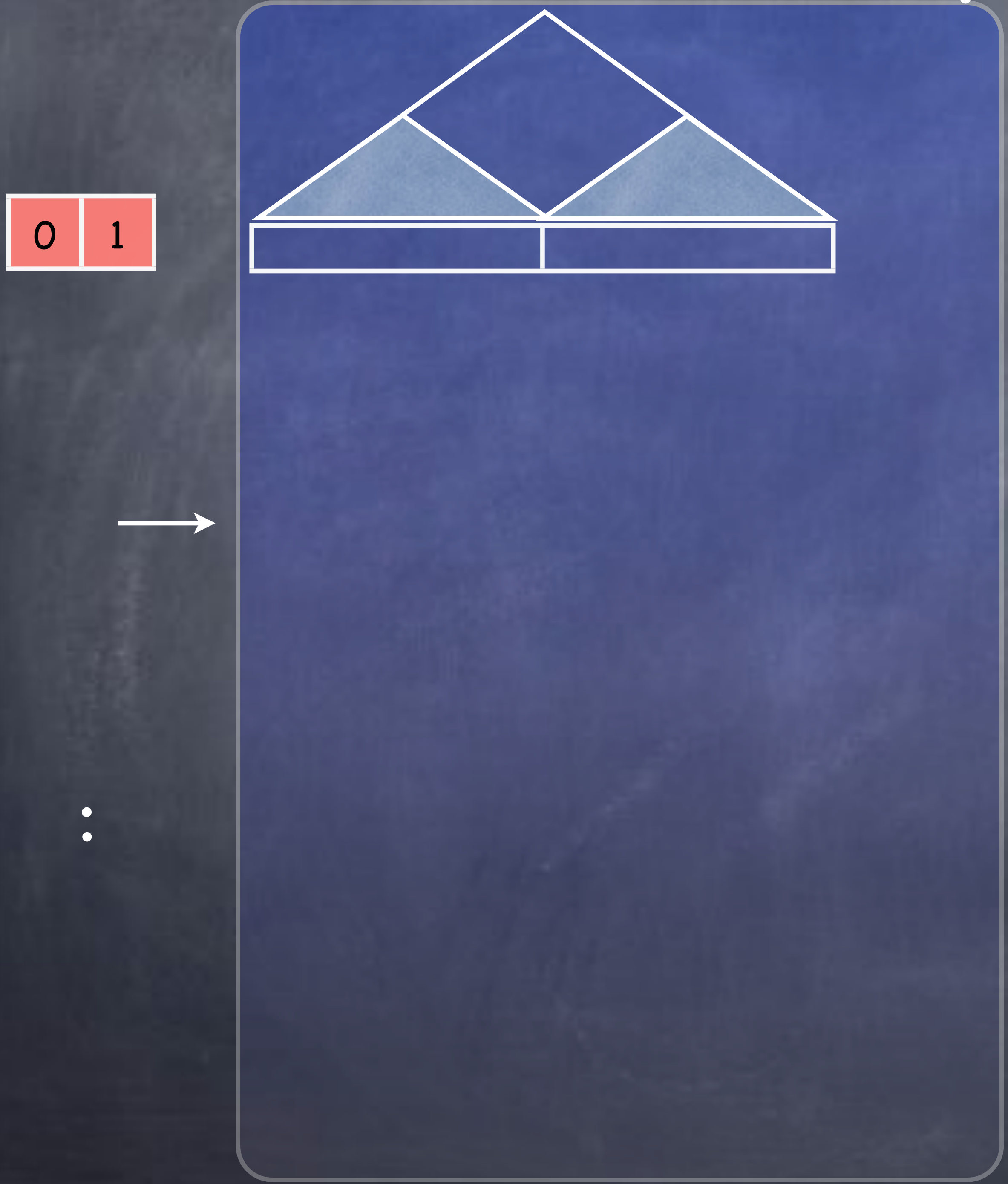

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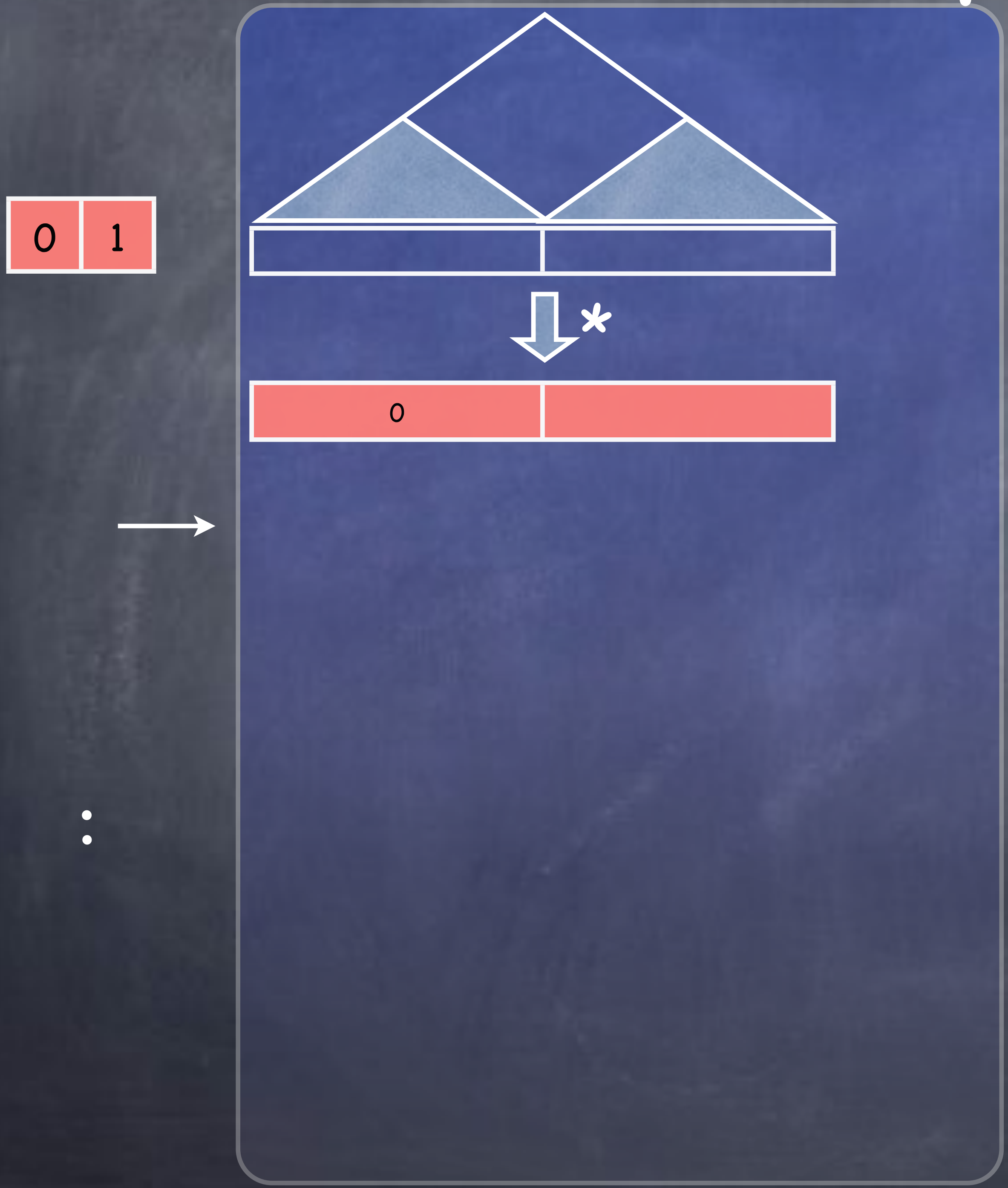

