Zero-Knowledge Proofs

Lecture 15

Interactive Proofs

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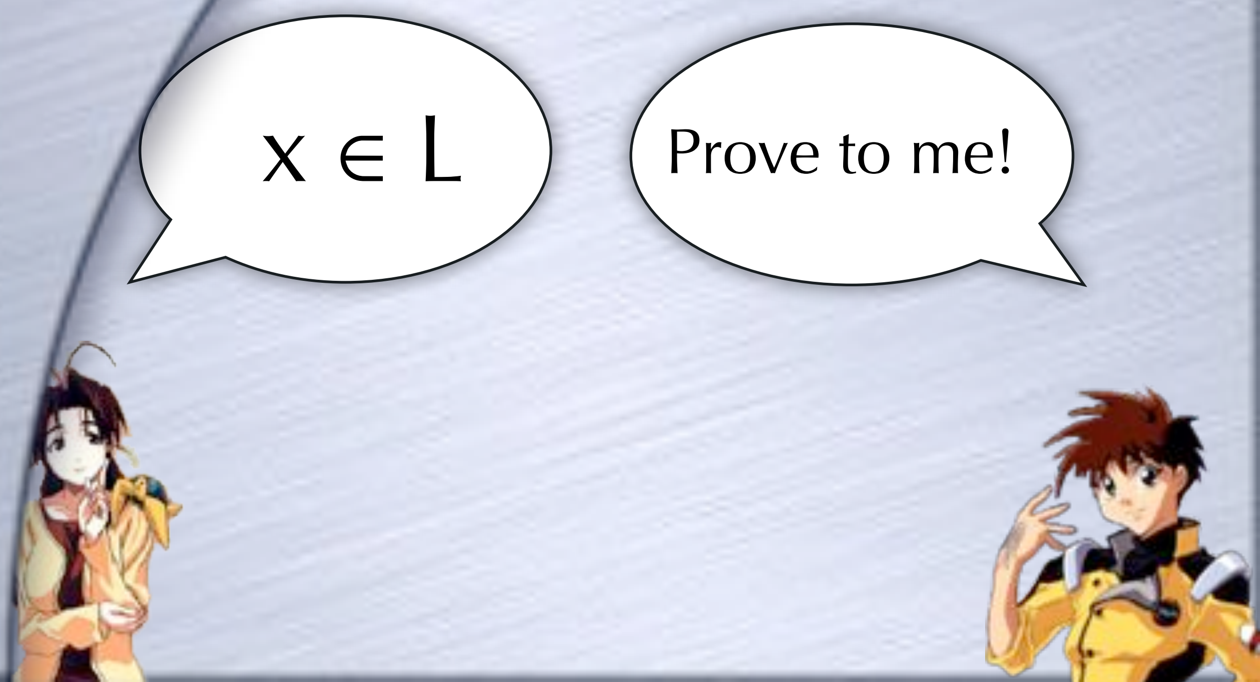

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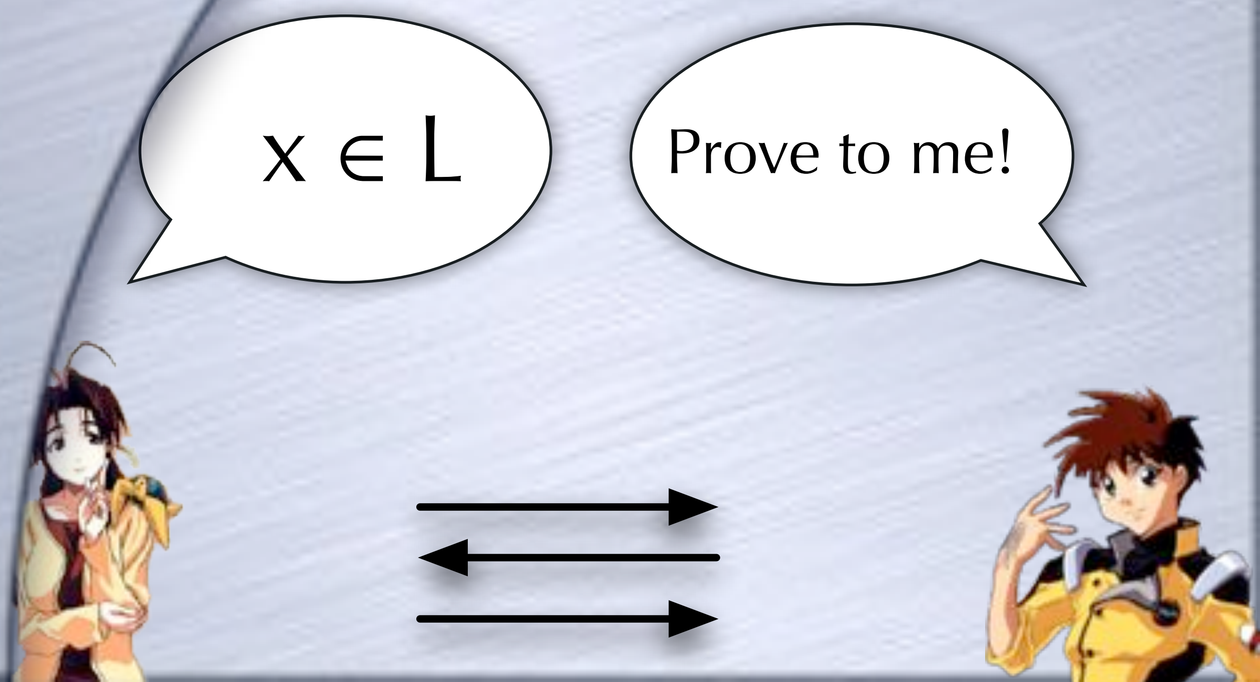

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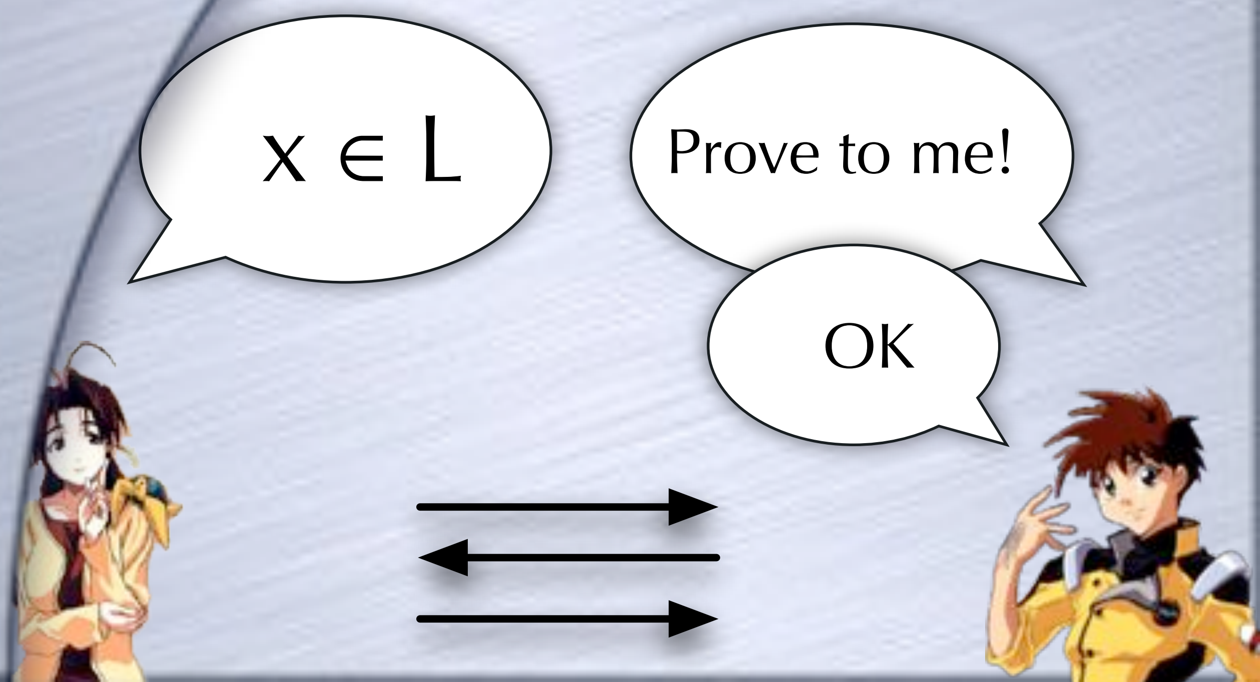

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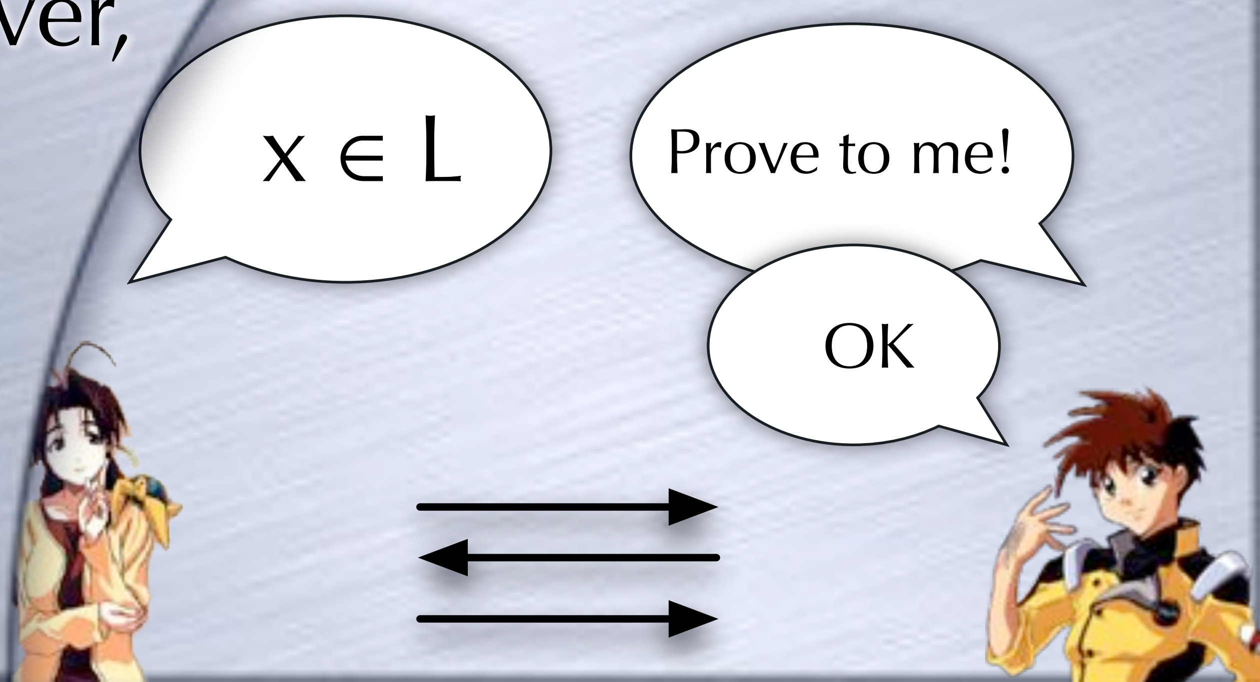
- ***Prover*** wants to convince ***verifier*** that x has some property
  - i.e. x is in “language” L
- All powerful prover, computationally bounded verifier (for now)



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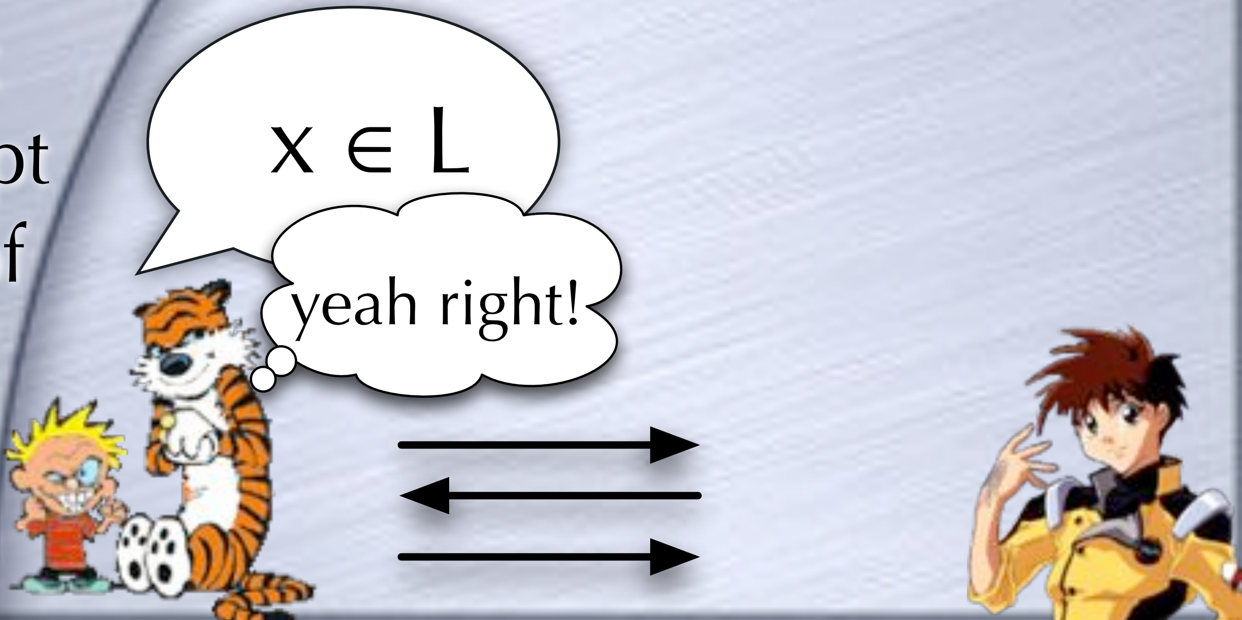

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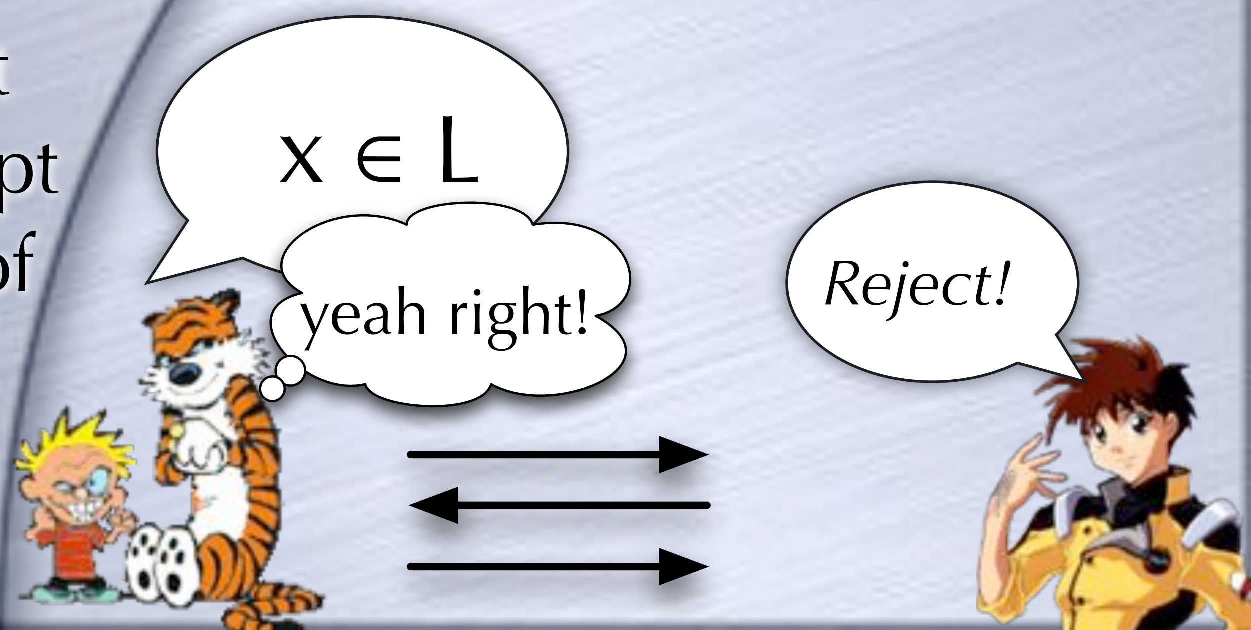

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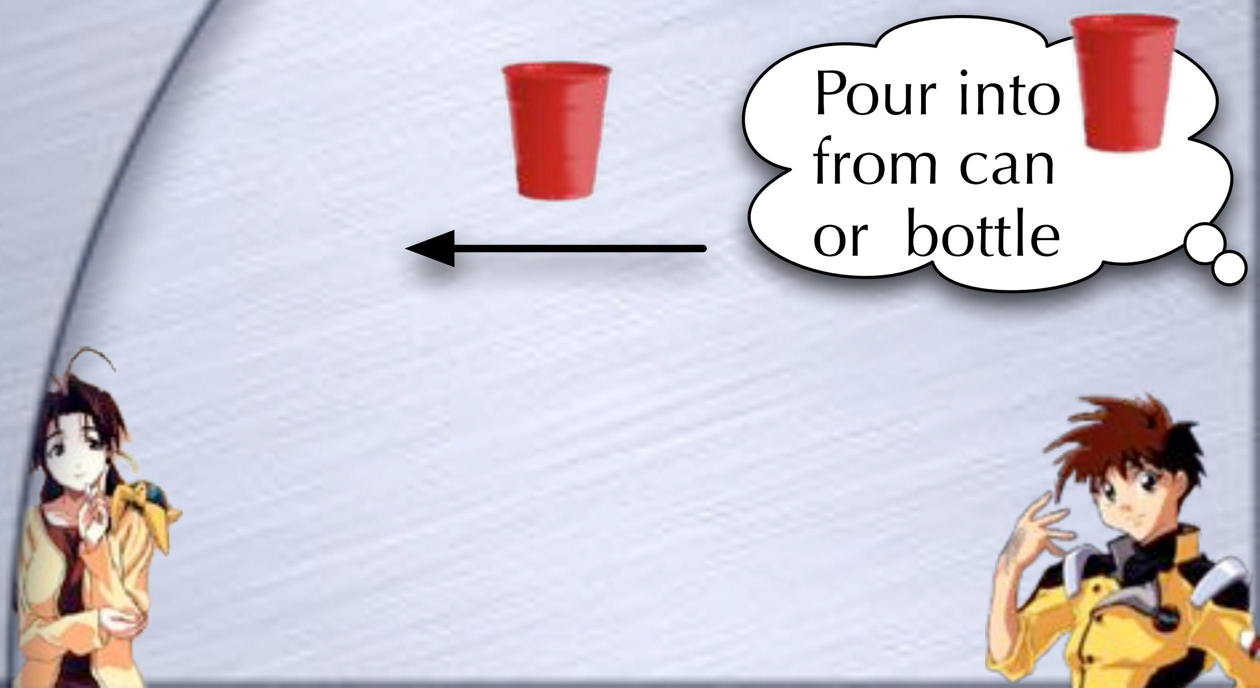

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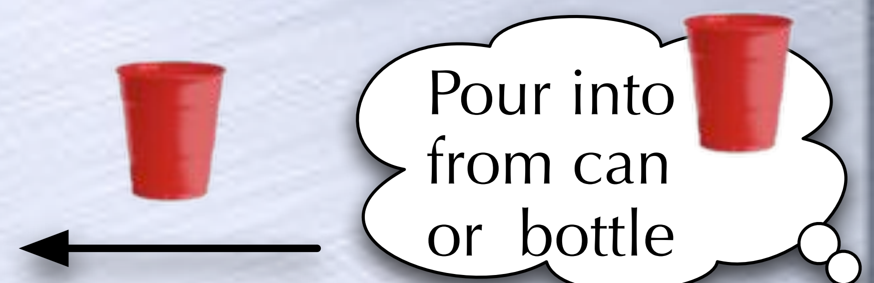

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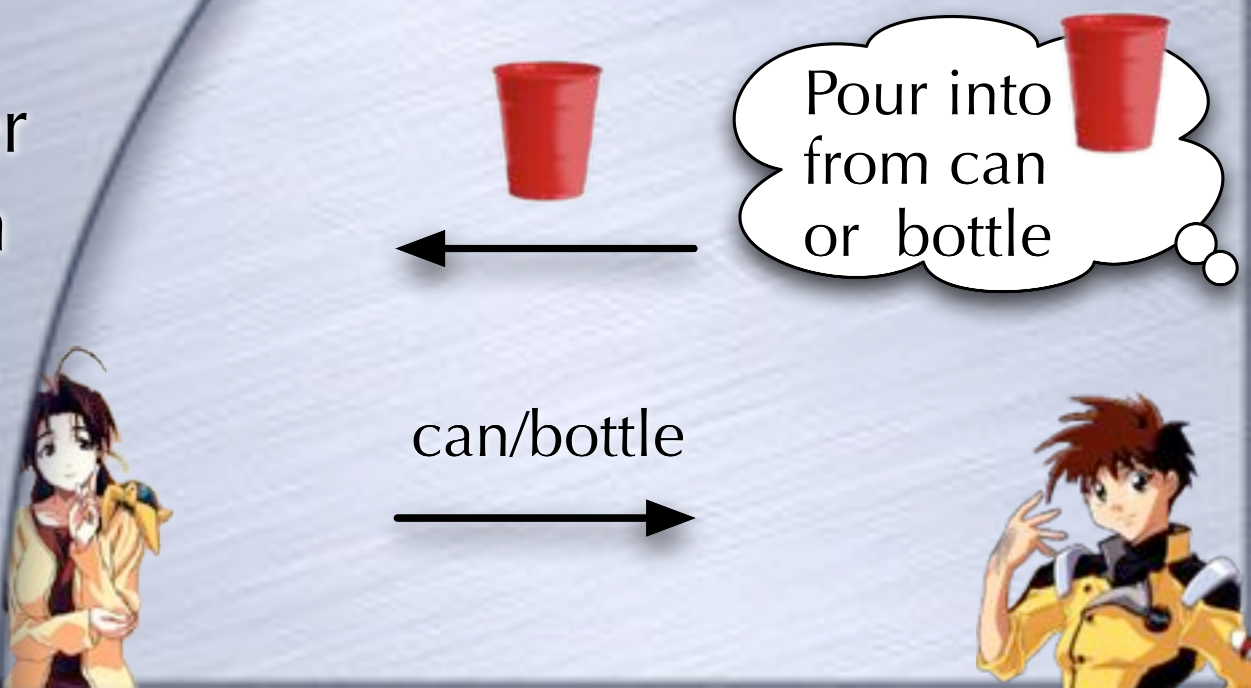