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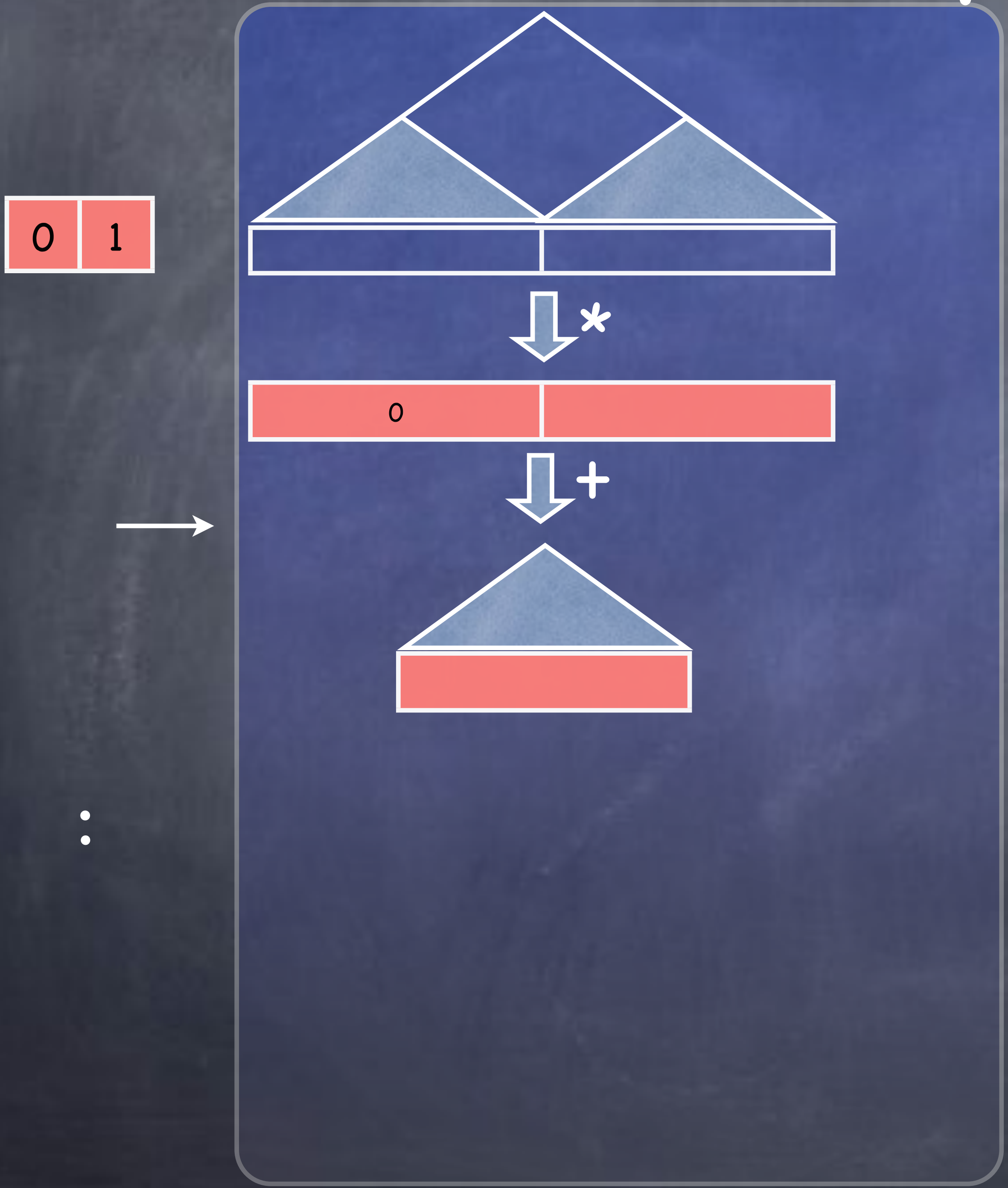

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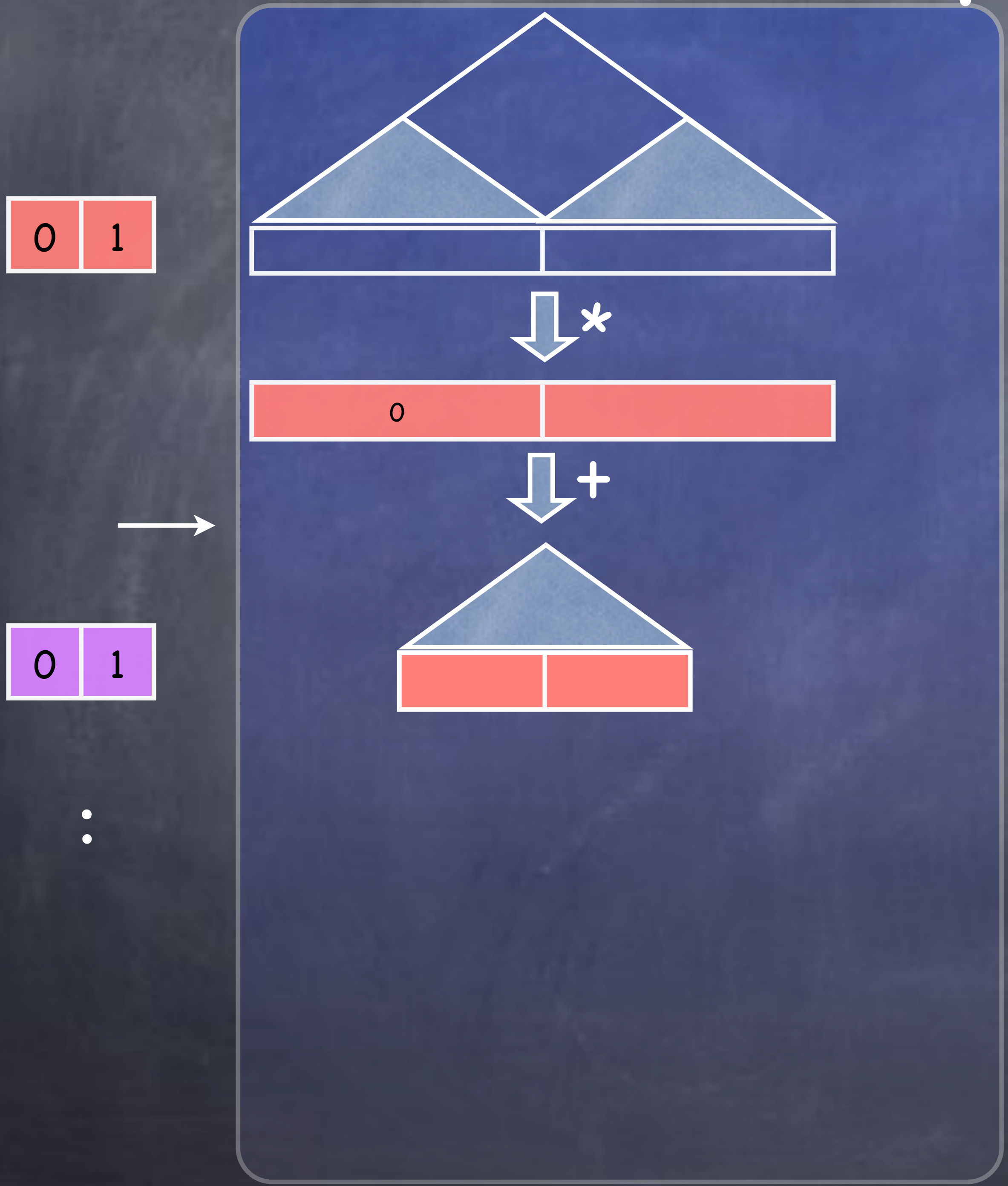

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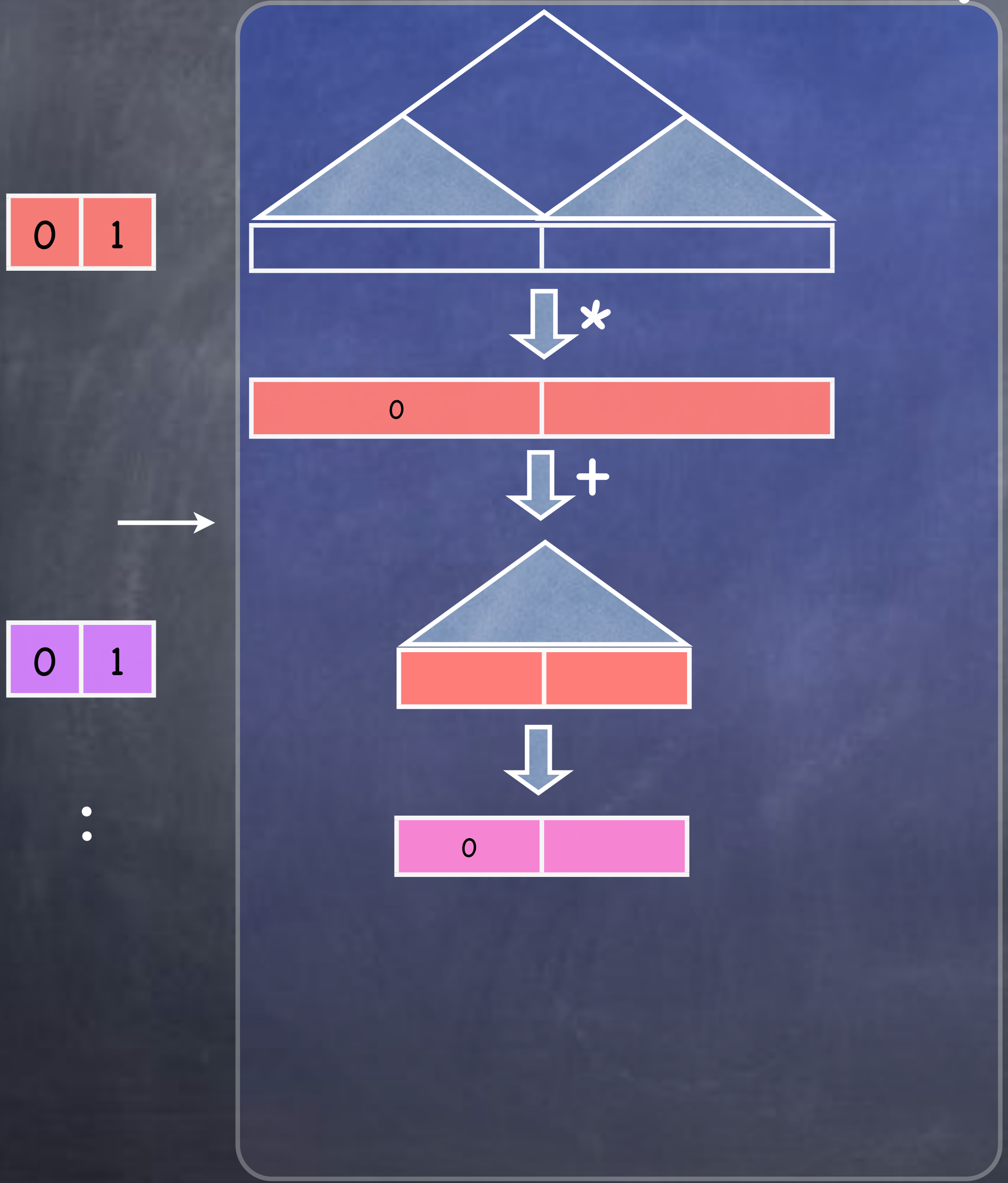

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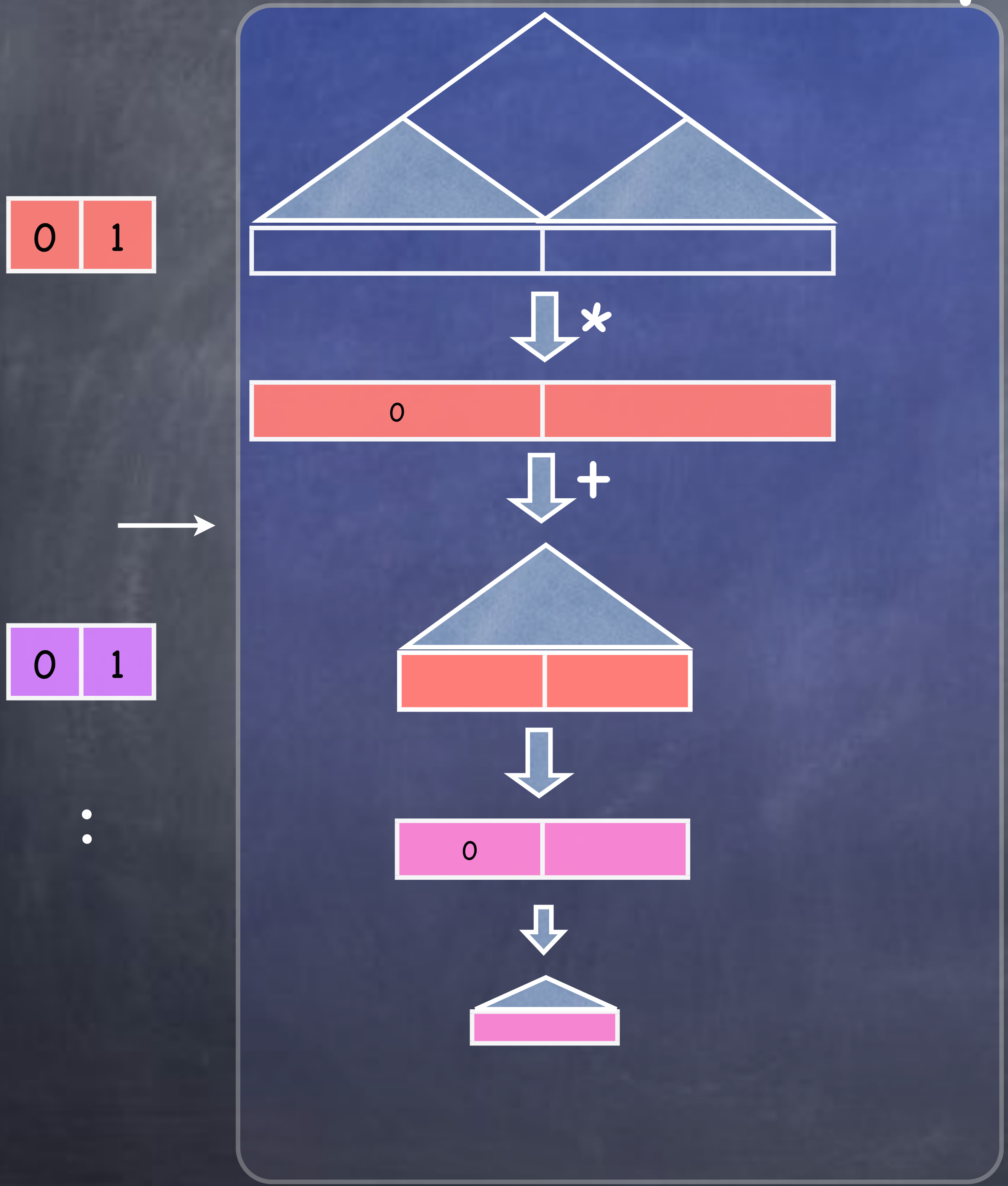