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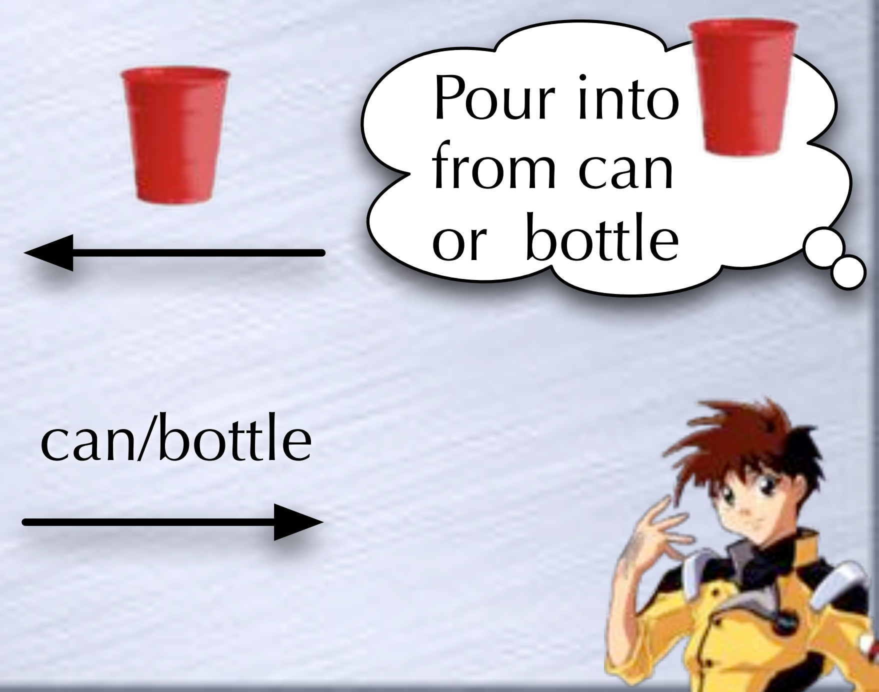

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Set G^* to be $\pi(G_0)$ or $\pi(G_1)$ (π random)

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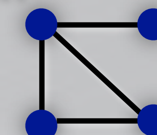
Isomorphism: Same graph can be represented as a matrix in different ways:

$$\text{e.g., } G_0 = \begin{matrix} 0&1&0&1 \\ 1&0&0&1 \\ 0&0&0&1 \\ 1&1&1&0 \end{matrix} \;\&\; G_1 = \begin{matrix} 0&1&0&1 \\ 1&0&1&1 \\ 0&1&0&0 \\ 1&1&0&0 \end{matrix}$$

both are isomorphic to the graph represented by the drawing



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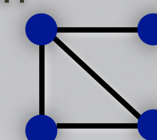

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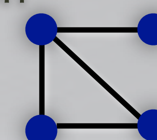

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$$\xleftarrow{\quad G^* \quad}$$

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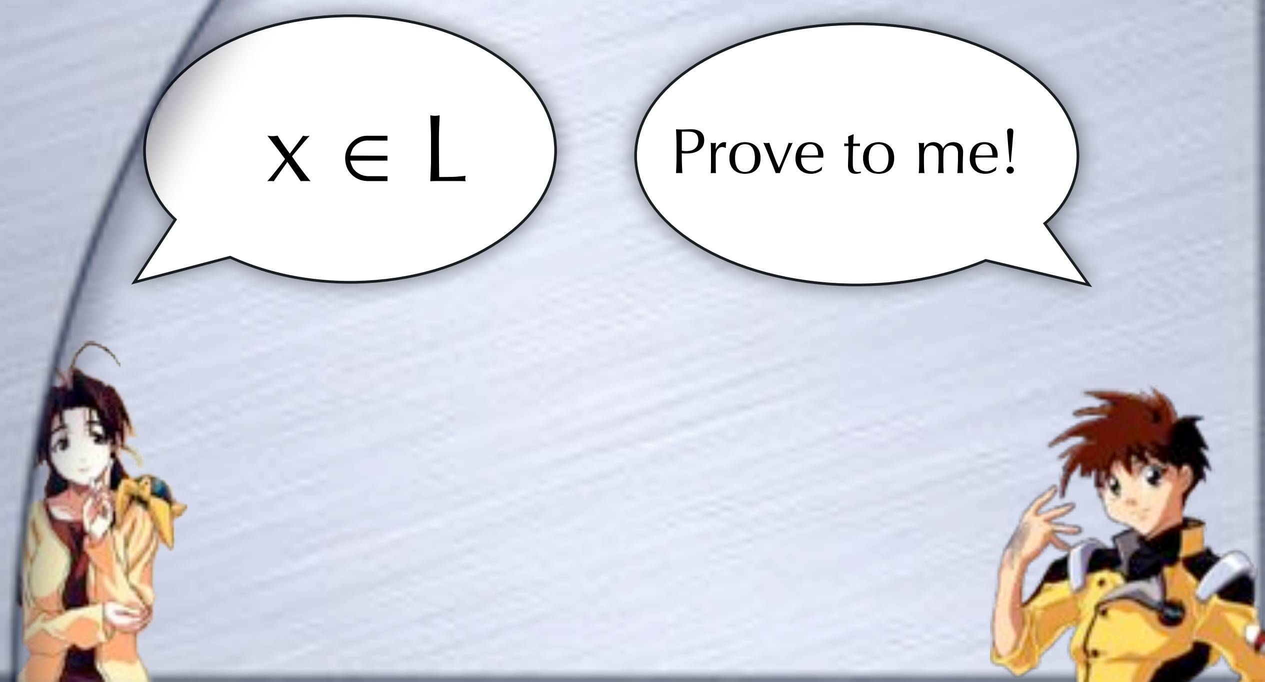
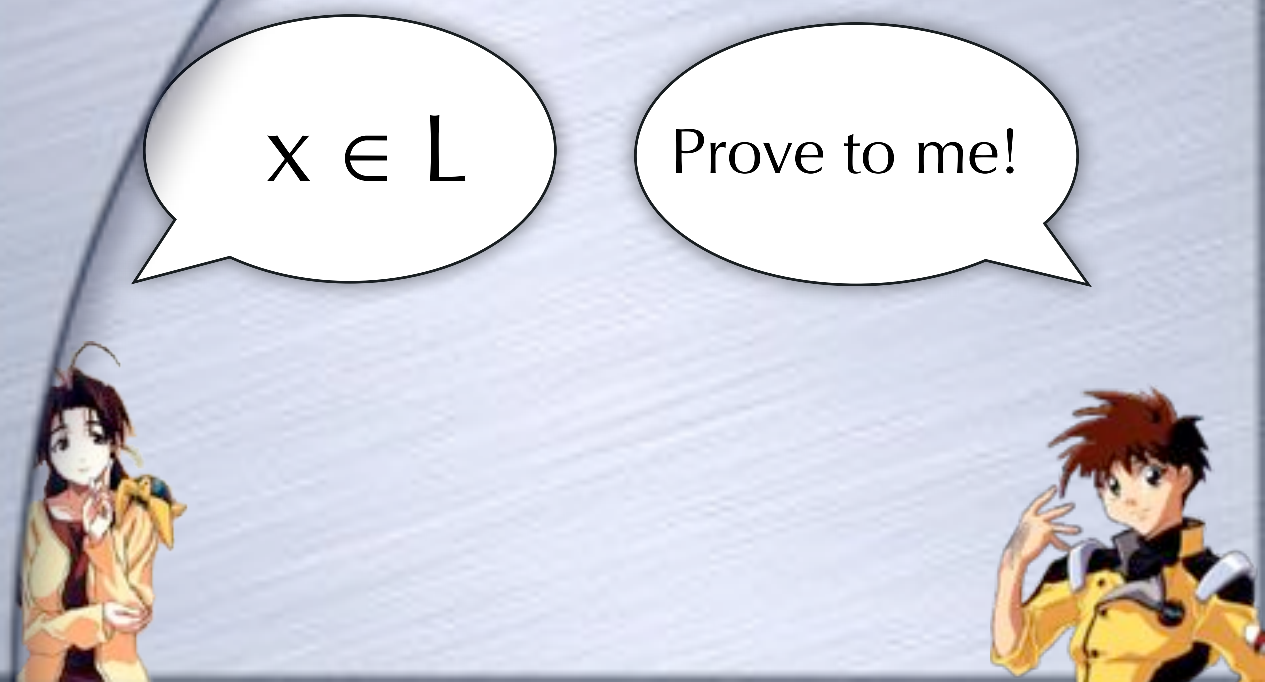
Proofs for NP languages



Proofs for NP languages

- Proving membership in an **NP** language L



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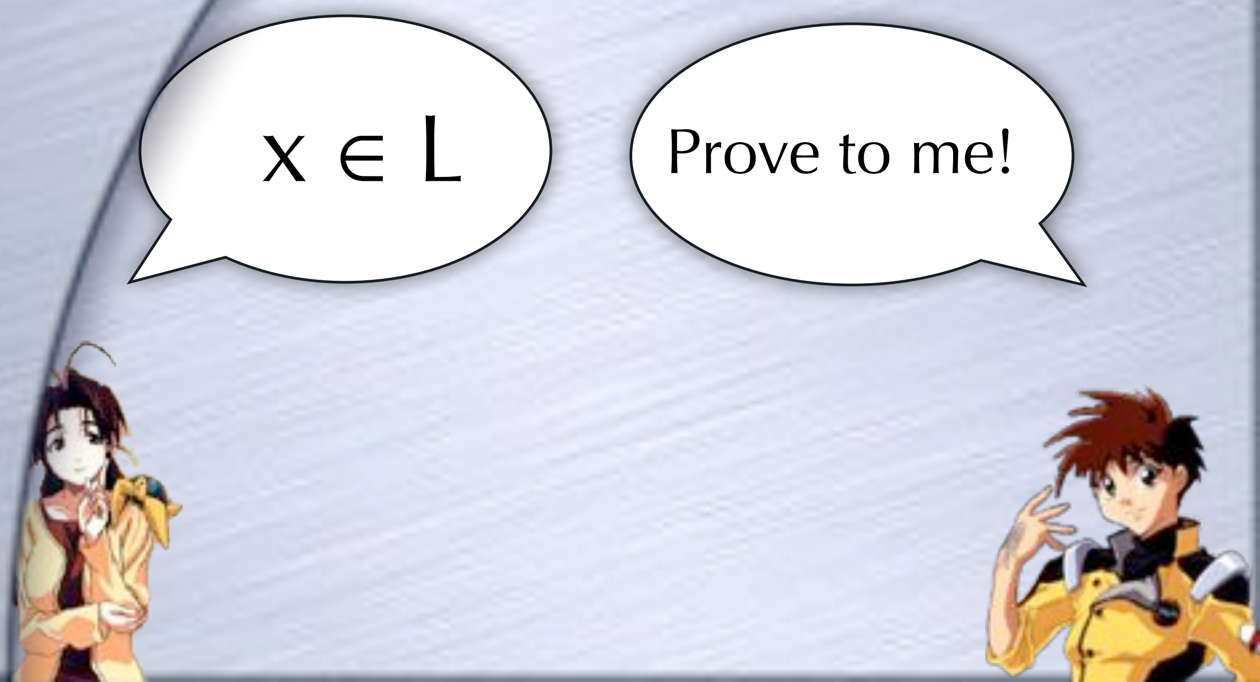

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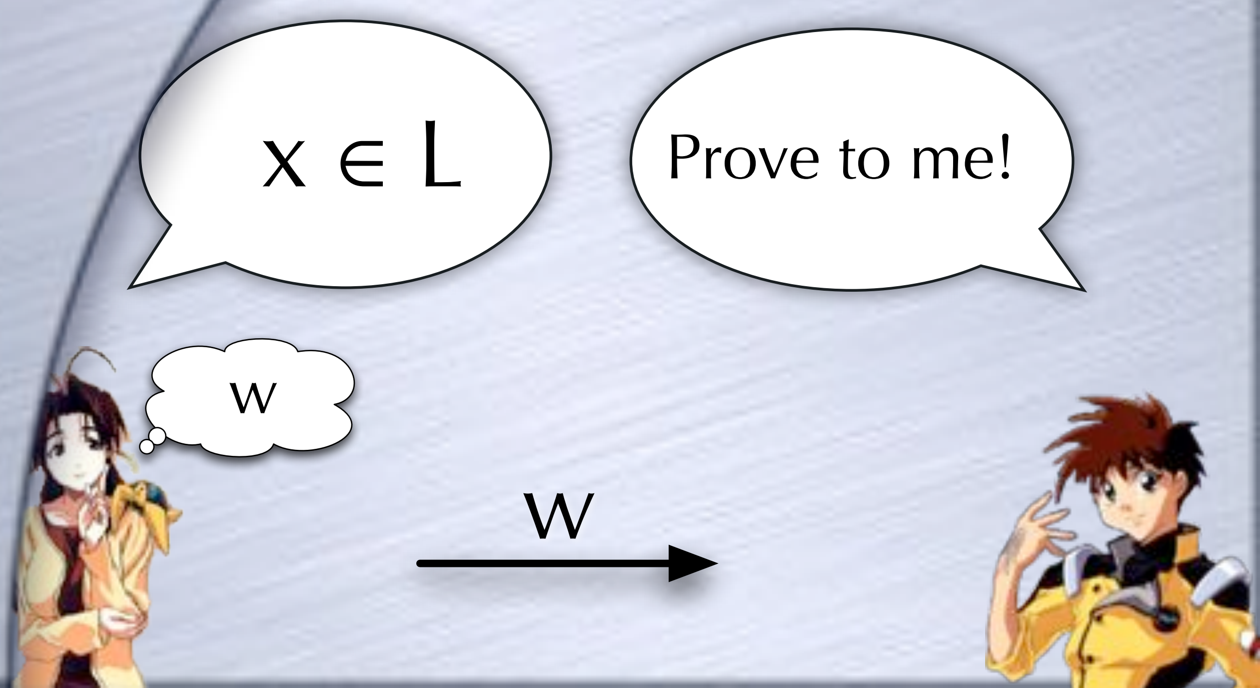

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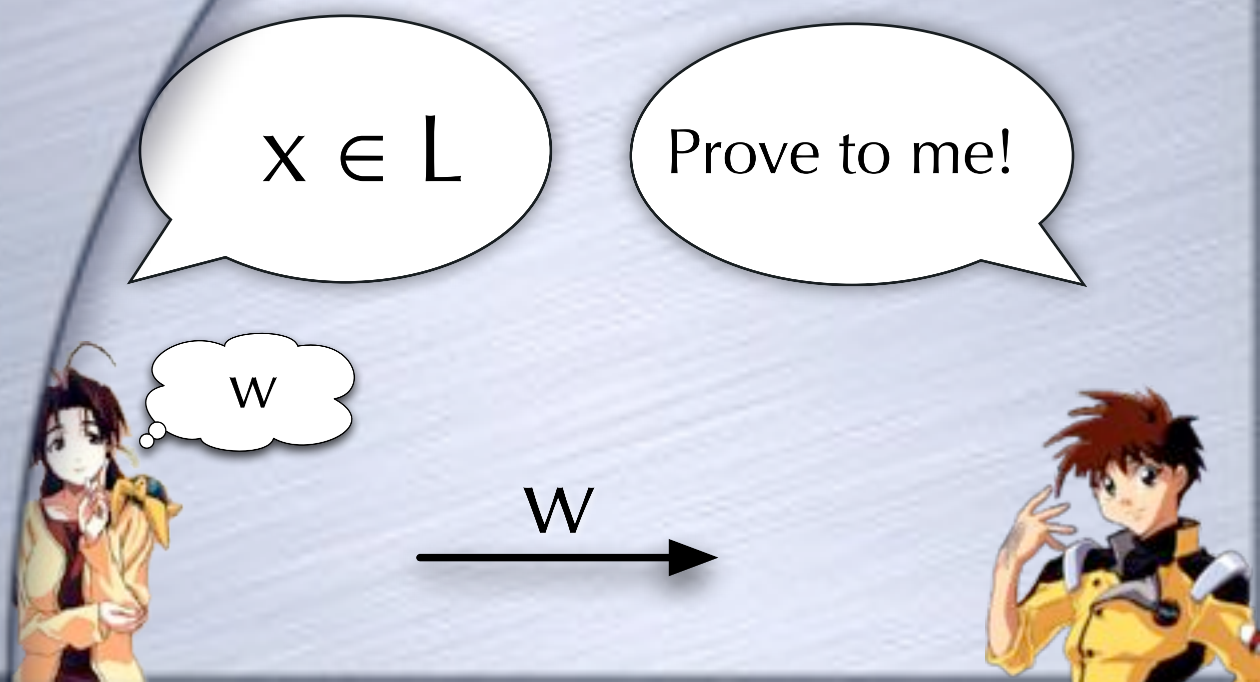

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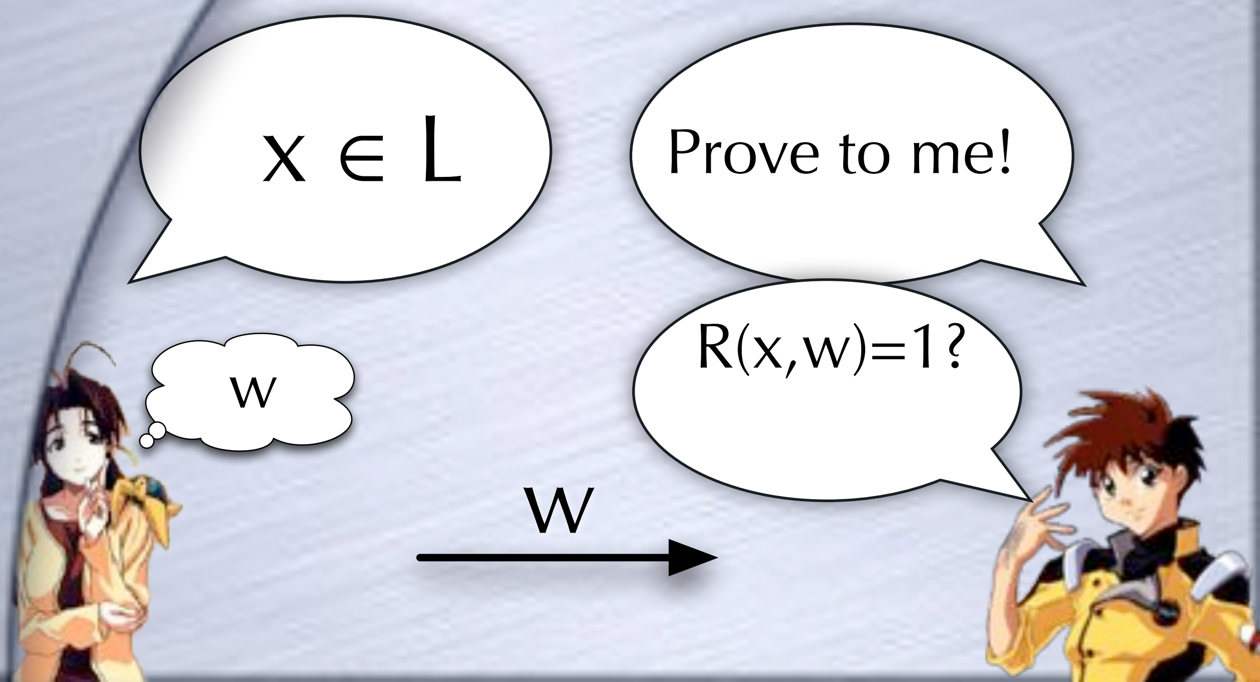