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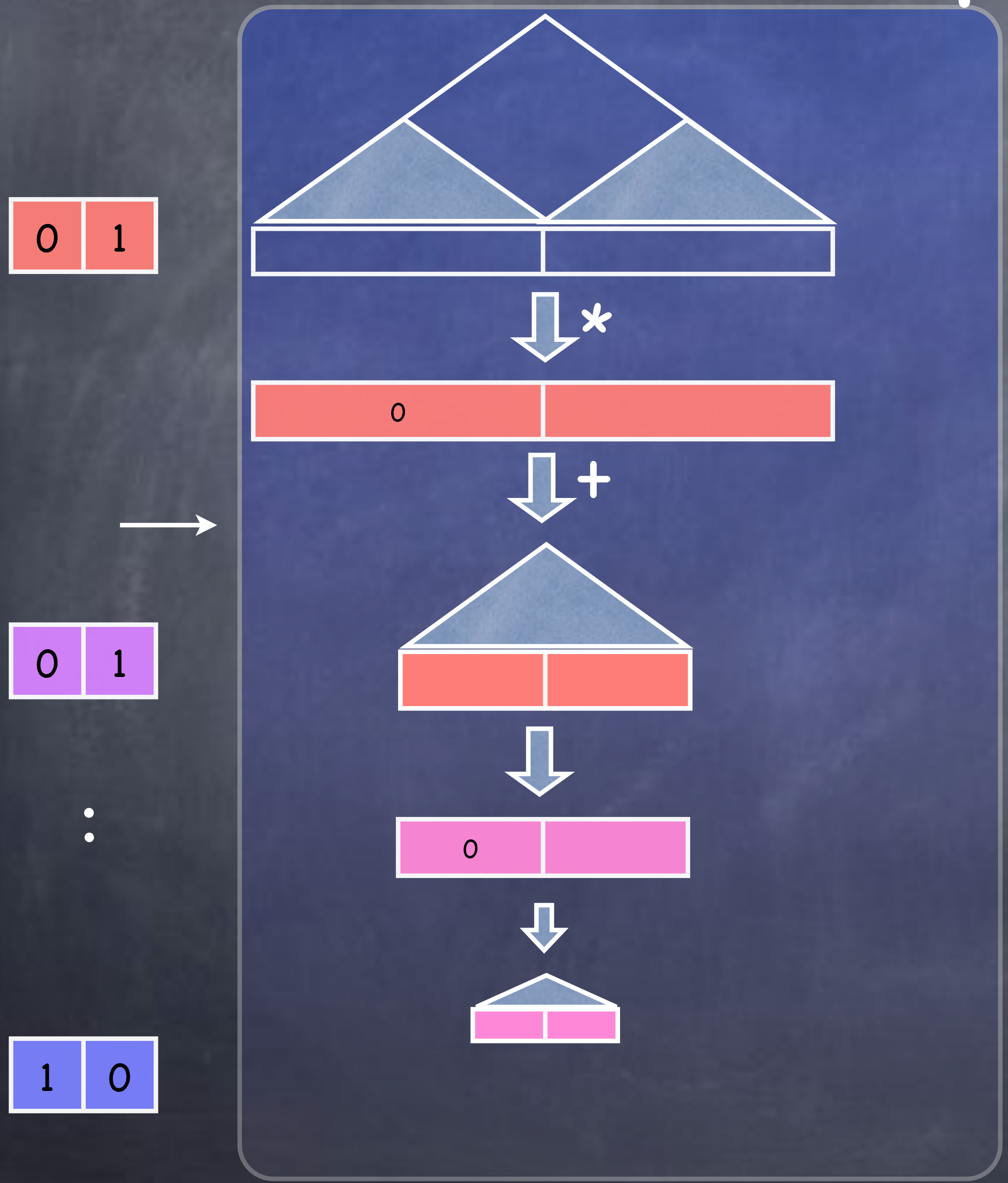

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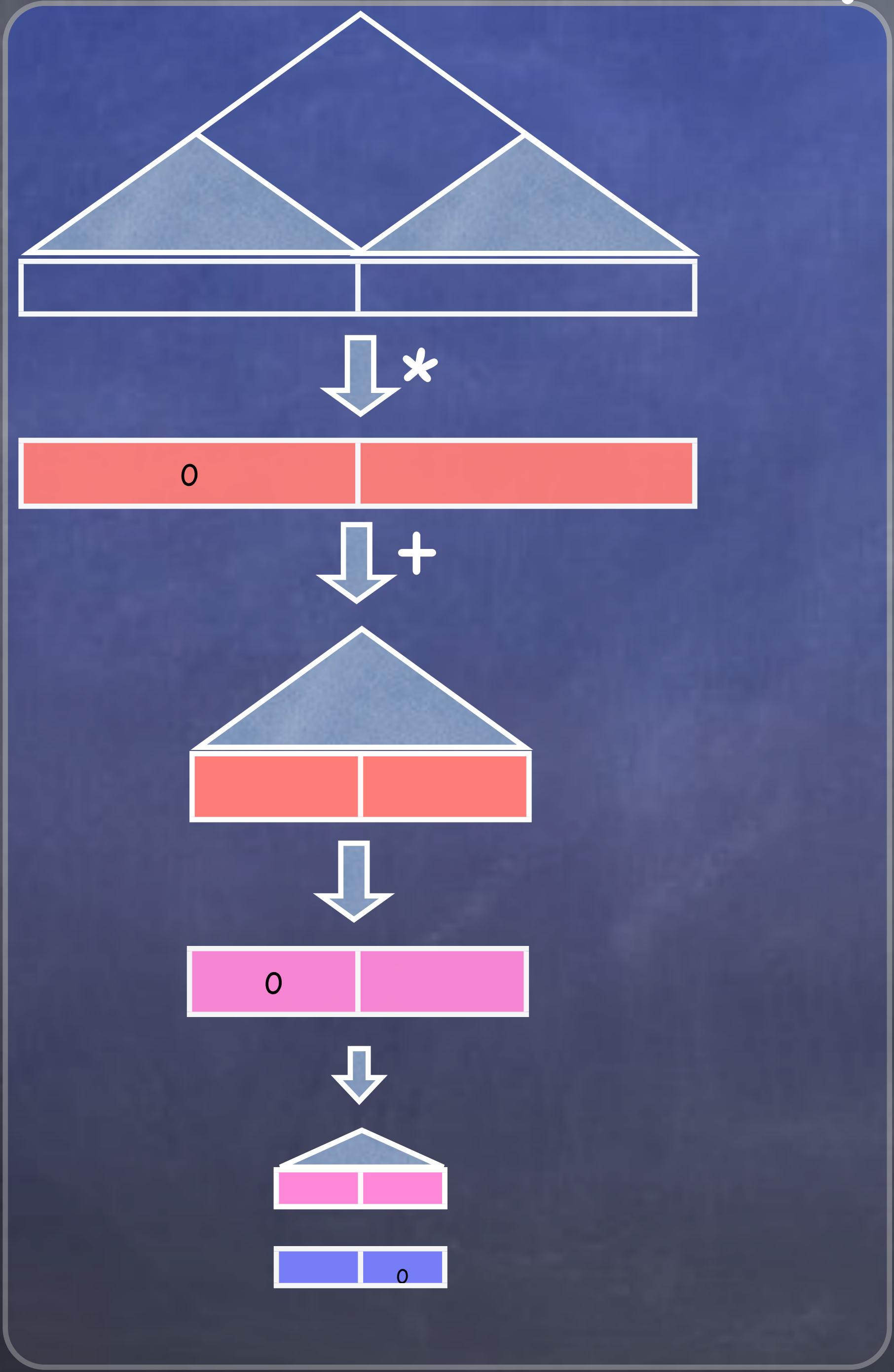

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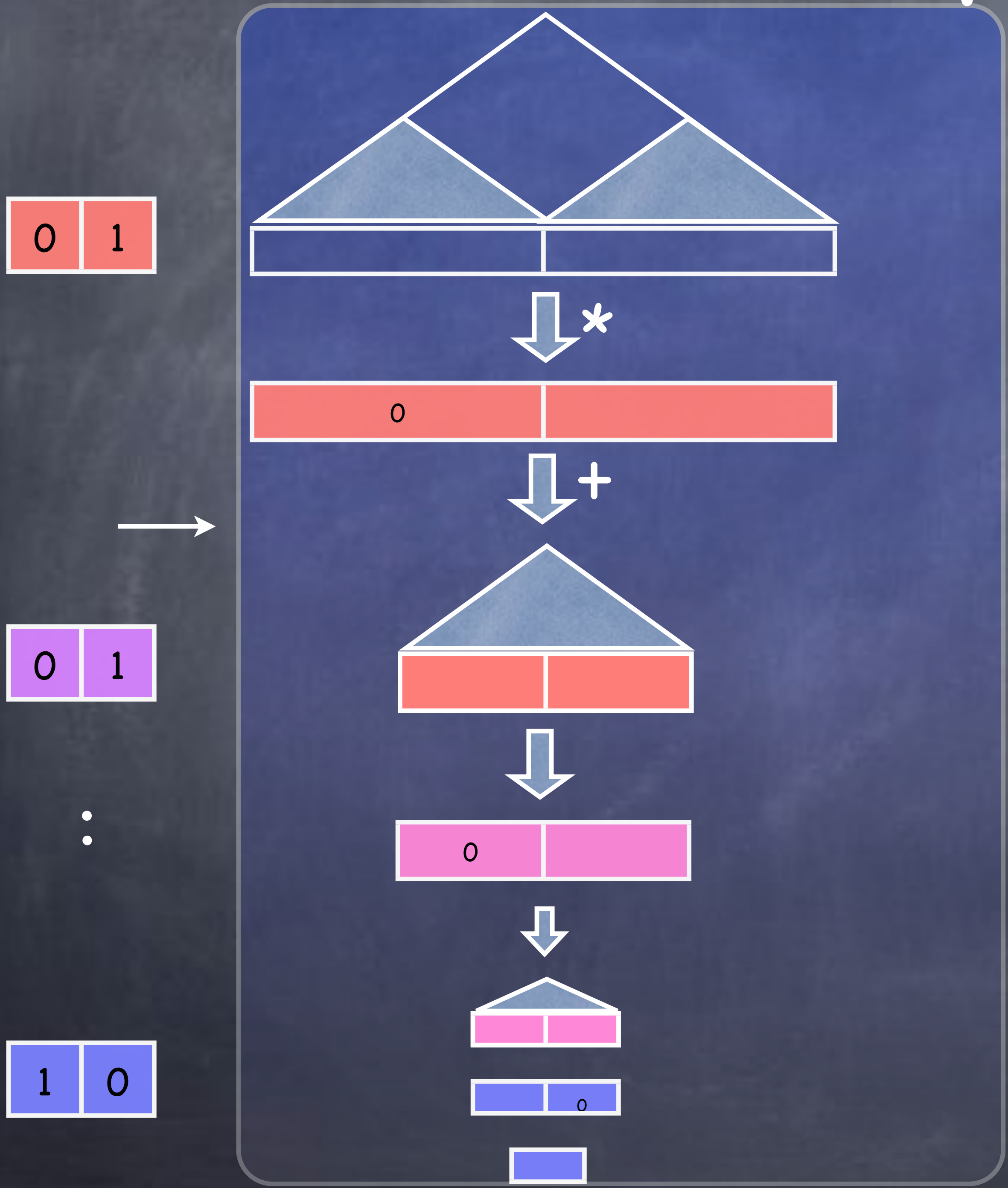