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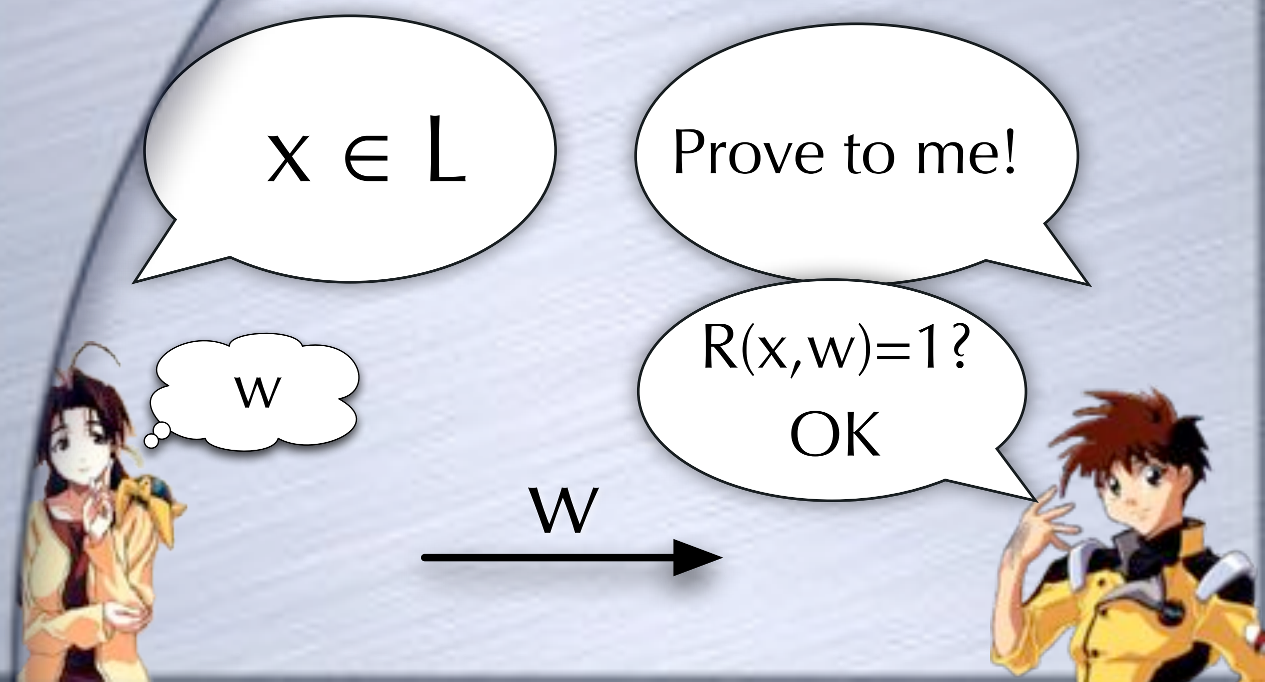

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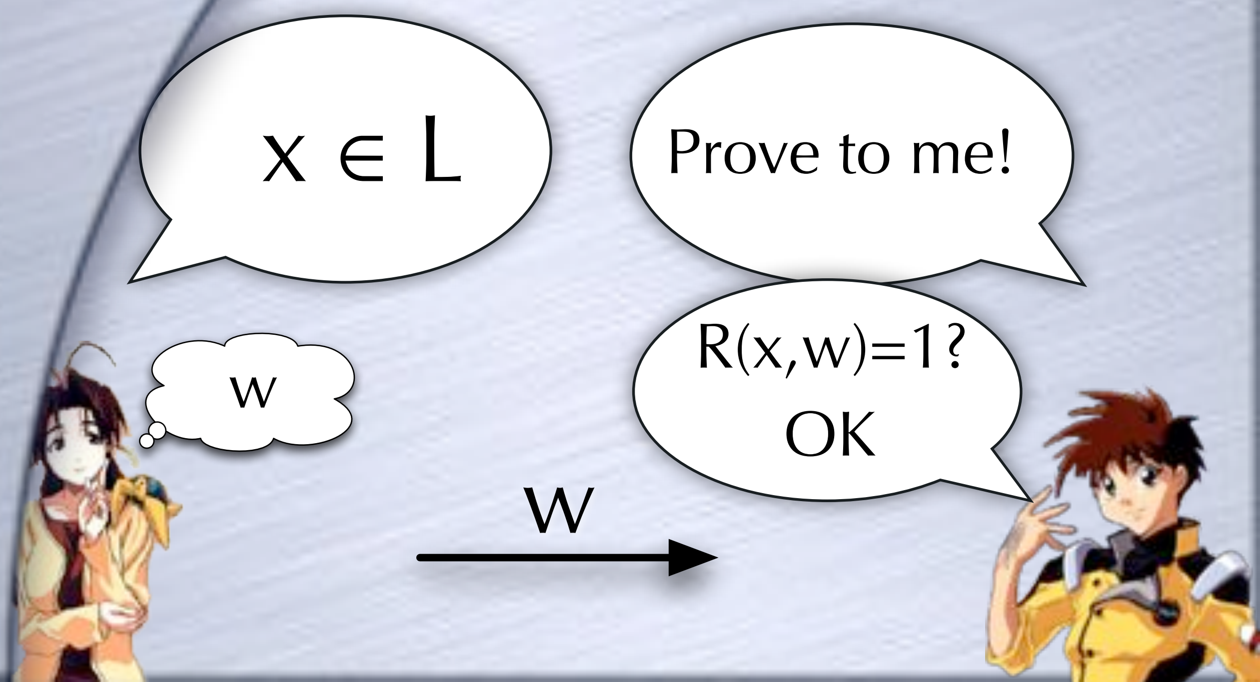

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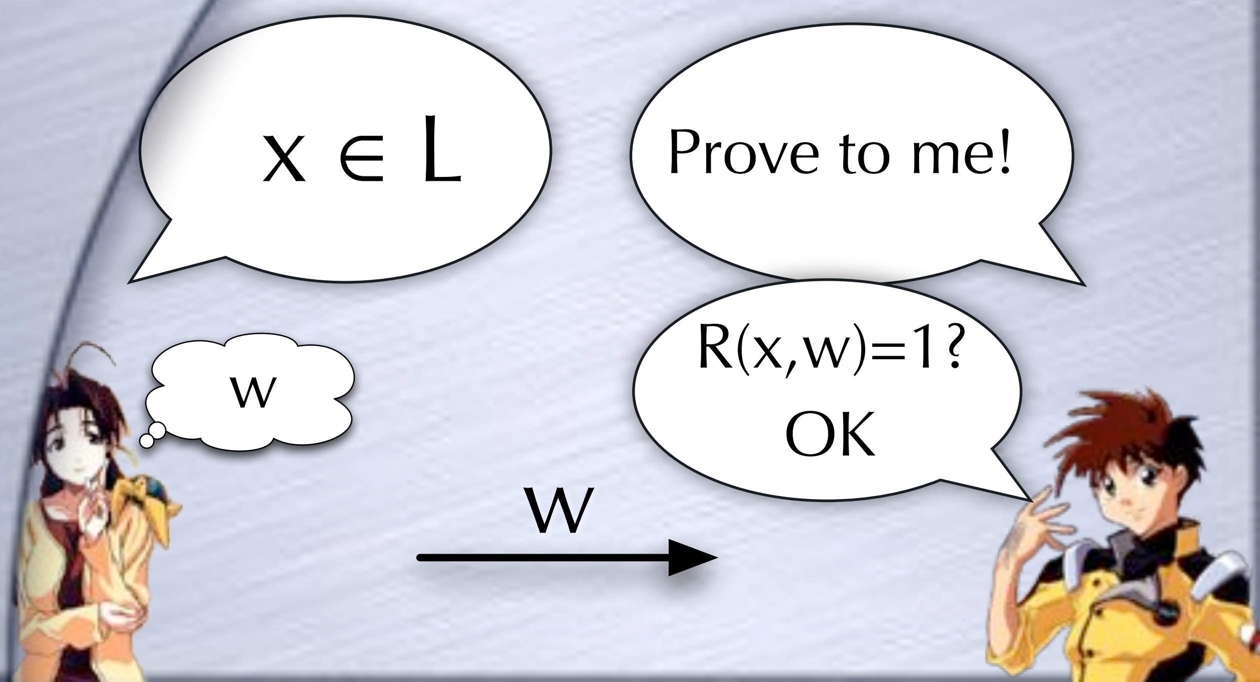

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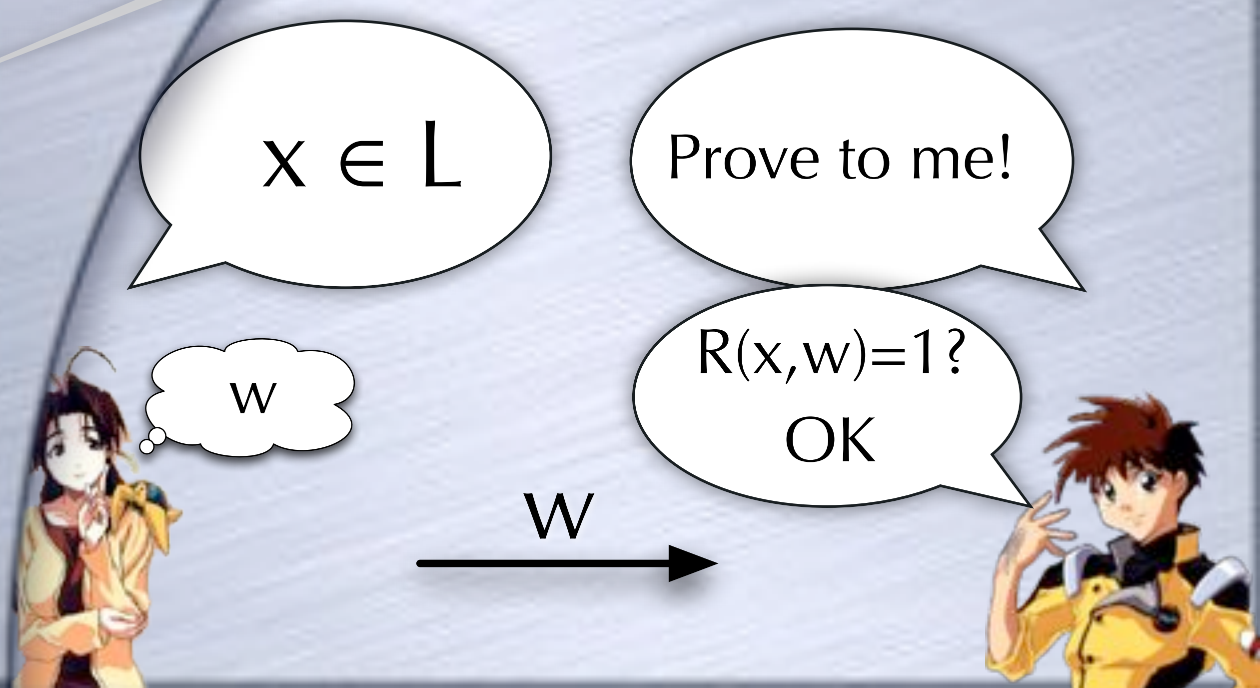
NP is the class of languages which have <u>non-interactive</u> and <u>deterministic</u> proof-systems



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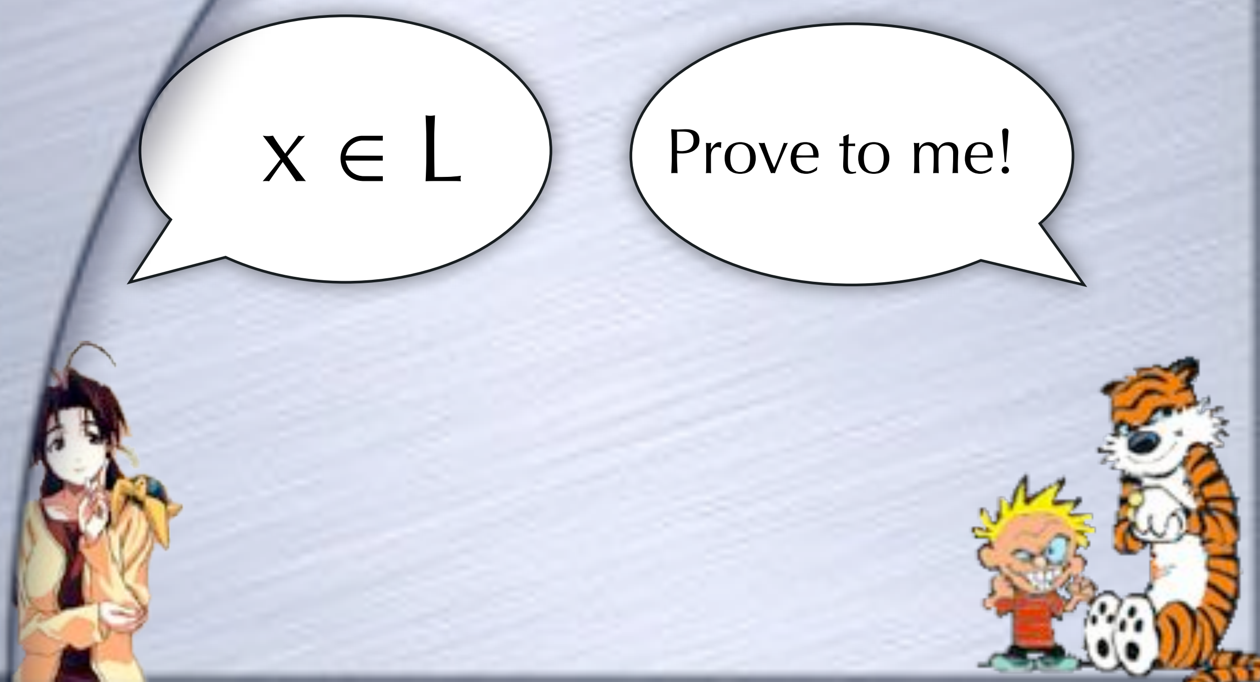

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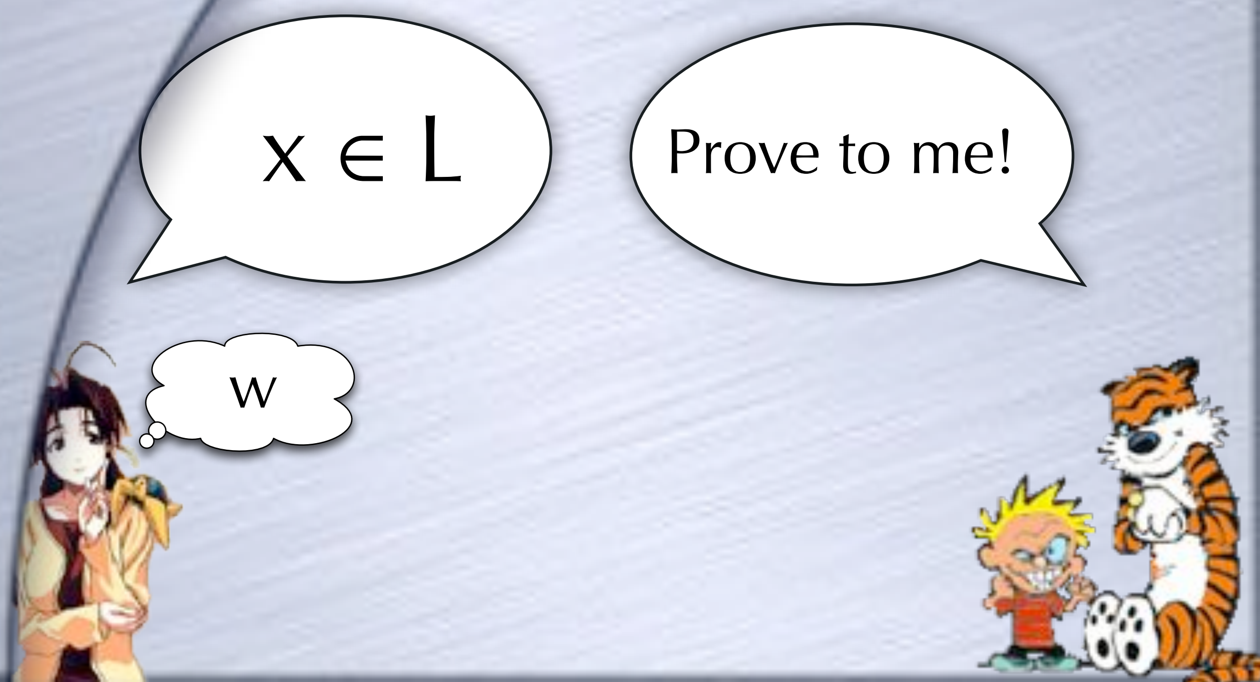

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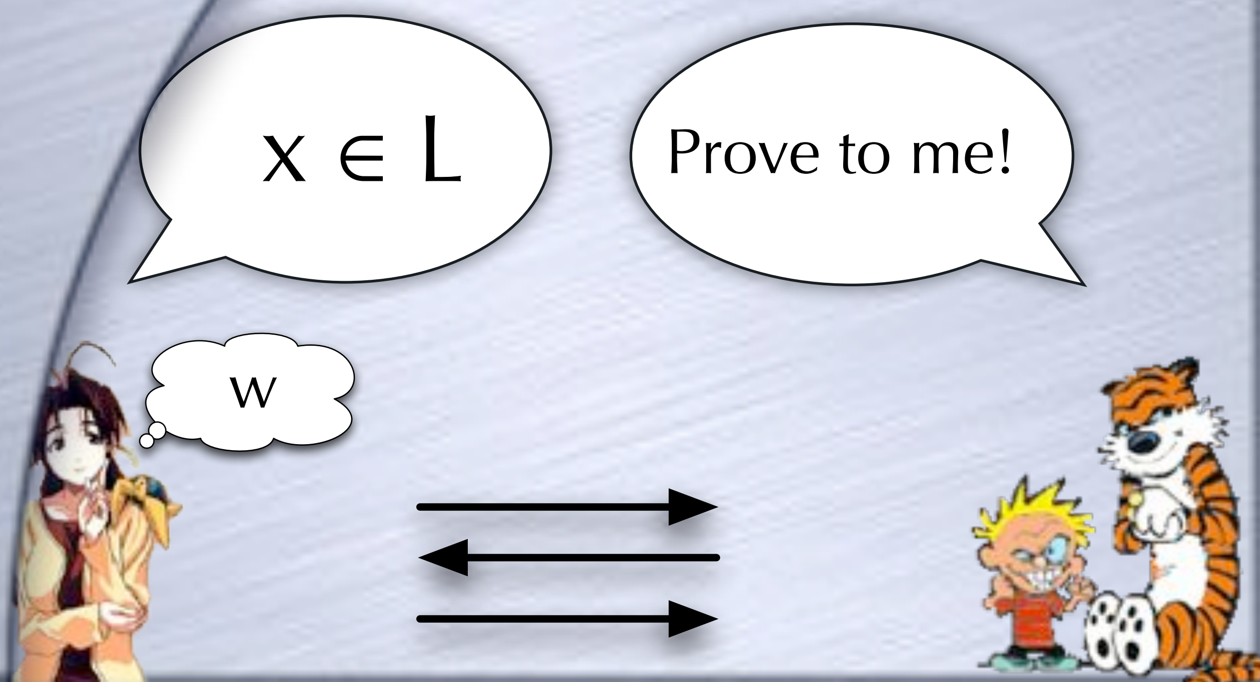

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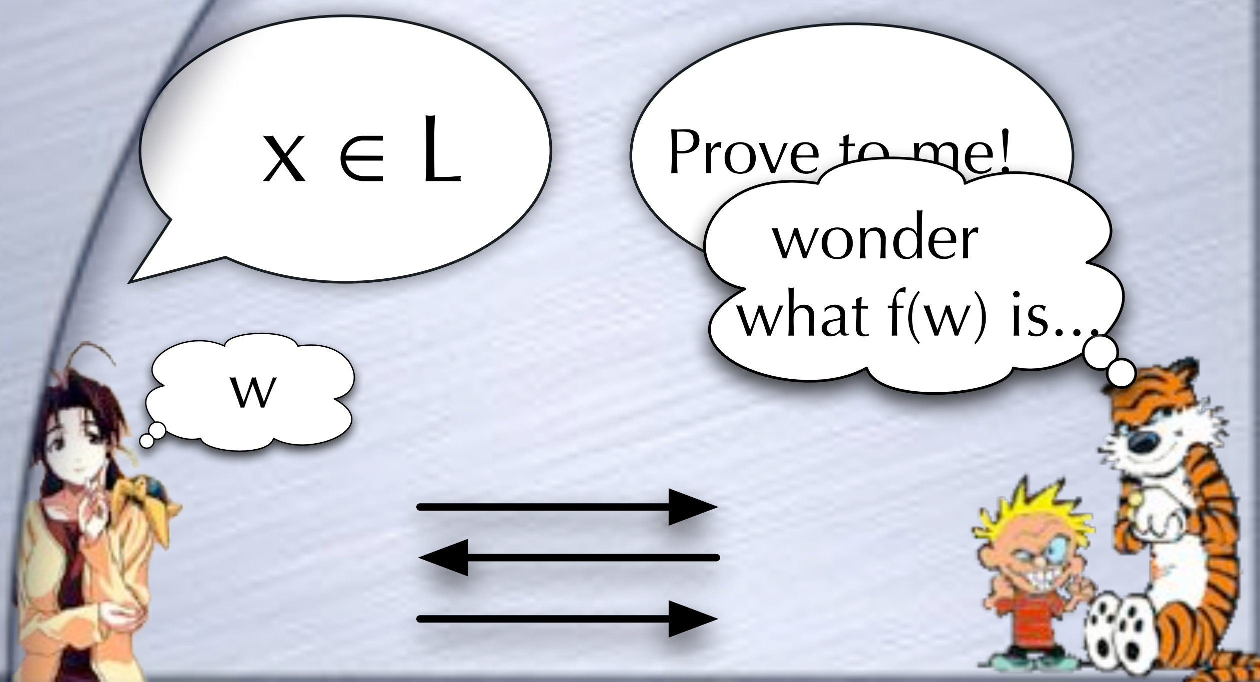

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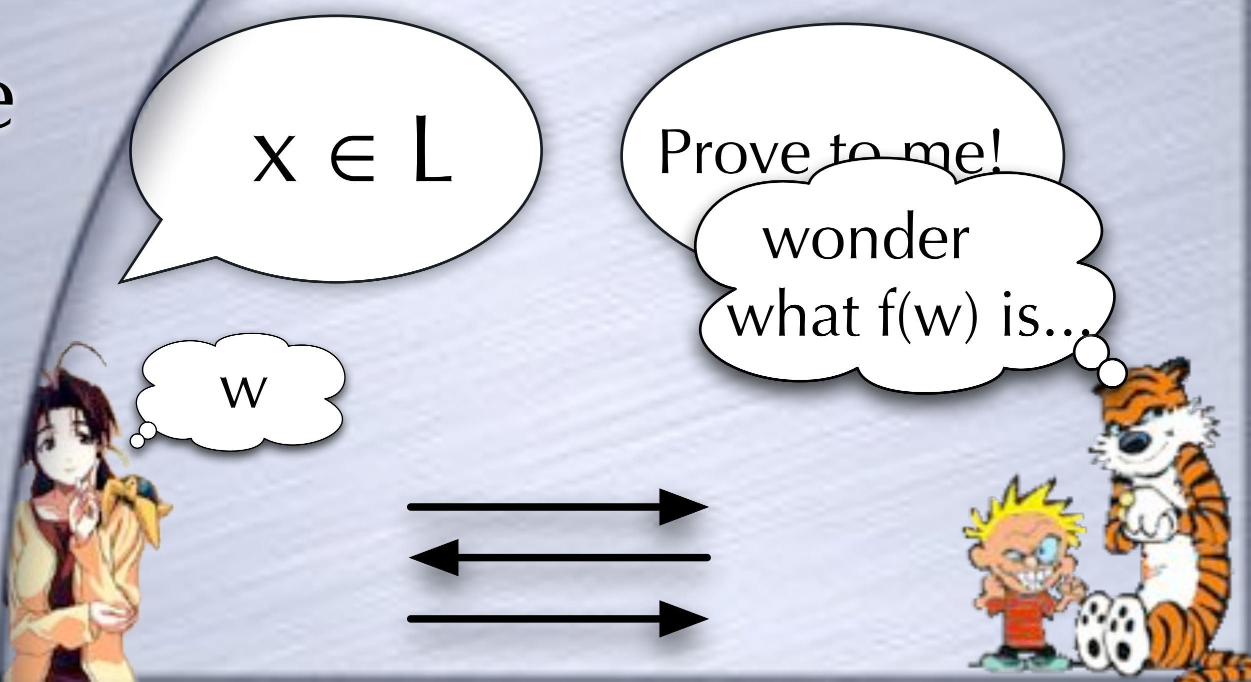

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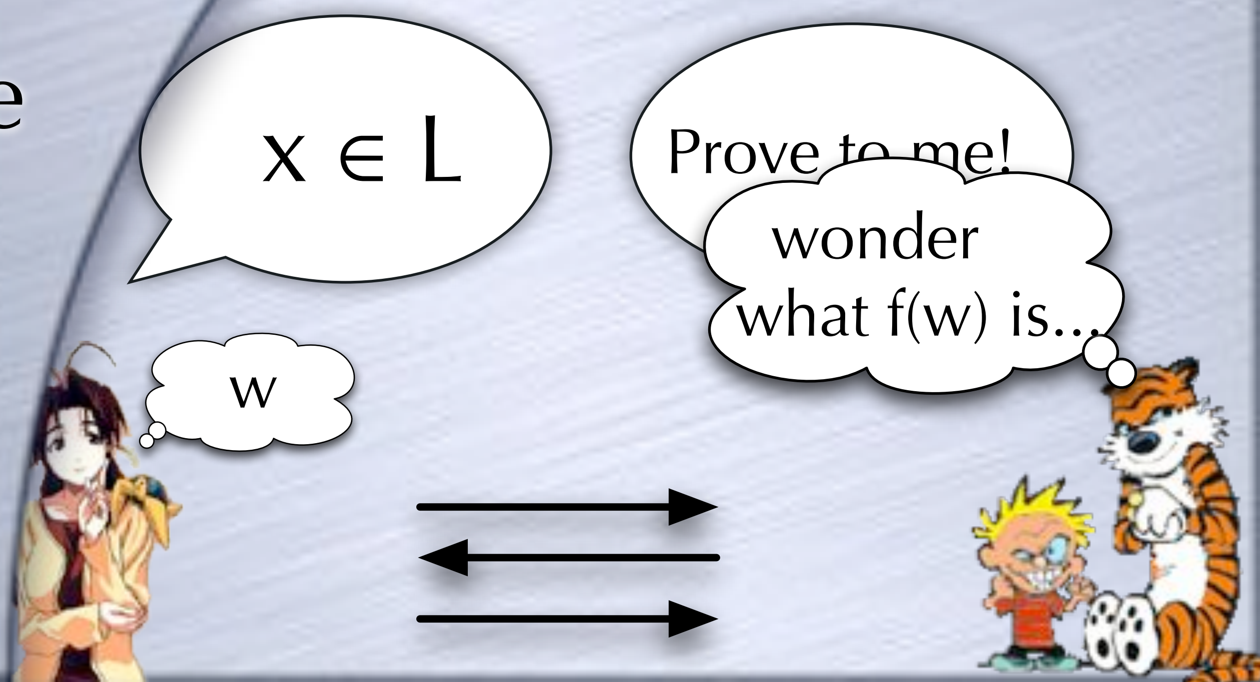
- How to formalize this?
    - Simulation!



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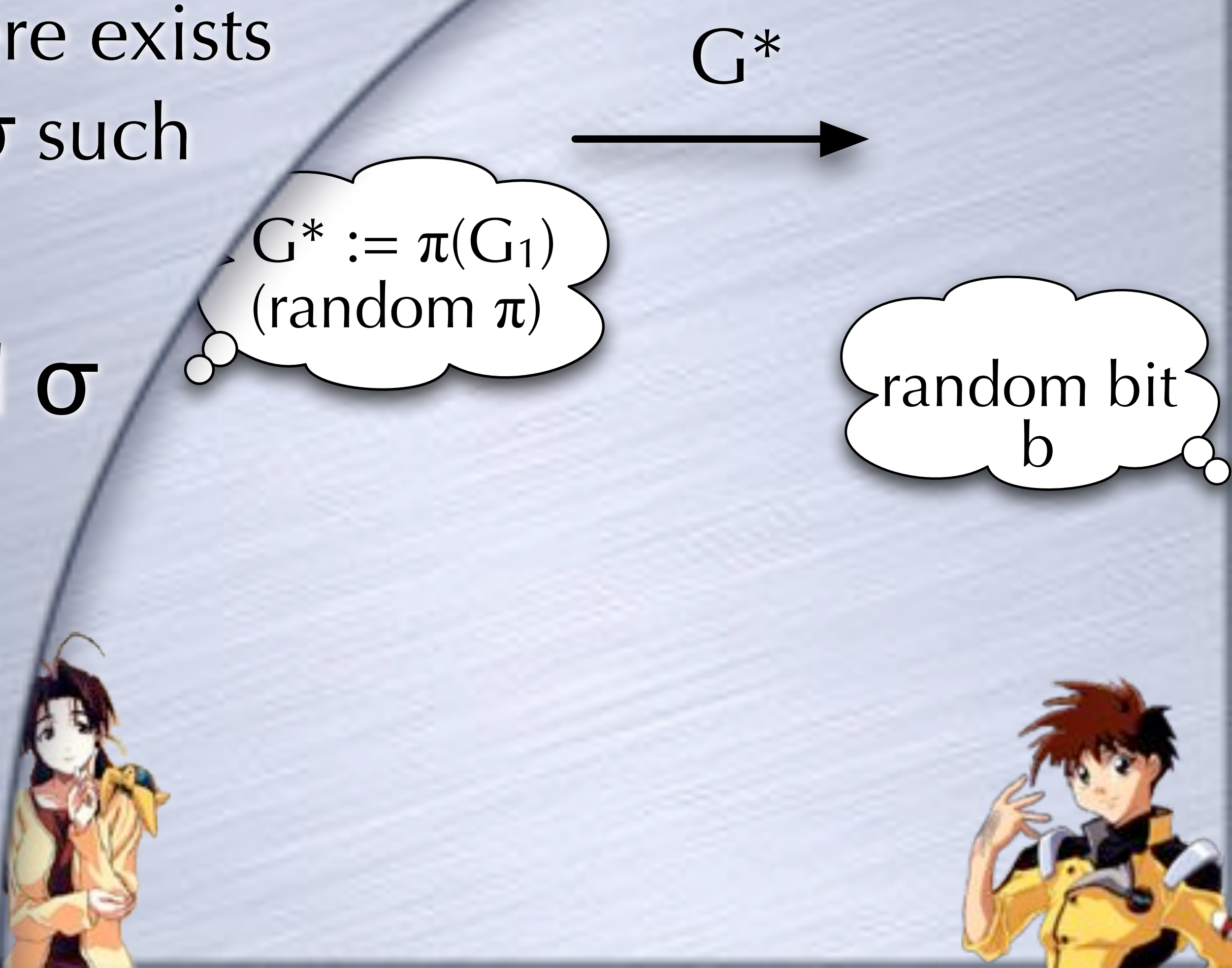

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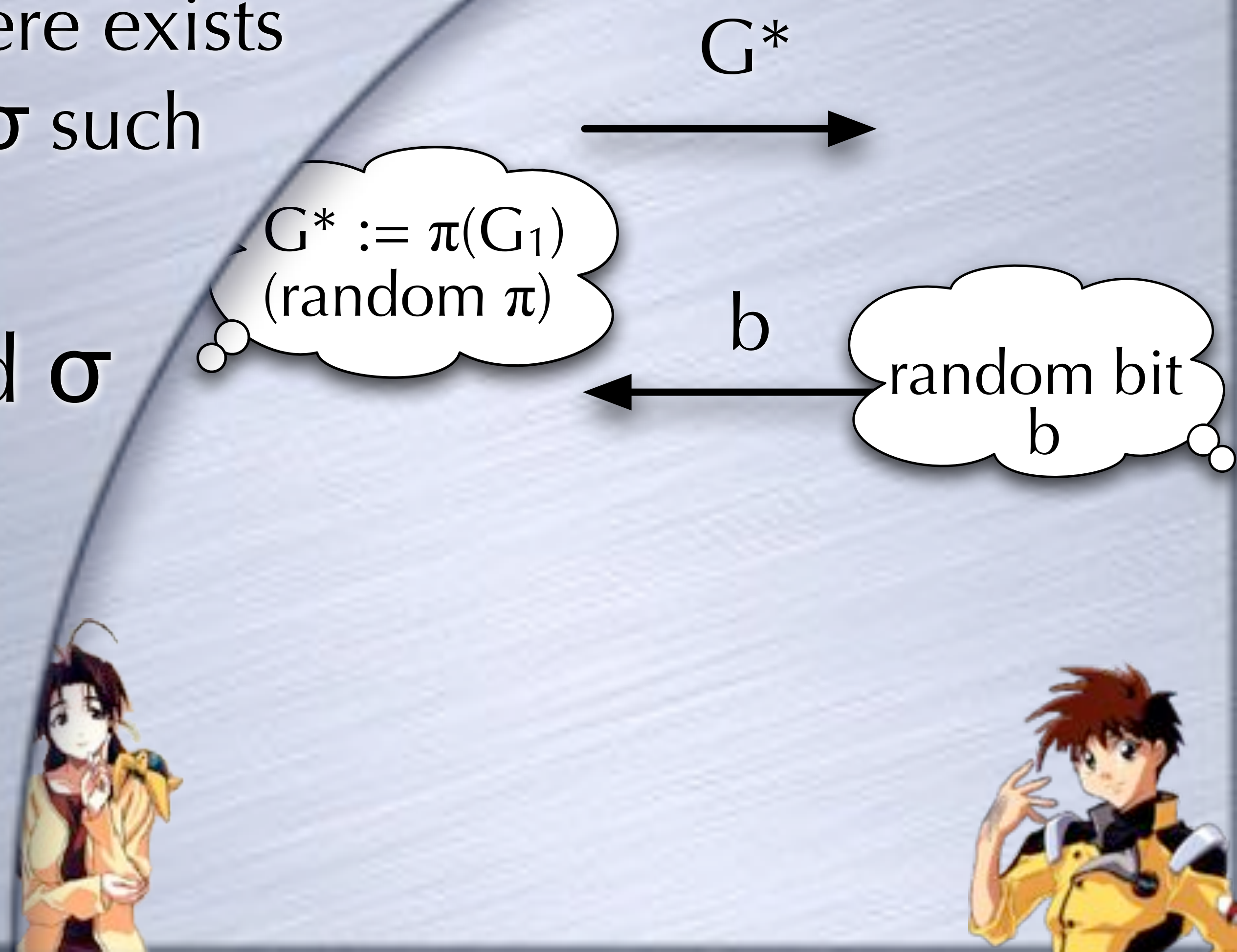

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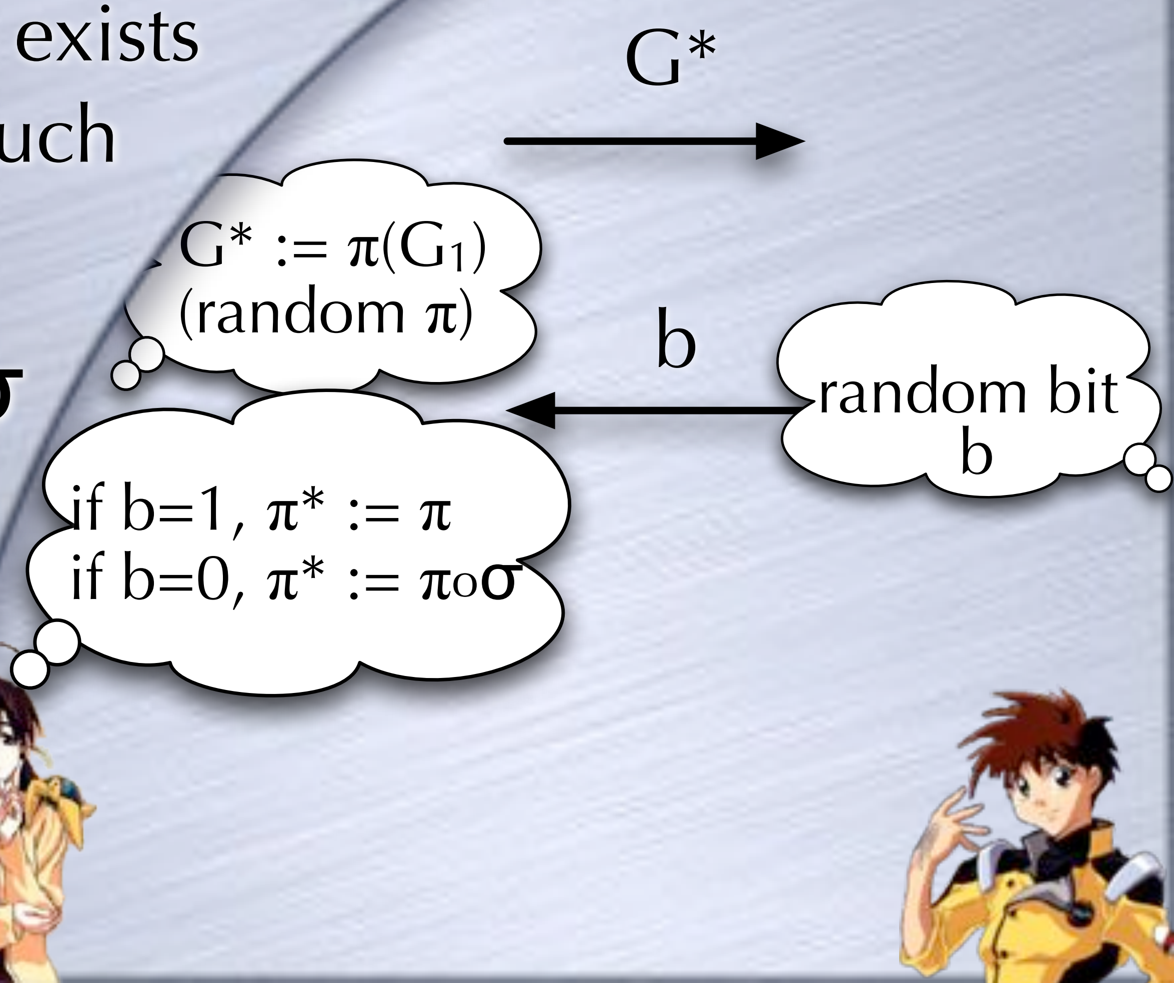

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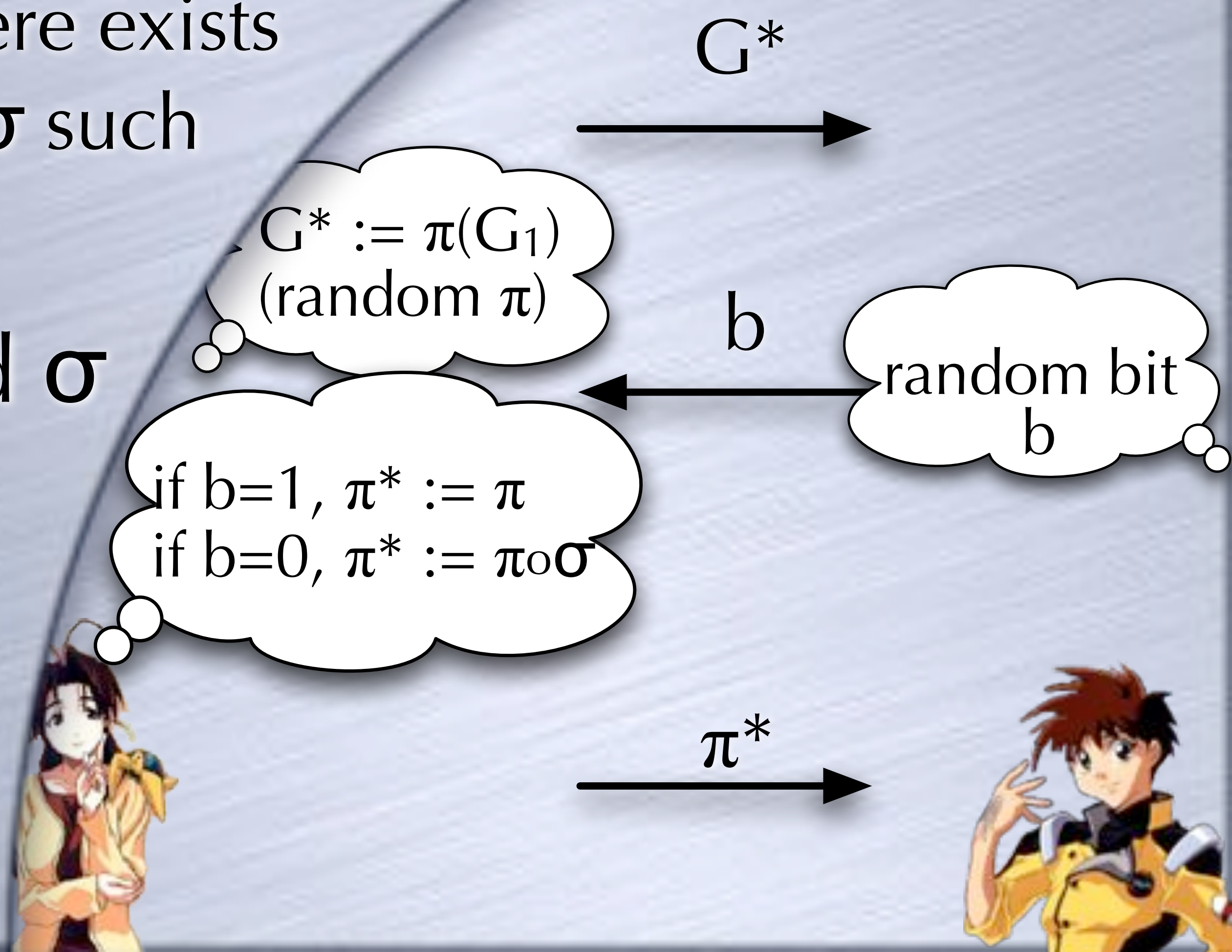