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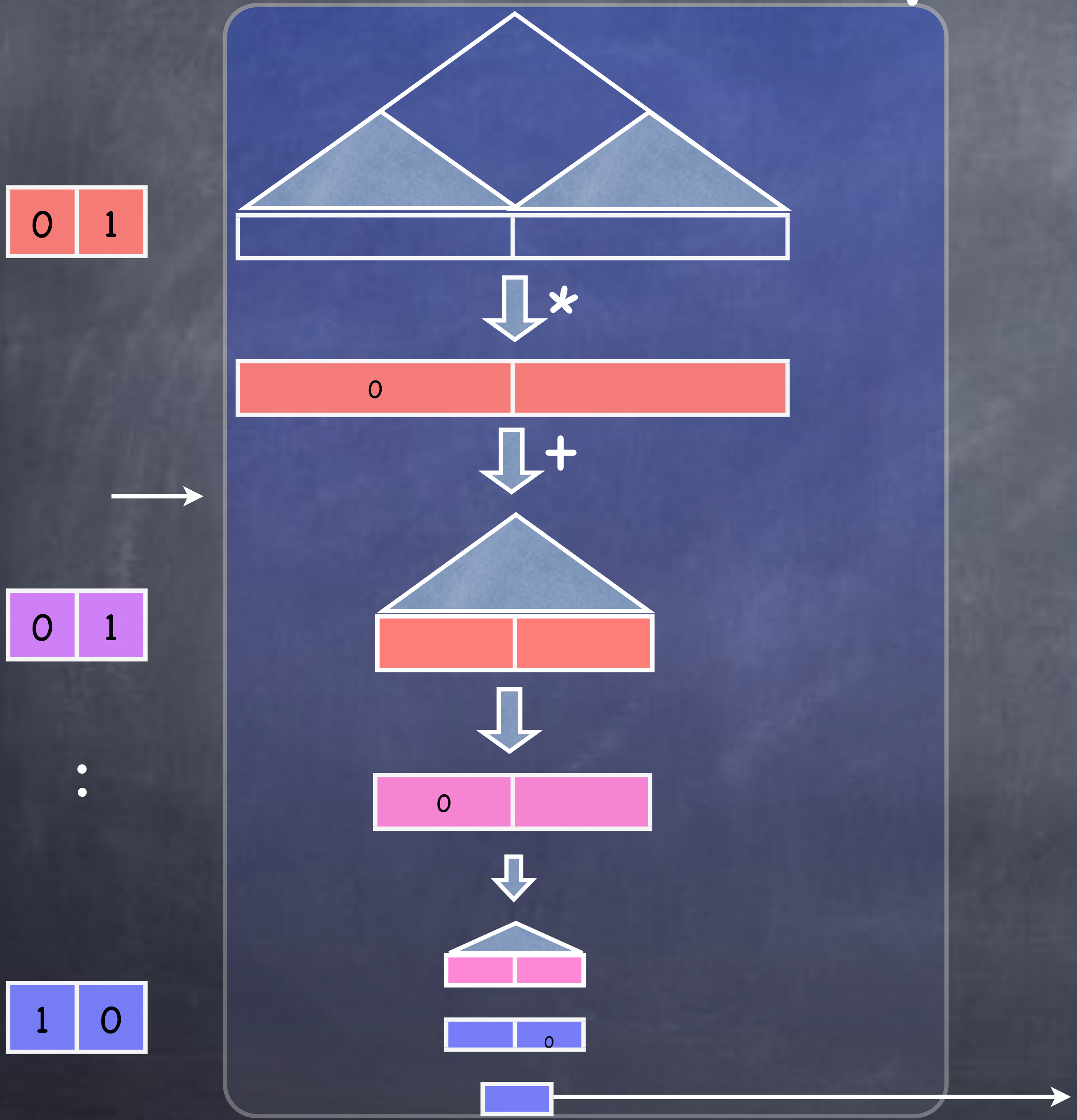

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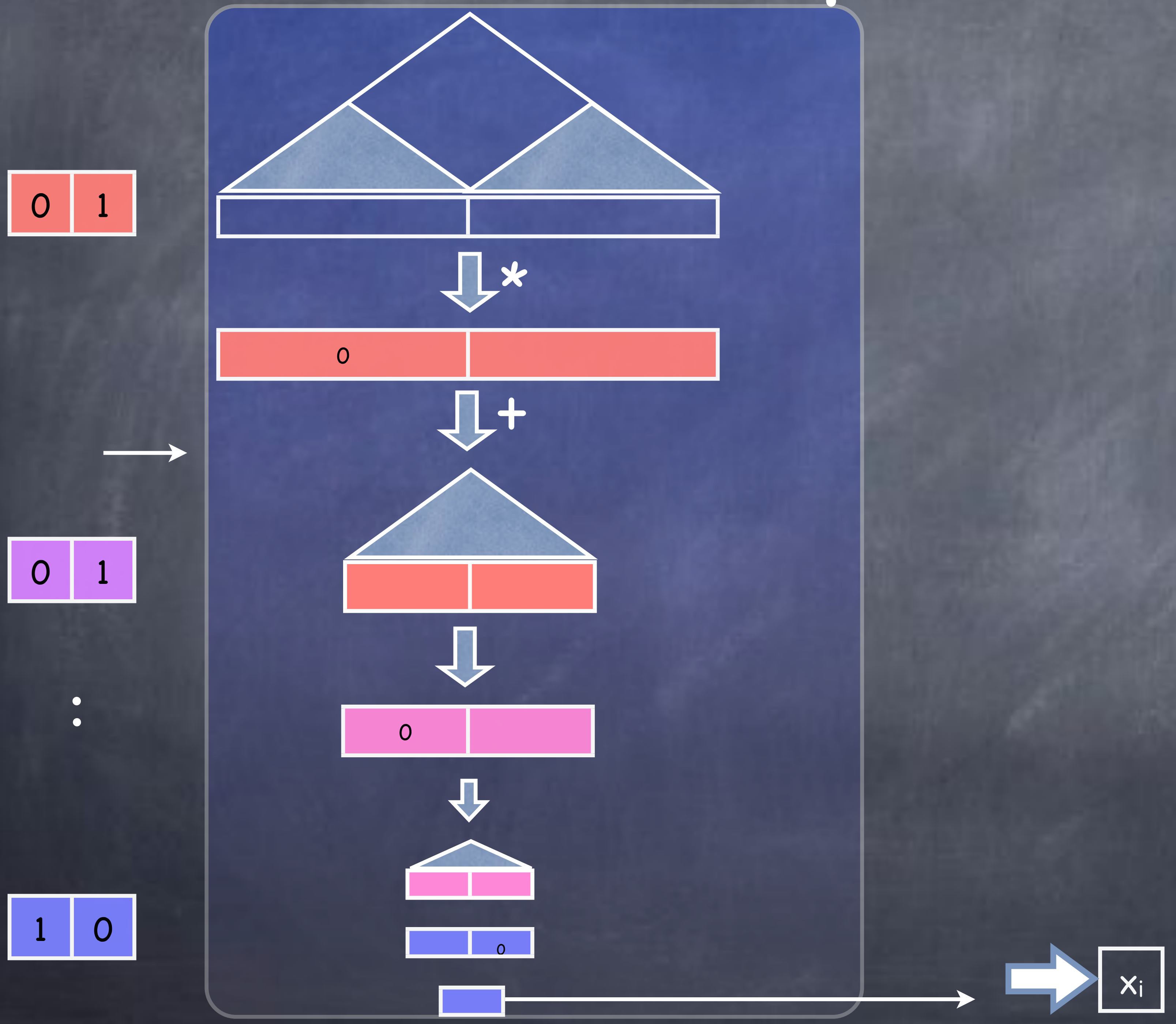