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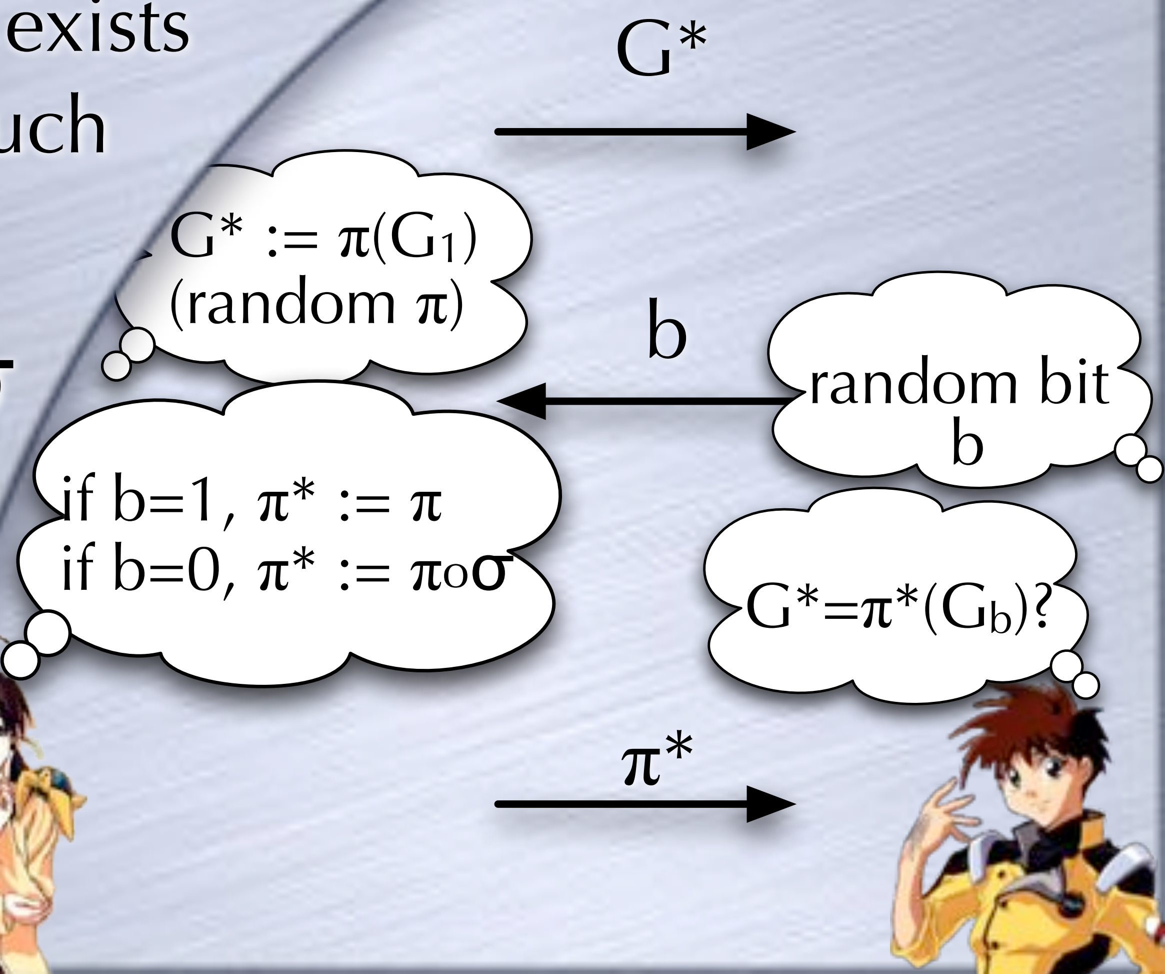

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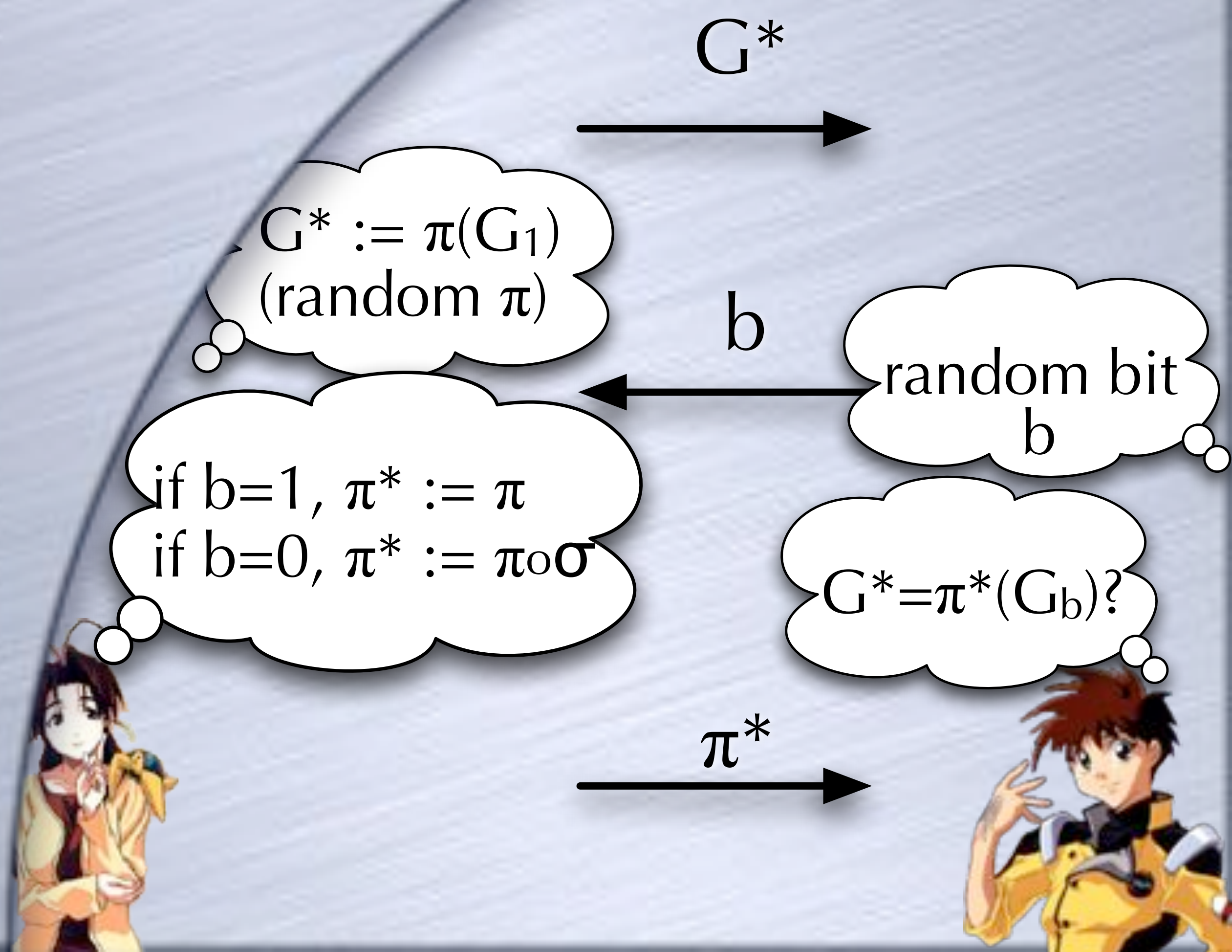

An Example

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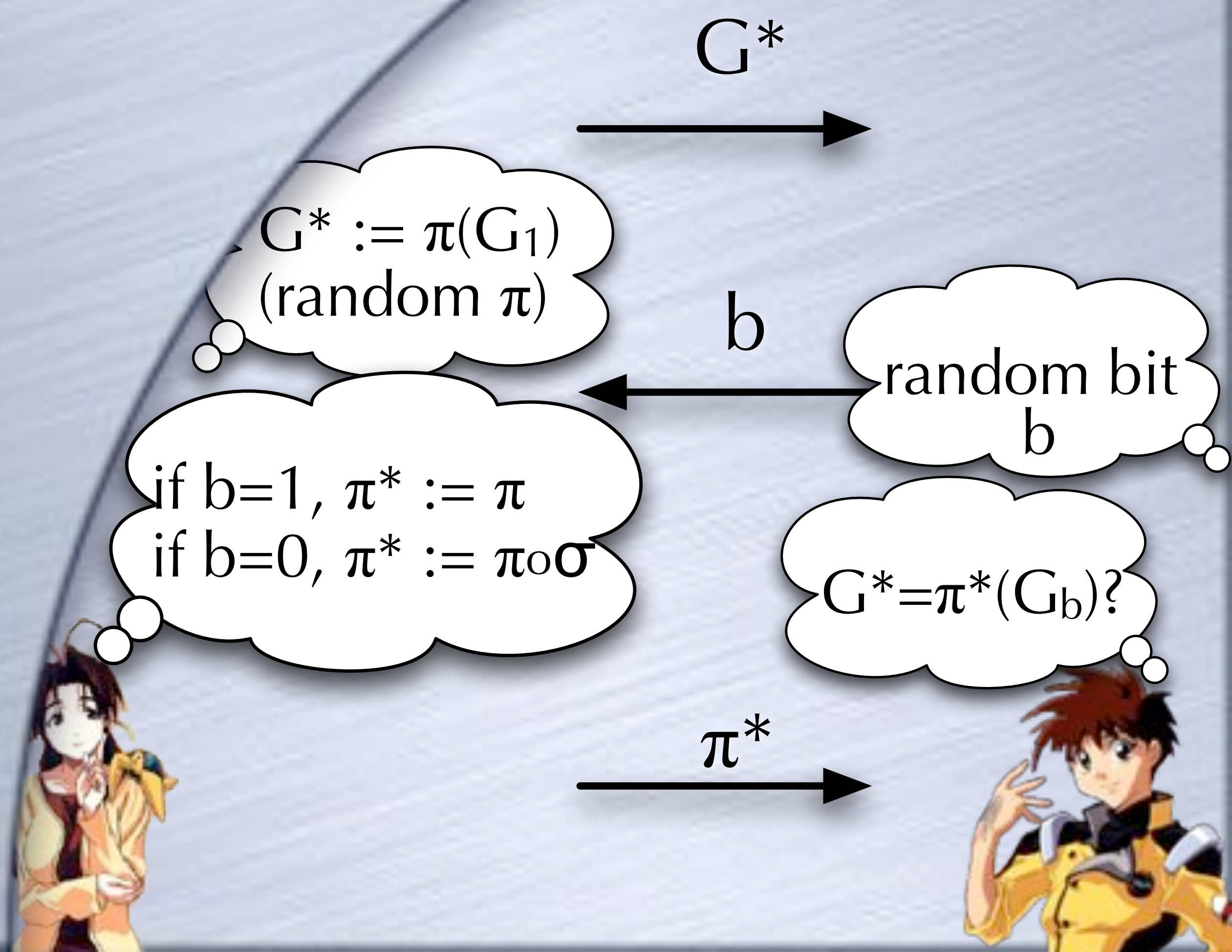

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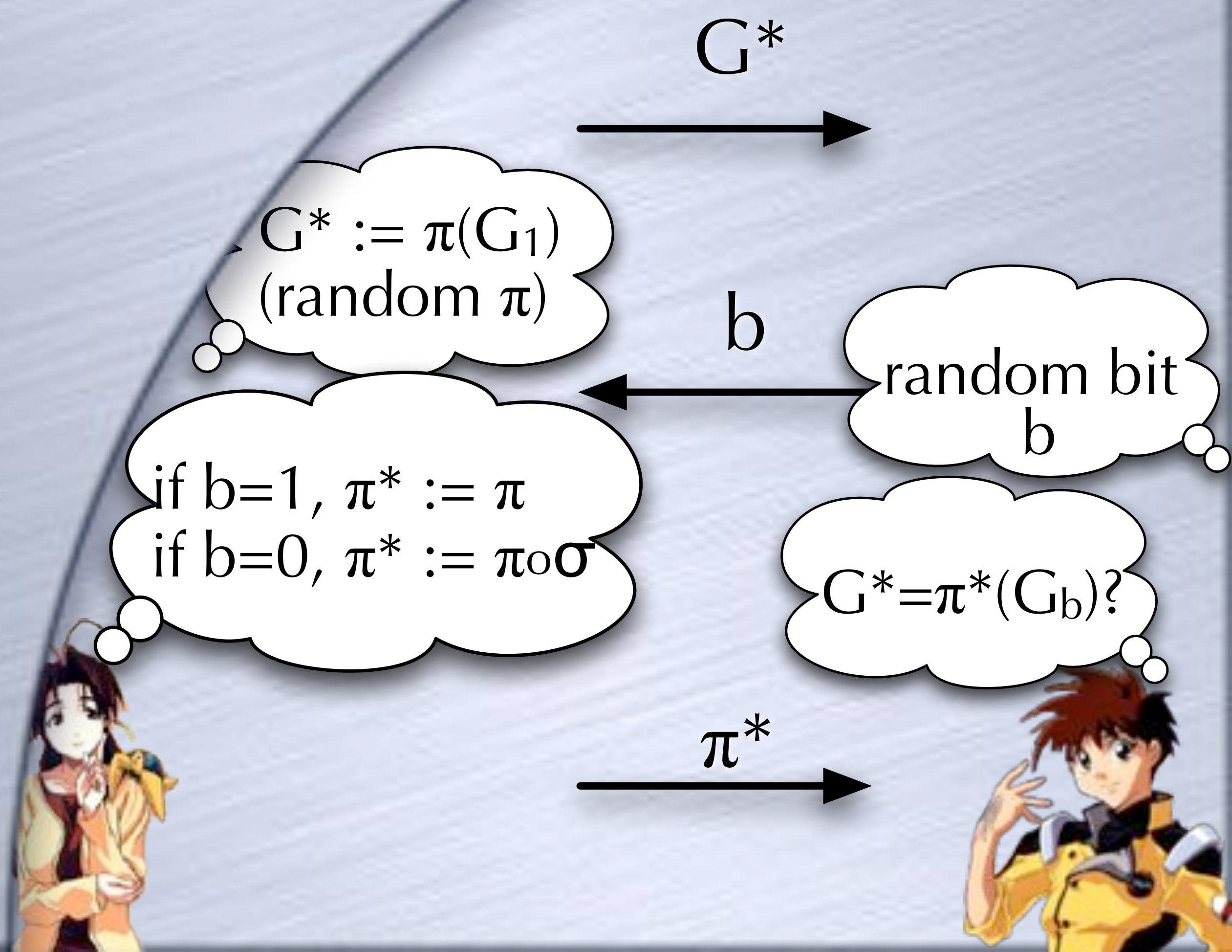

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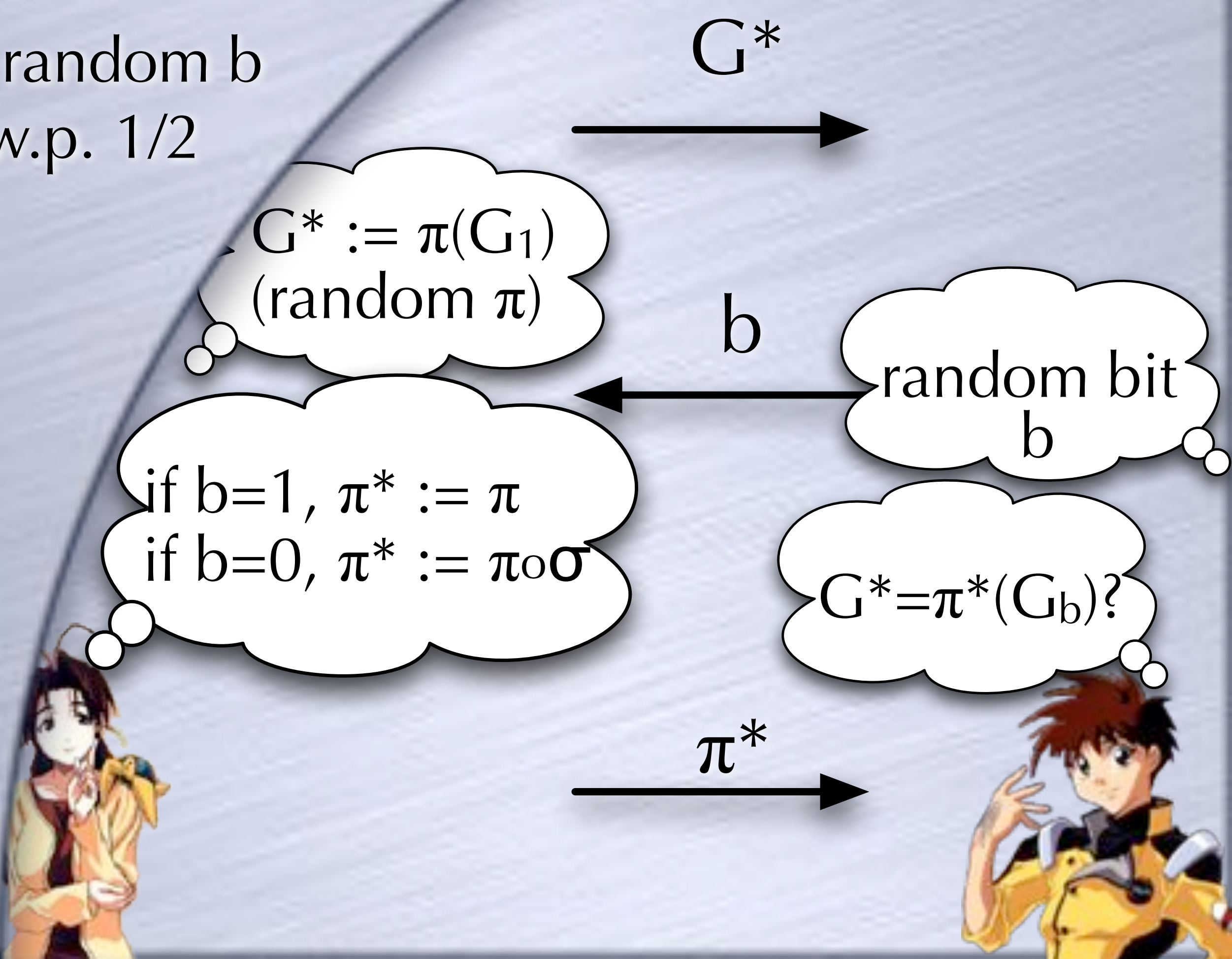

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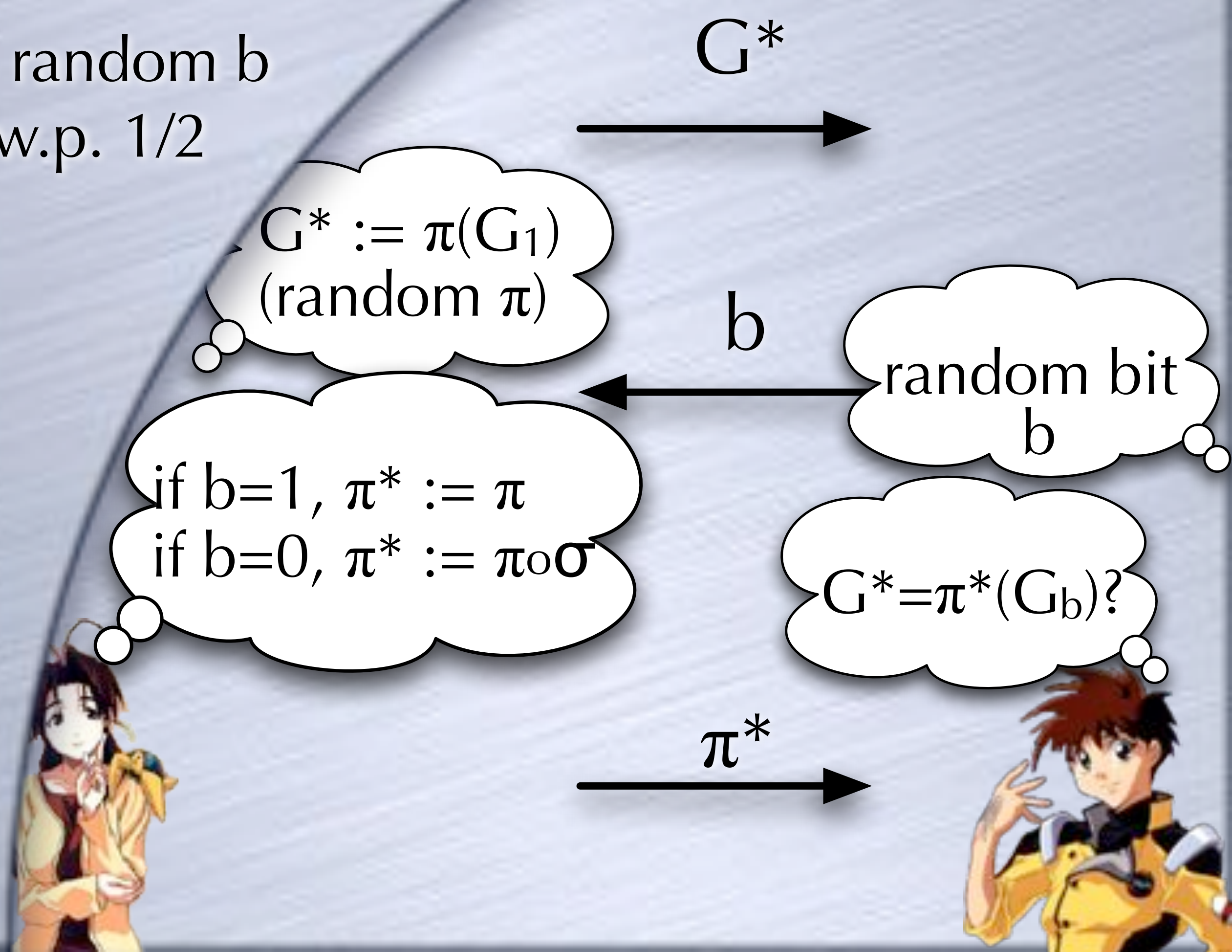

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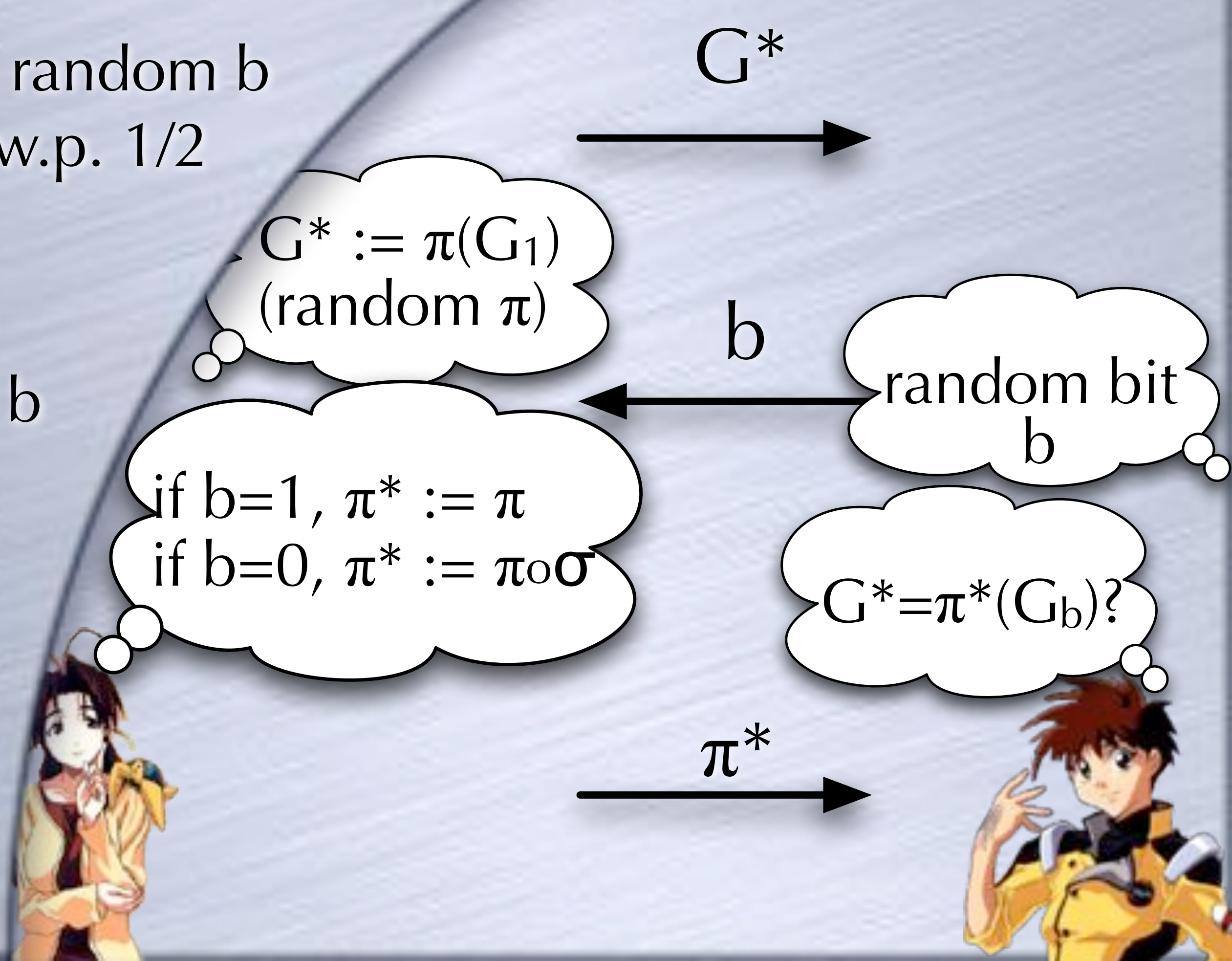

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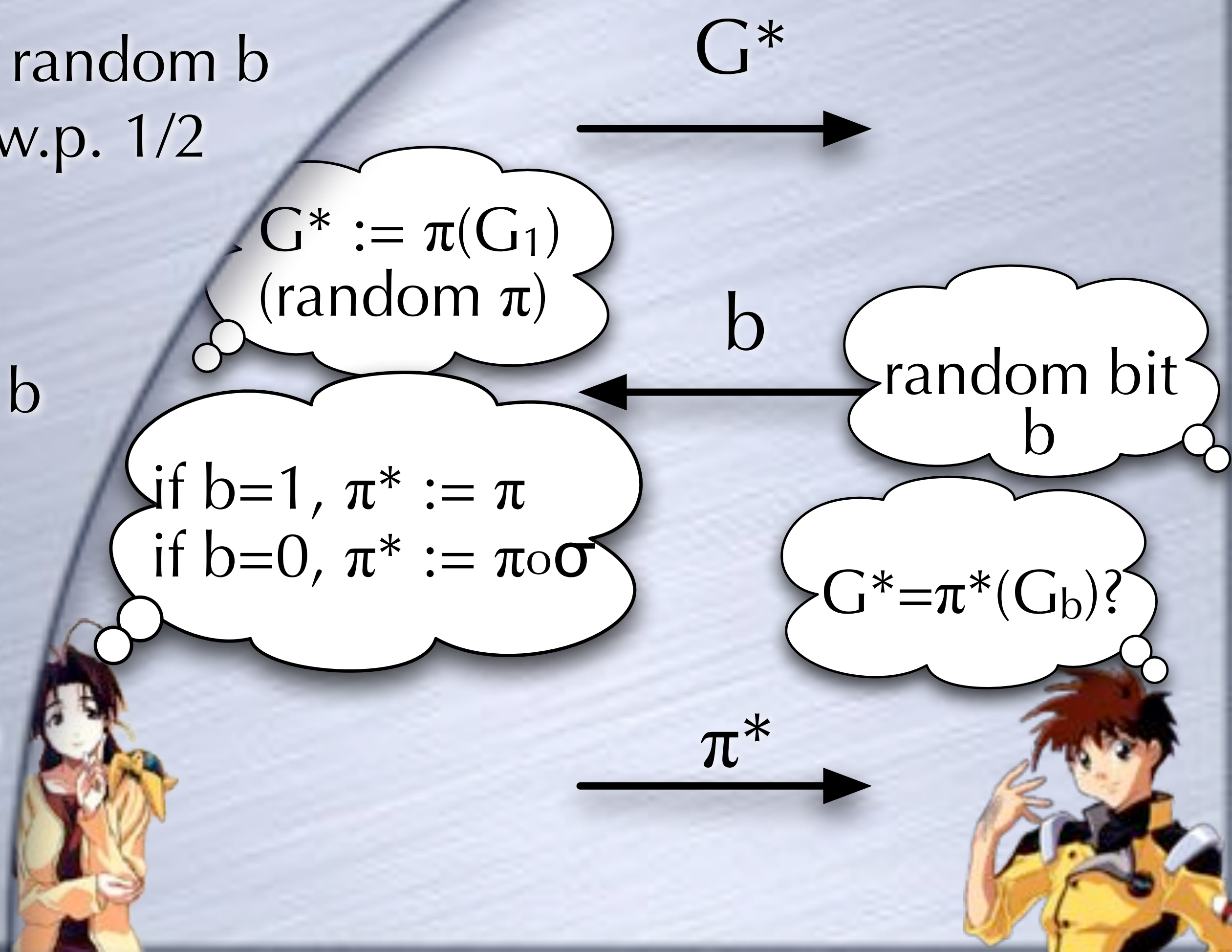
  - Verifier's view: random b and π^* s.t. $G^* = \pi^*(G_b)$
  - Which he could have generated by himself (whether $G_0 \sim G_1$ or not)



Zero-Knowledge Proofs

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- Interactive Proof

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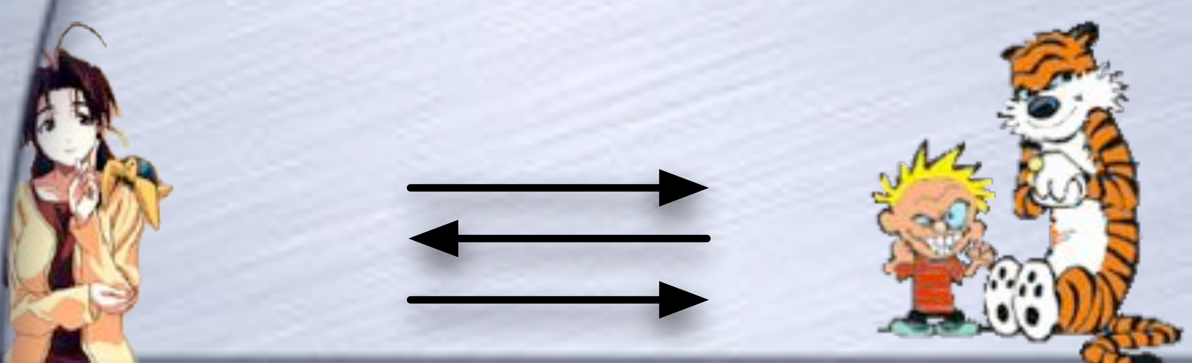

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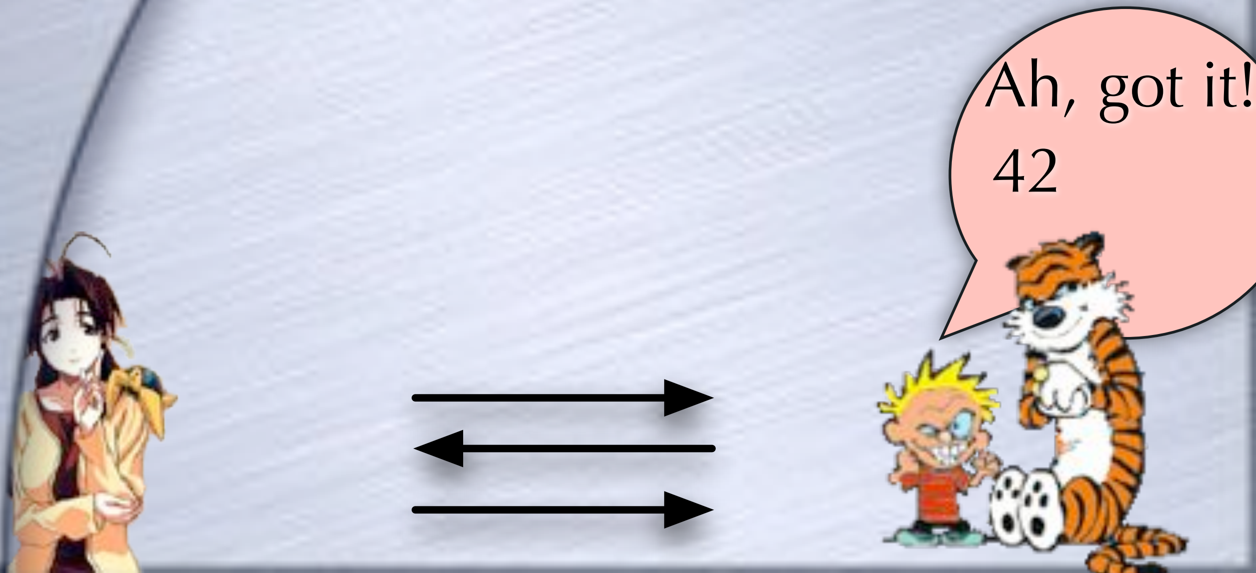

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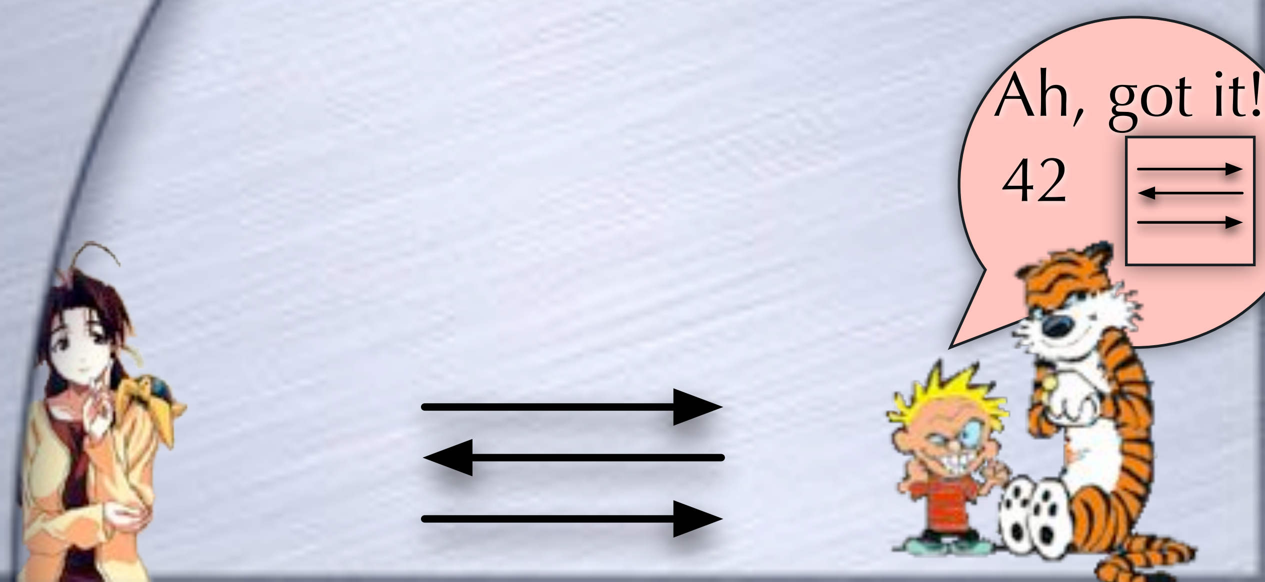

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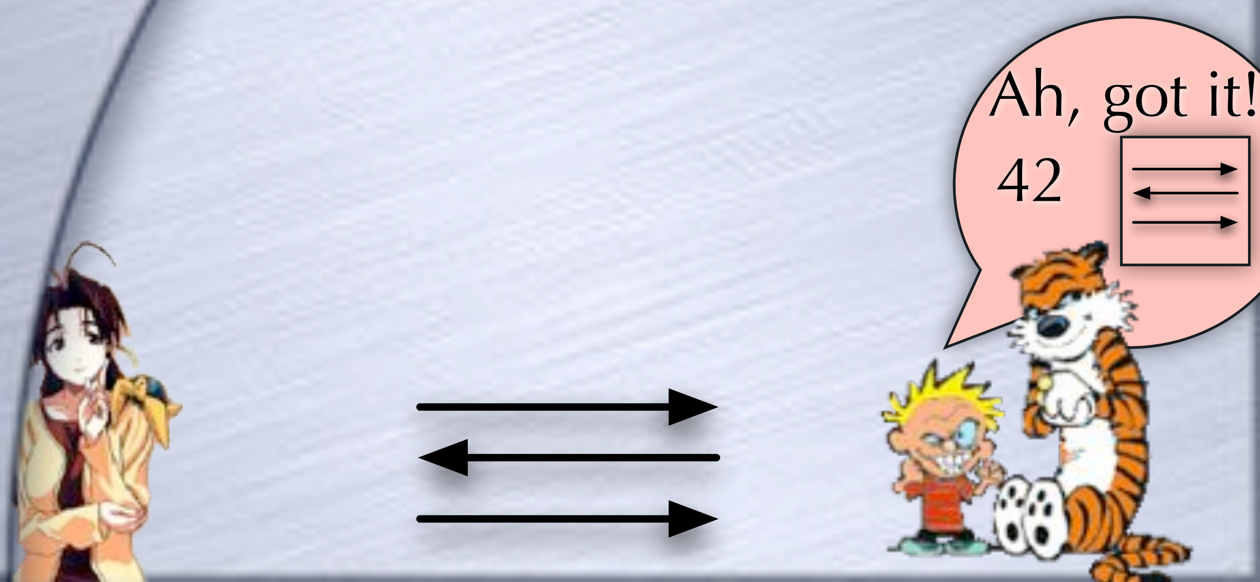

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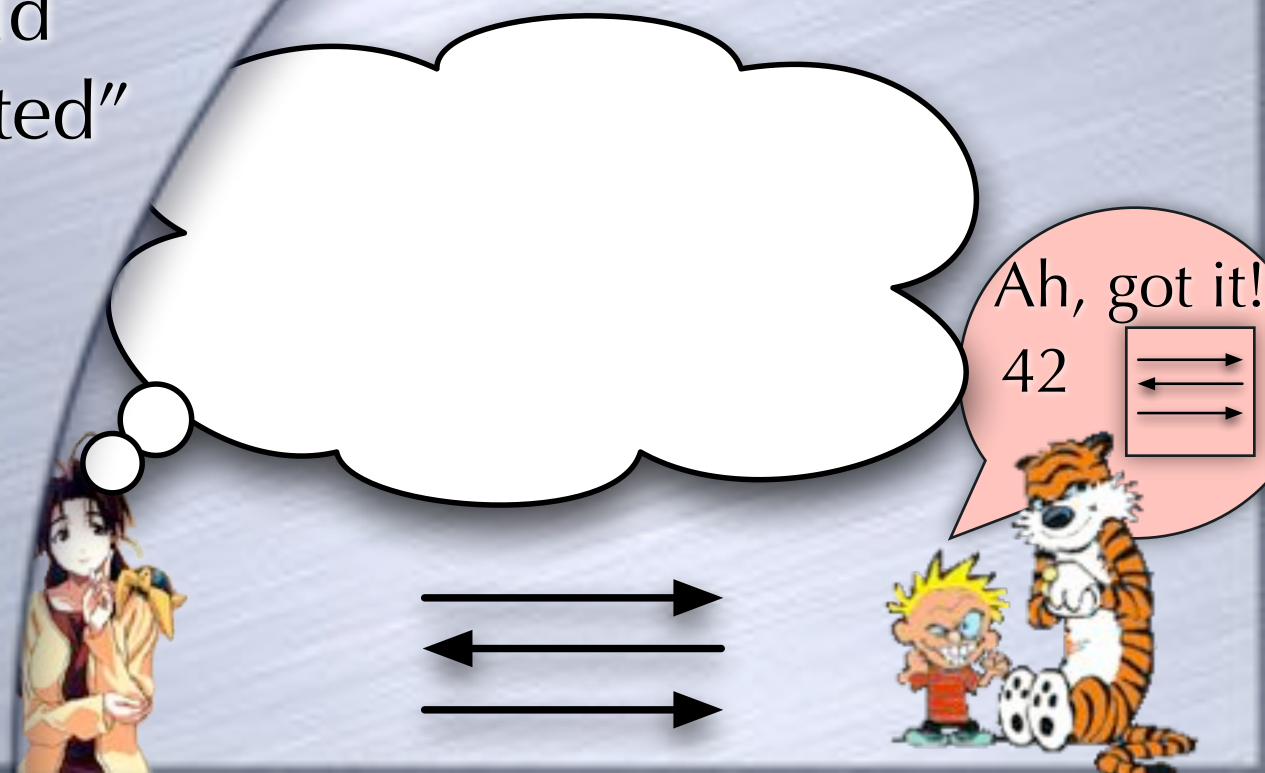

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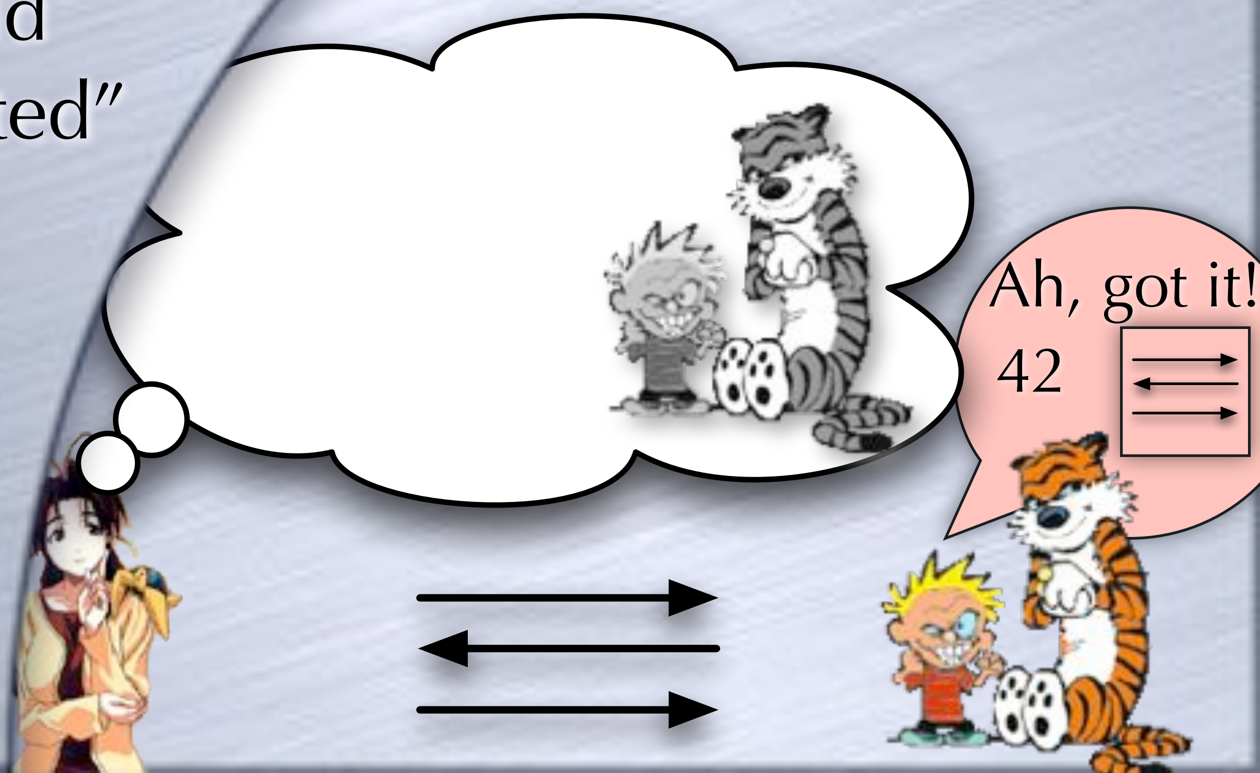