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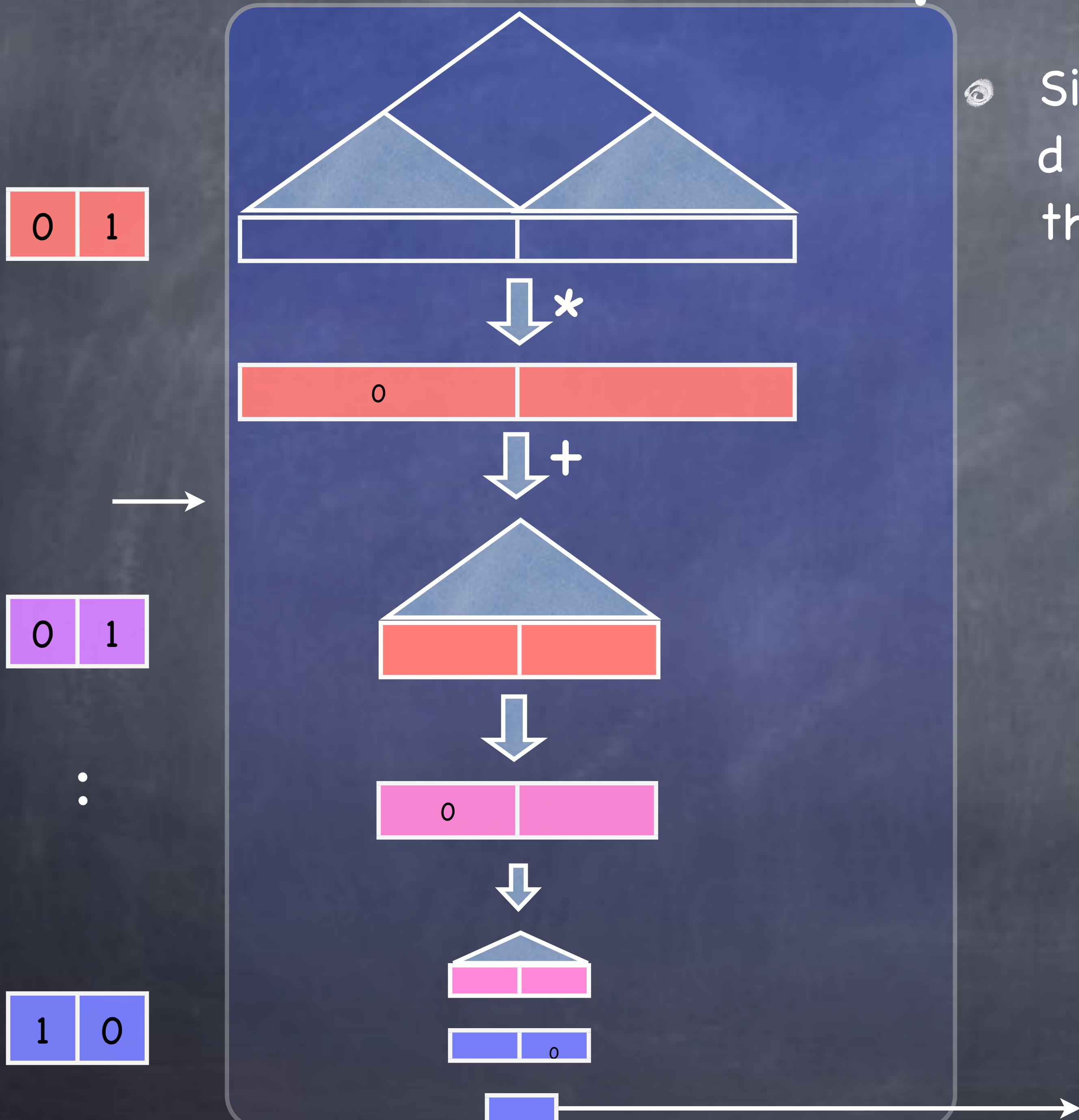

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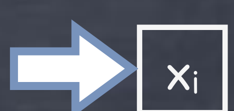

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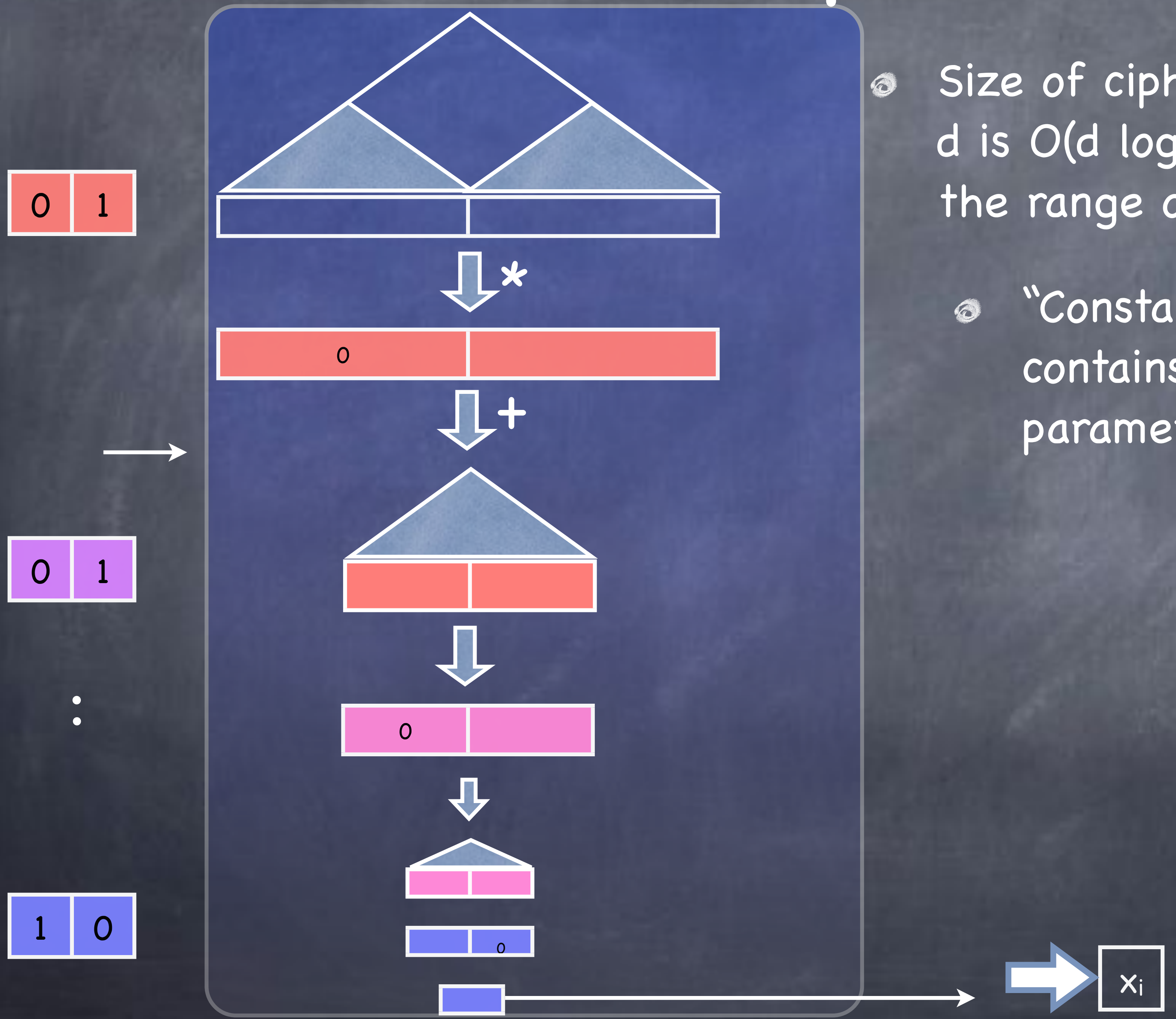

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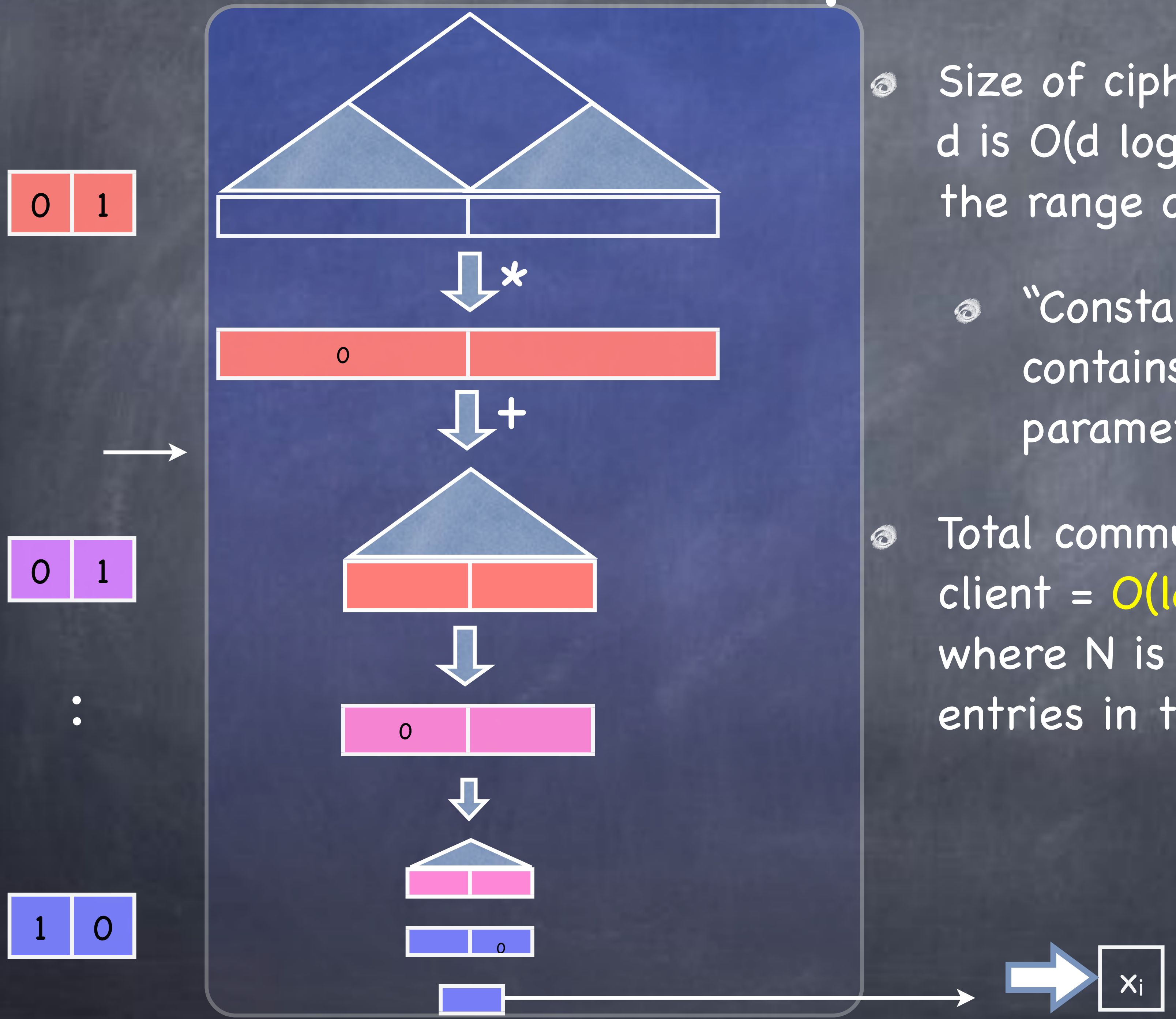

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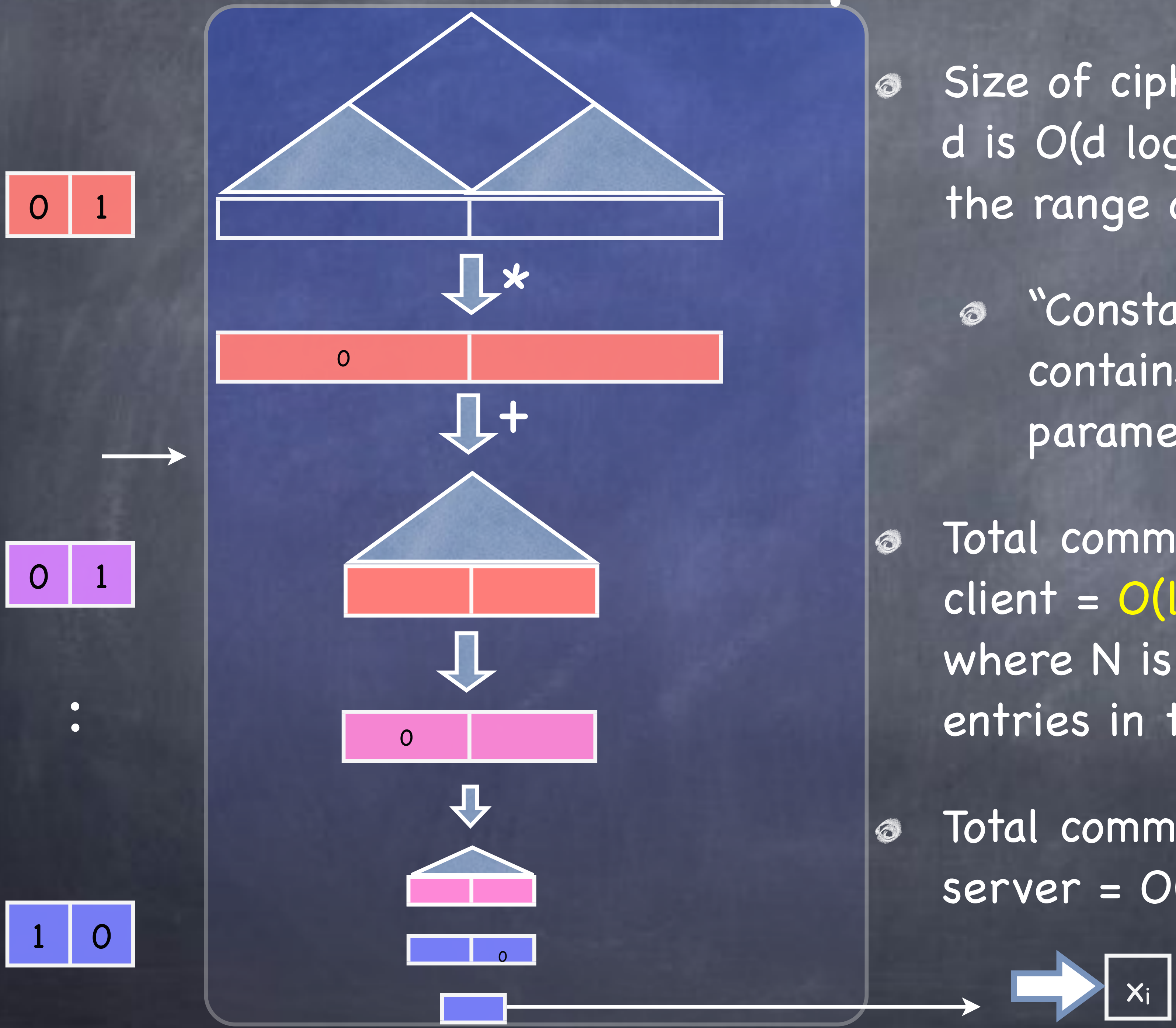

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- Total communication from client = $O(\log^2 N \log m)$, where N is the number of entries in the db
- Total communication from server = $O(\log N \log m)$

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- But not good for homomorphic encryption: say, an application needs to use addition modulo 10; can we use Paillier?

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    - Each time add a large random multiple of 10 (but not large enough to cause overflow): 9+3+10r and 2+10r are statistically close if r drawn from a large range

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- Coming up: more applications - in voting