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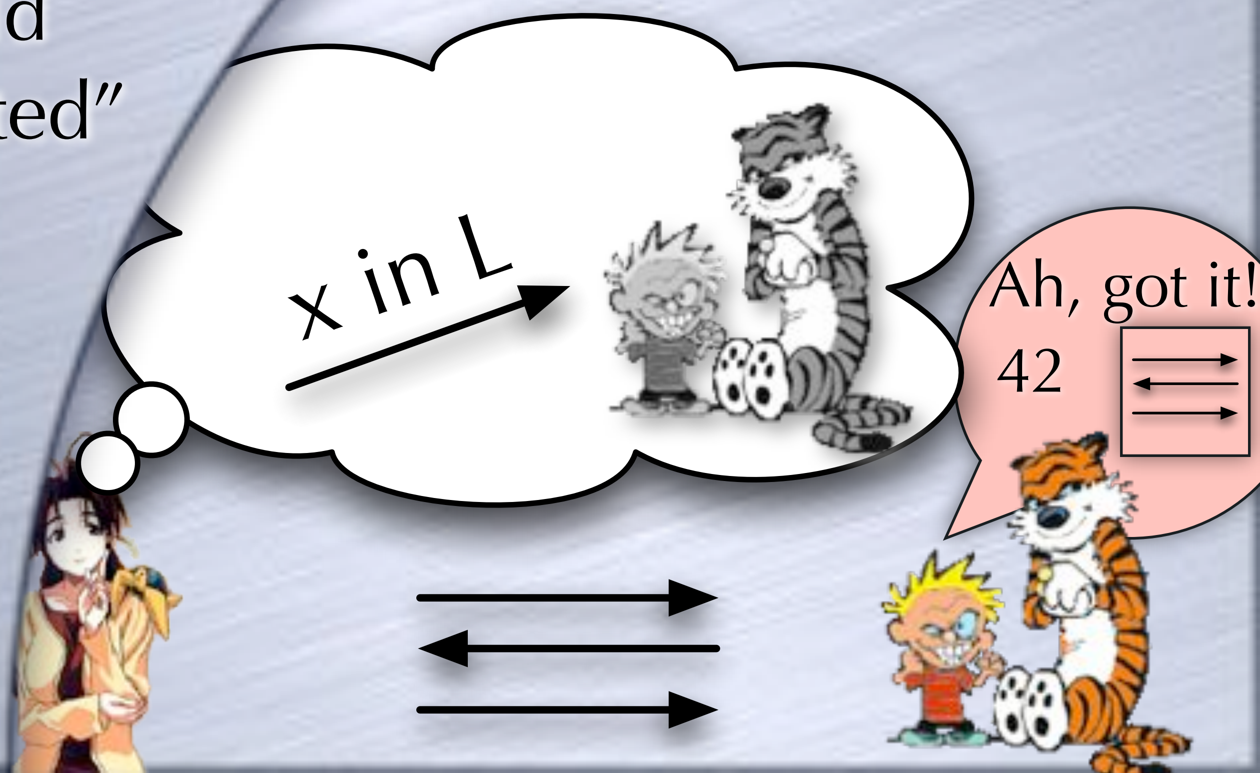

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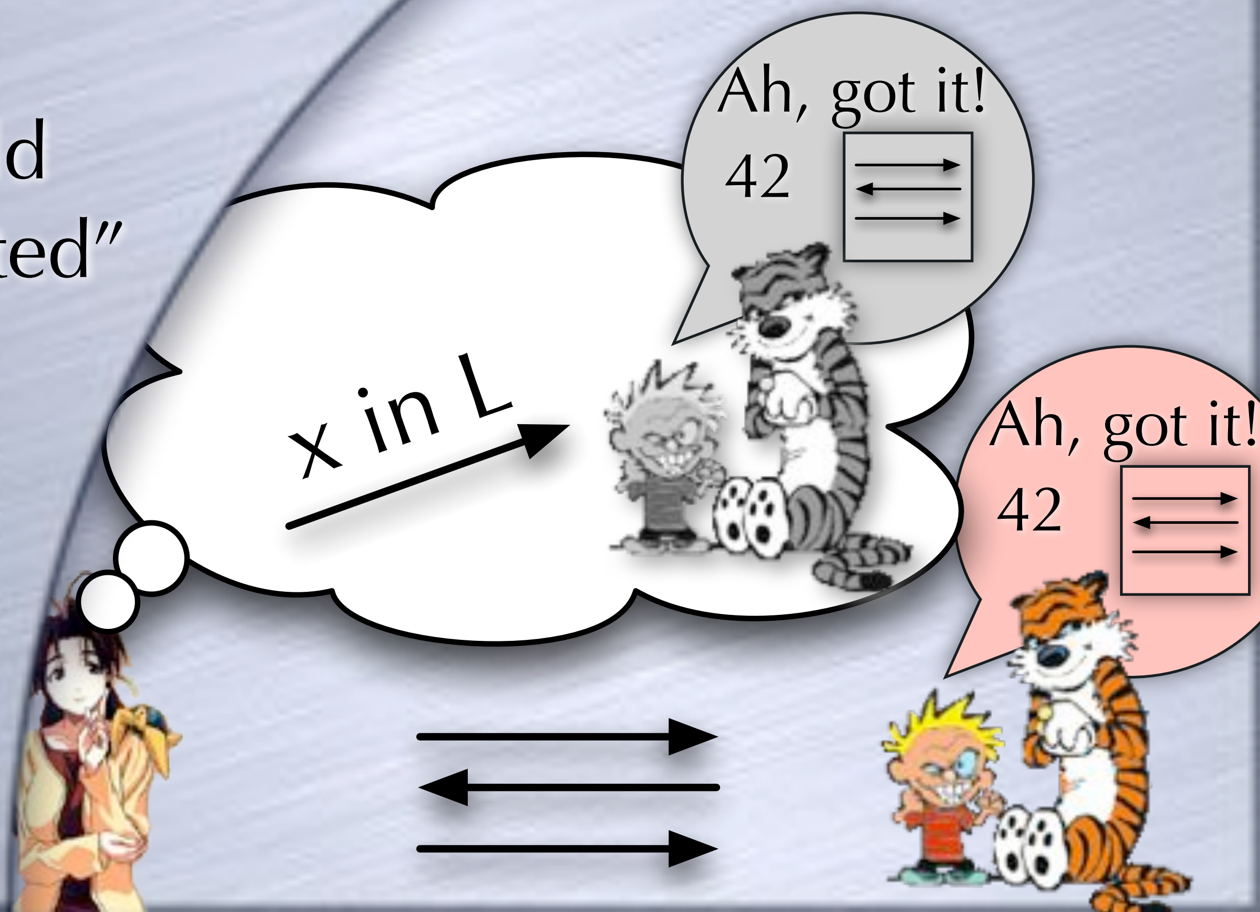

Zero-Knowledge Proofs

- Interactive Proof
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- ZK Property:
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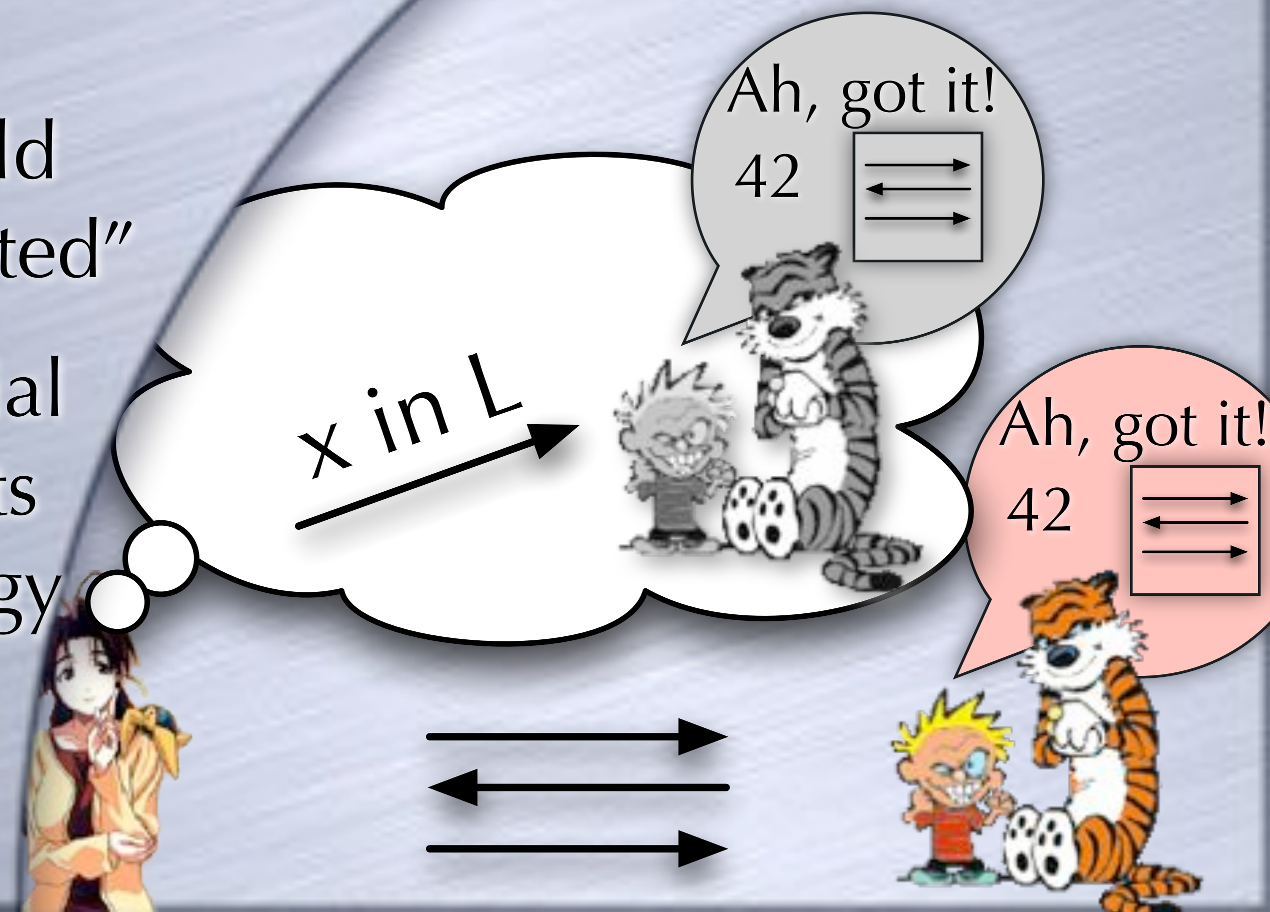
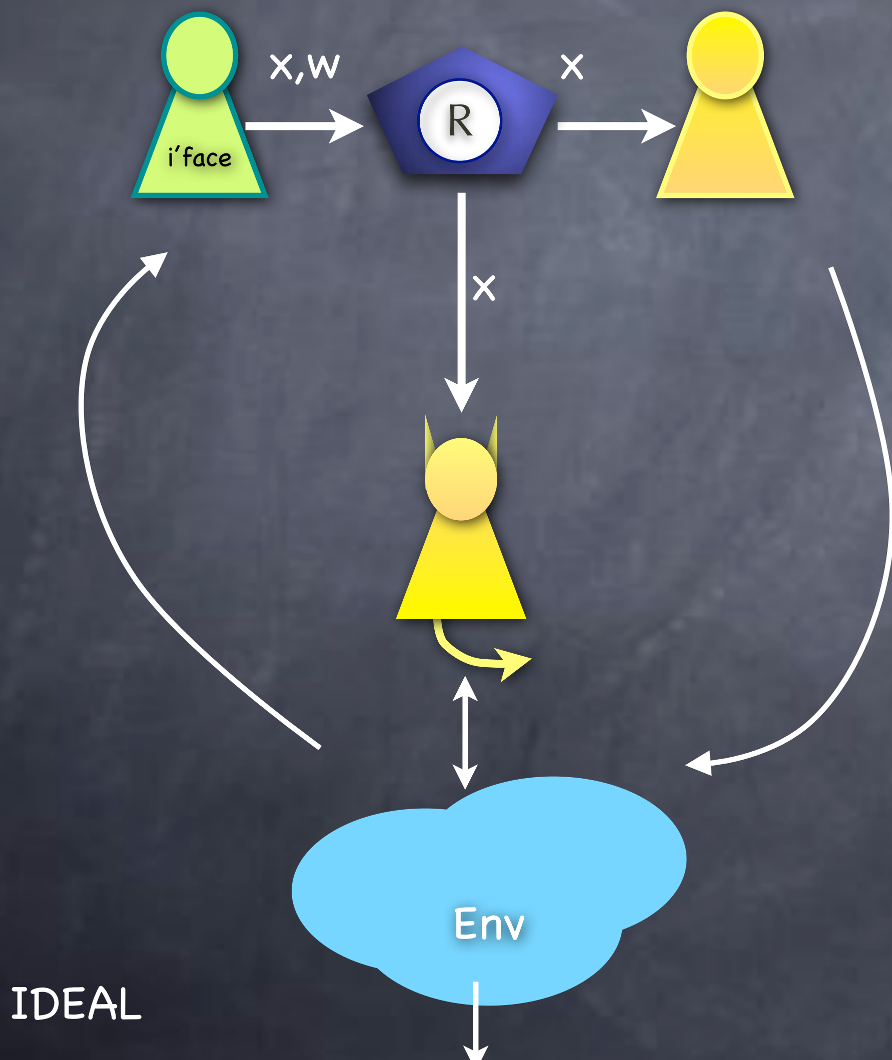
  - For every adversarial strategy, there exists a simulation strategy



ZK Property (in other pict's)



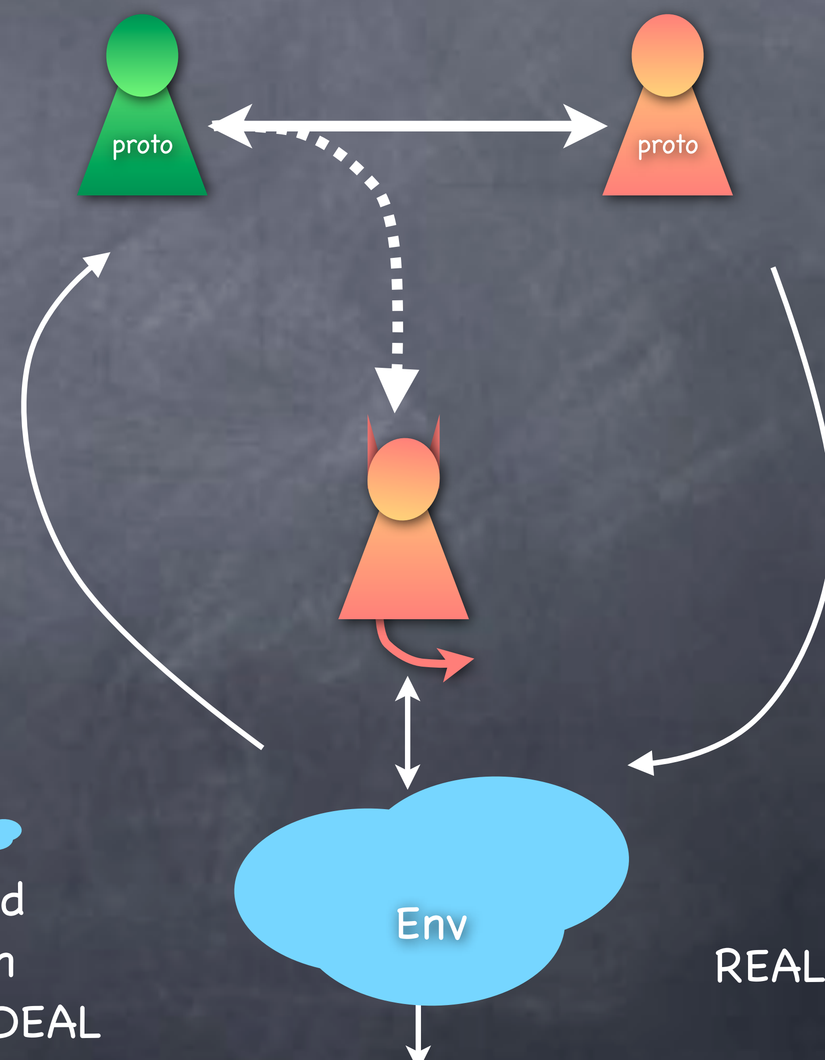
Secure (and correct) if:

∀ 

∃  s.t.

∀ 

output of  is distributed identically in REAL and IDEAL



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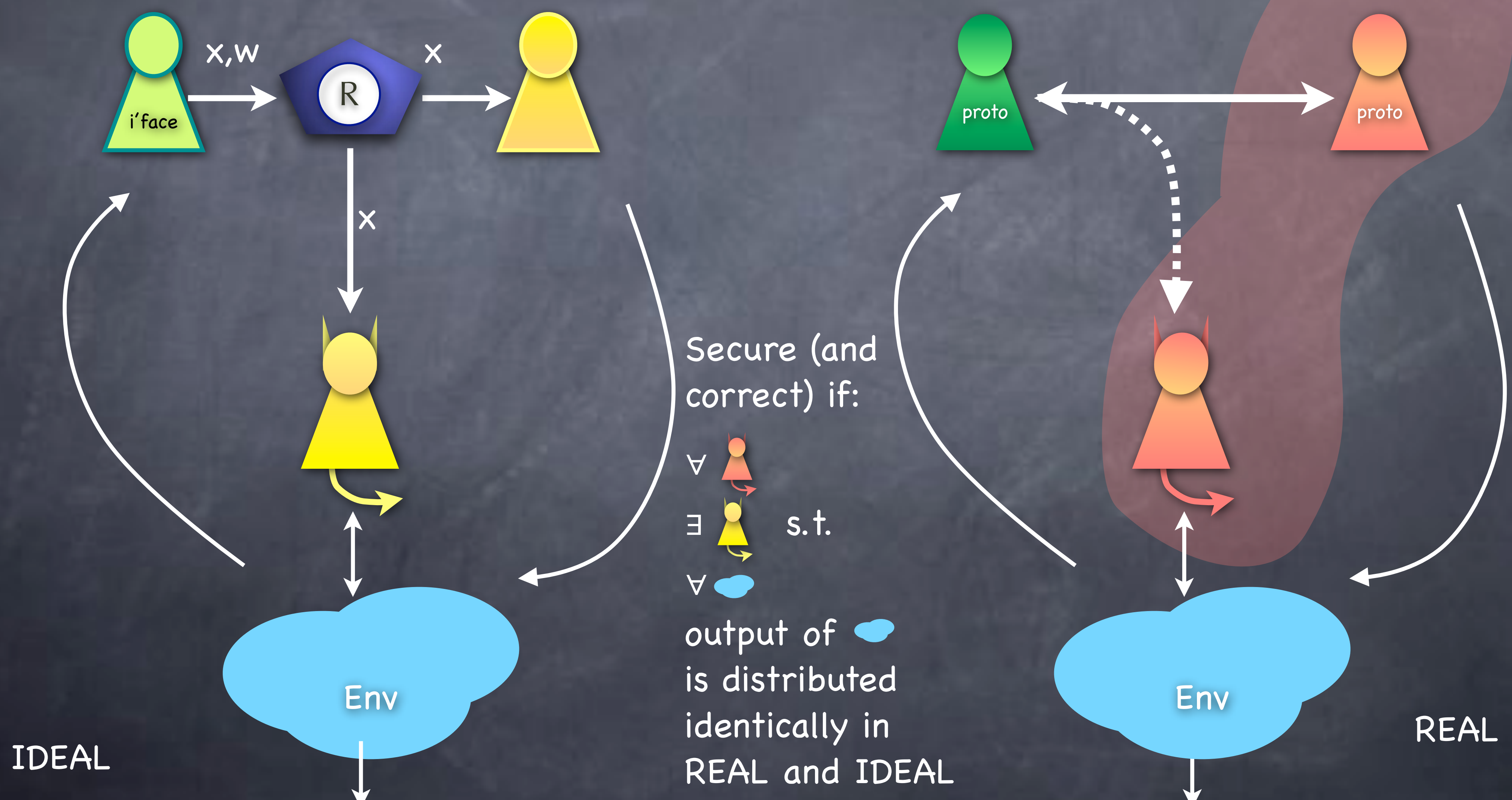

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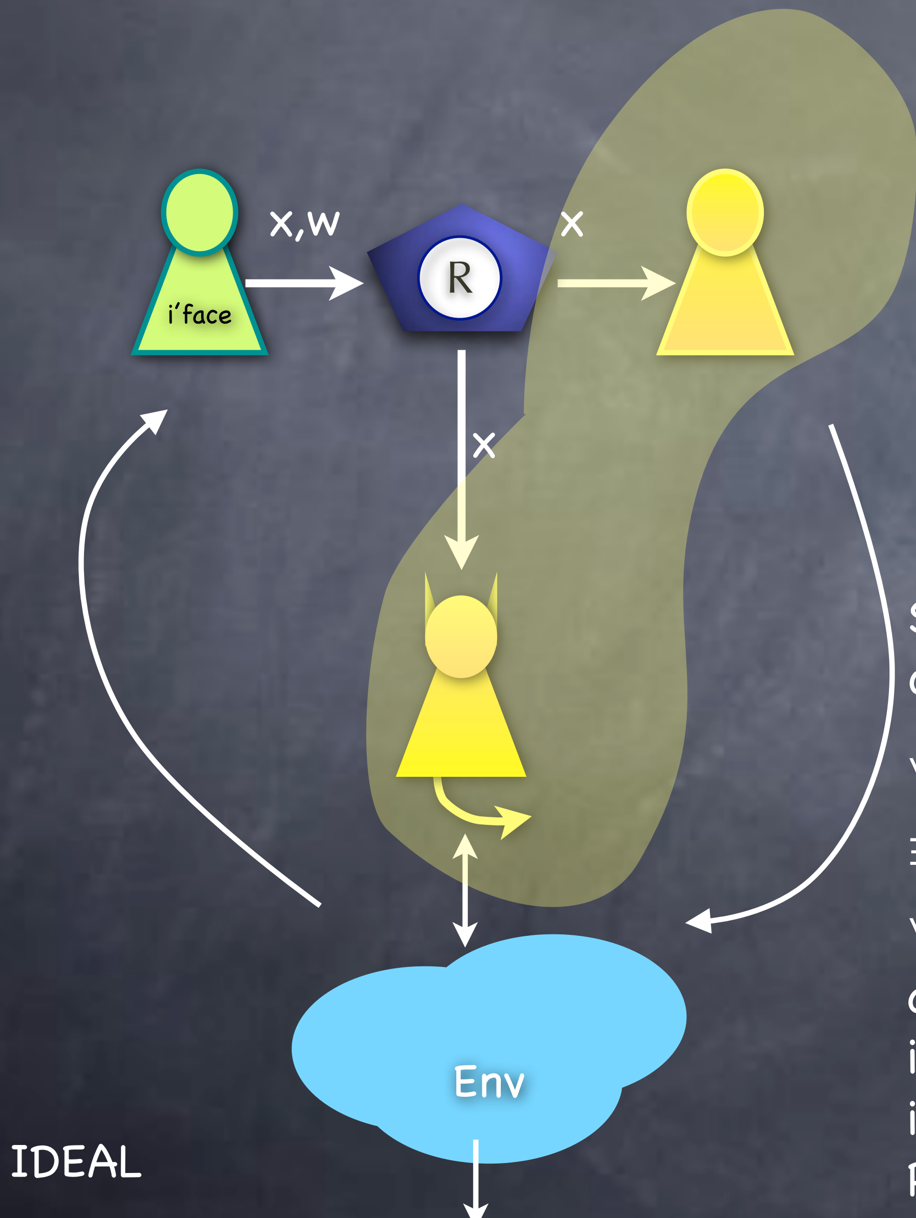

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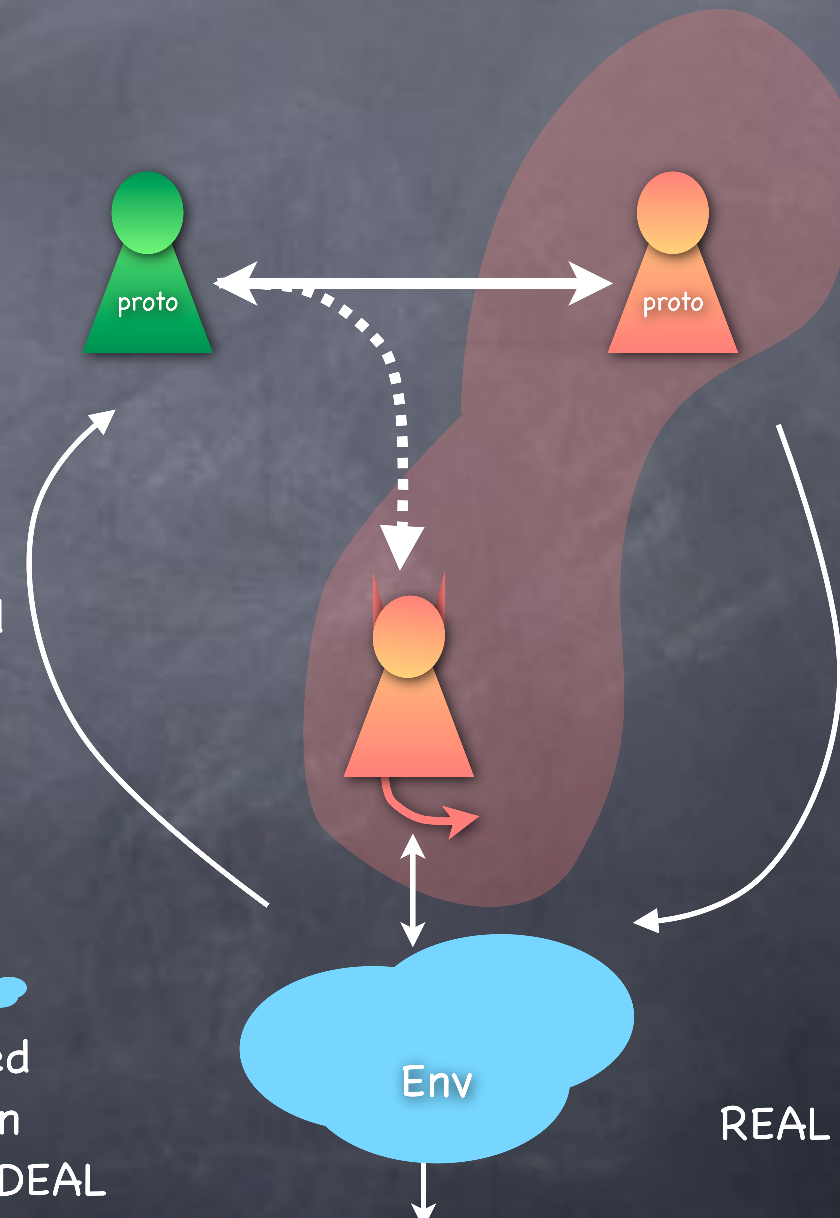
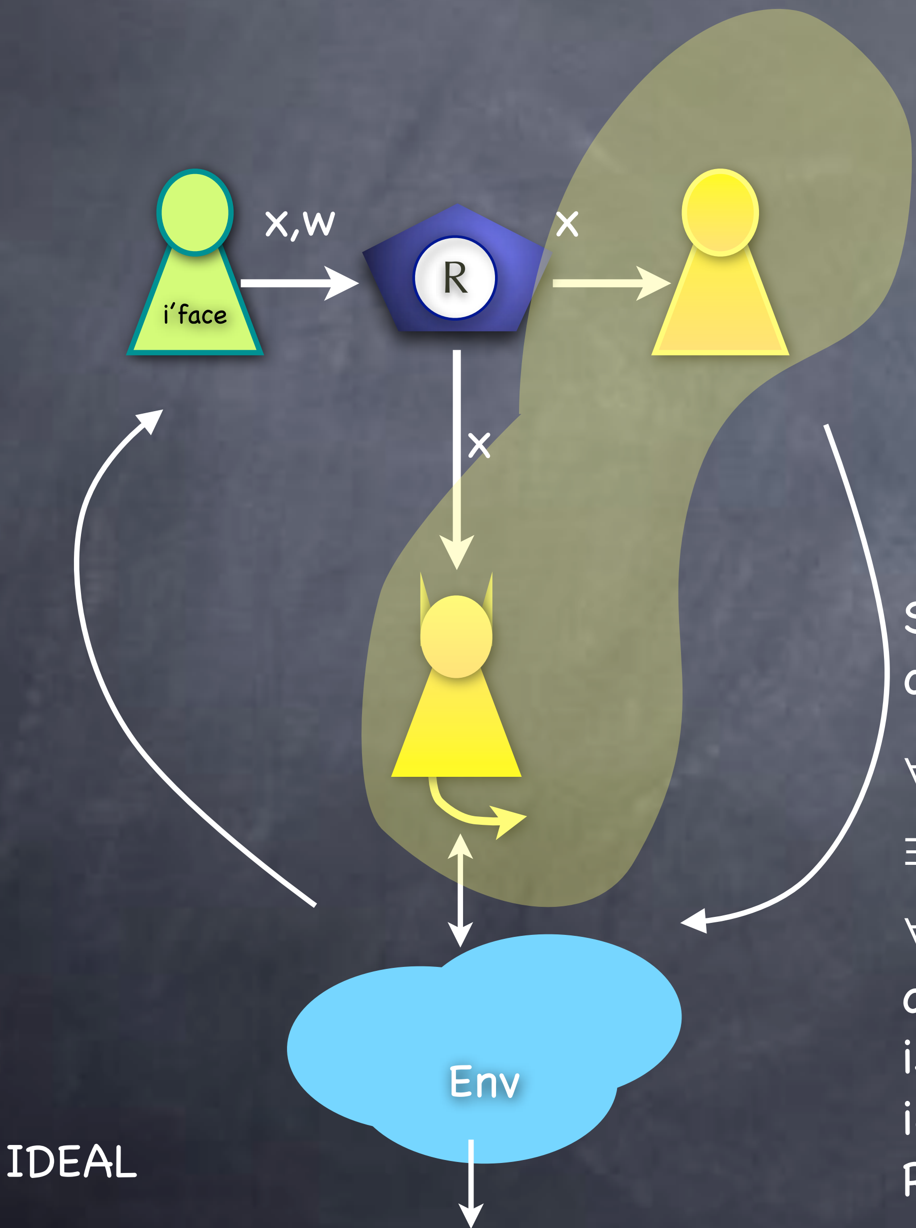


ZK Property (in other pict's)

Classical definition uses simulation only for corrupt receiver;



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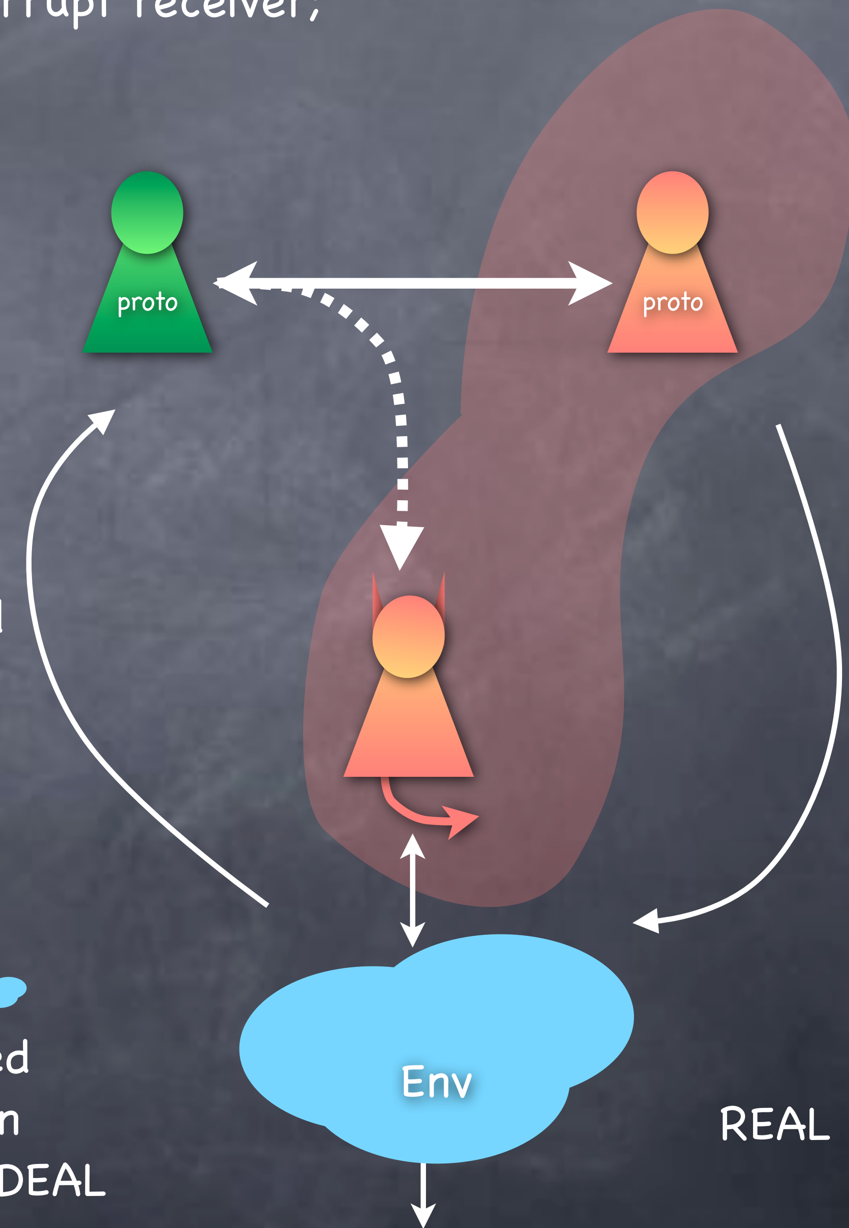
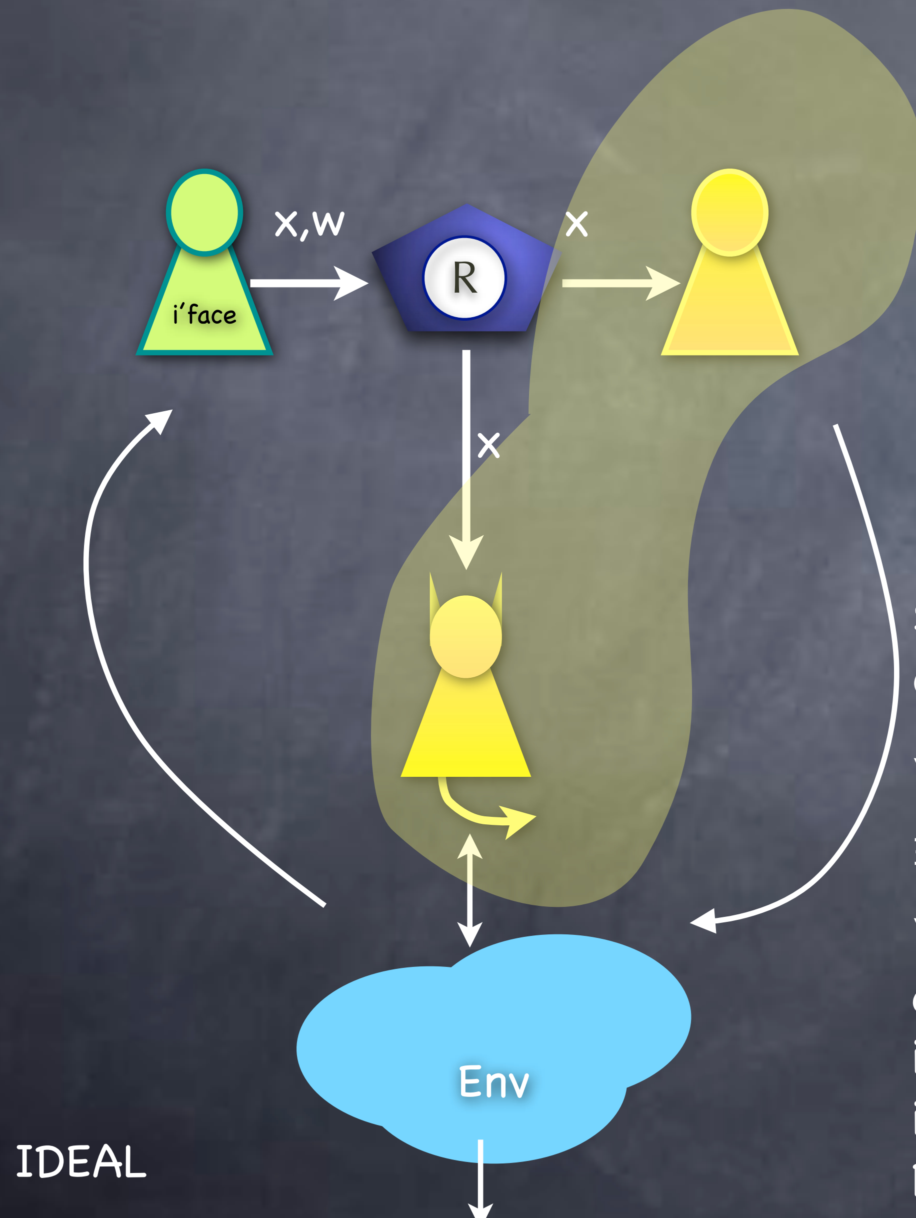


ZK Property (in other pict's)

Classical definition uses simulation only for corrupt receiver; and uses only standalone security: Environment gets only a transcript at the end



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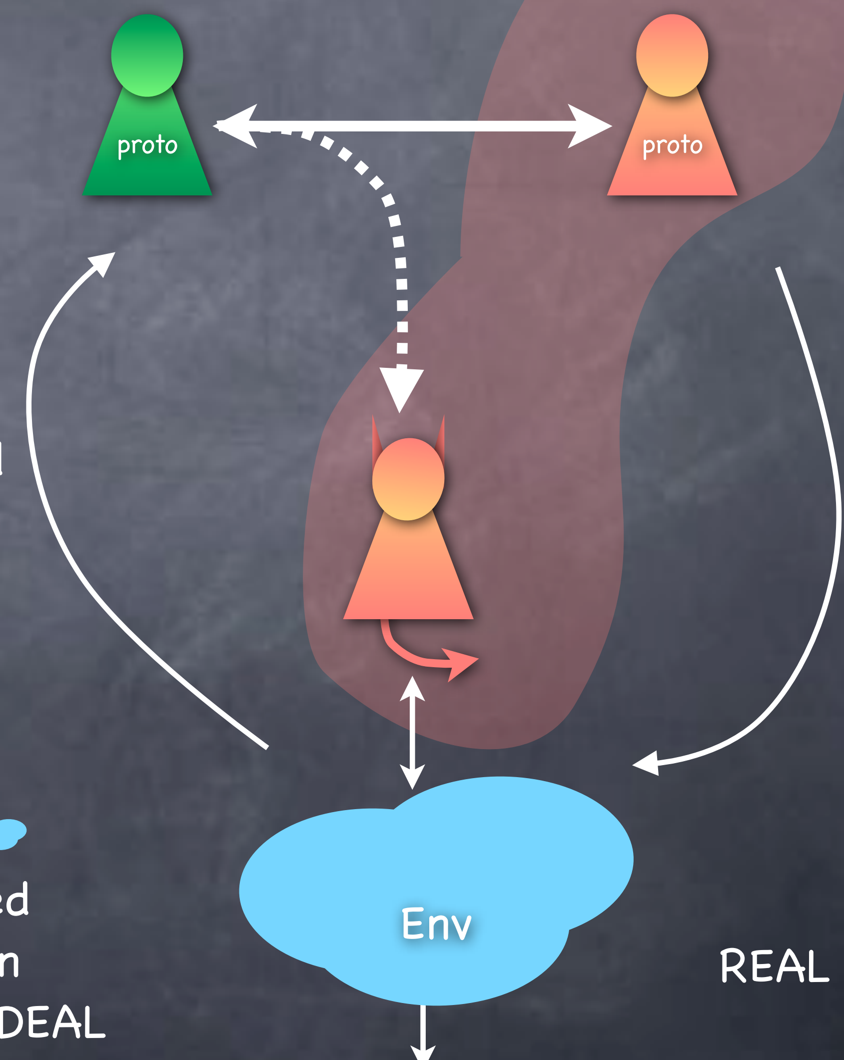
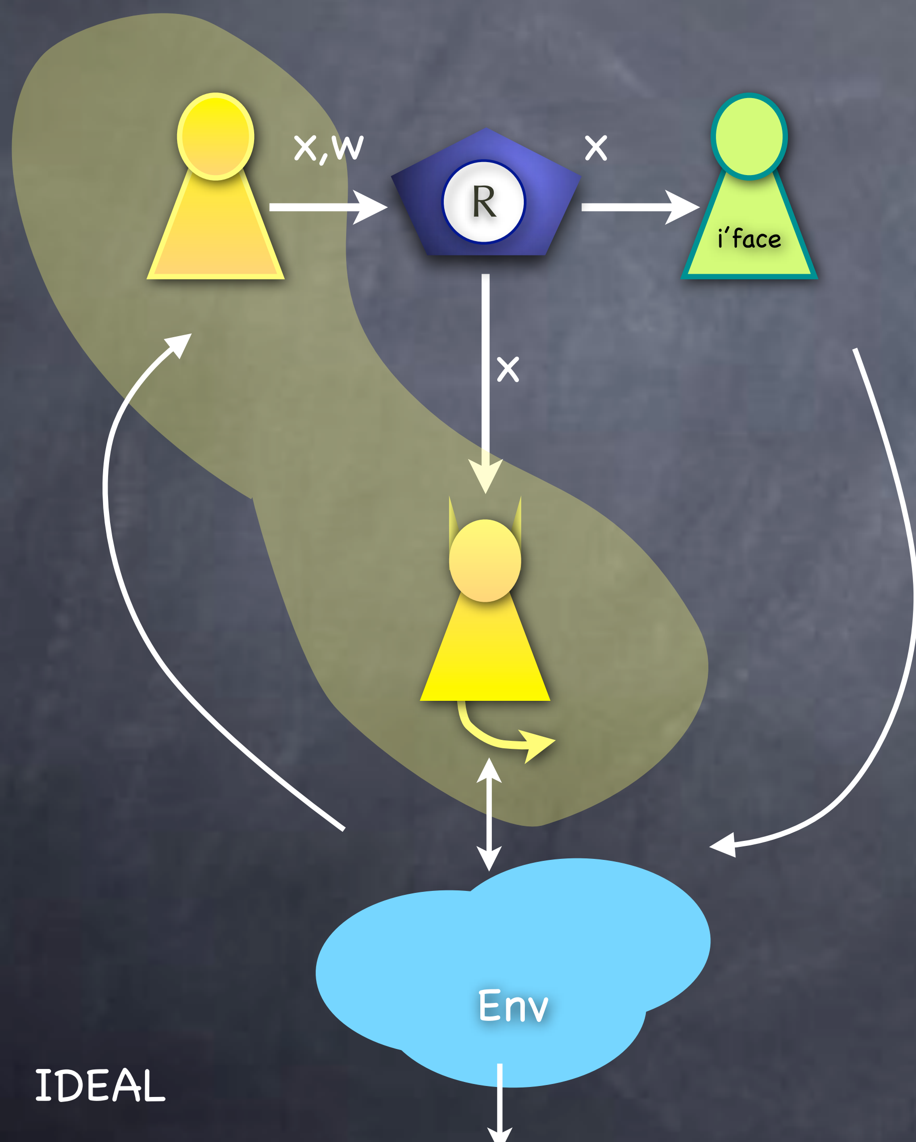


SIM ZK



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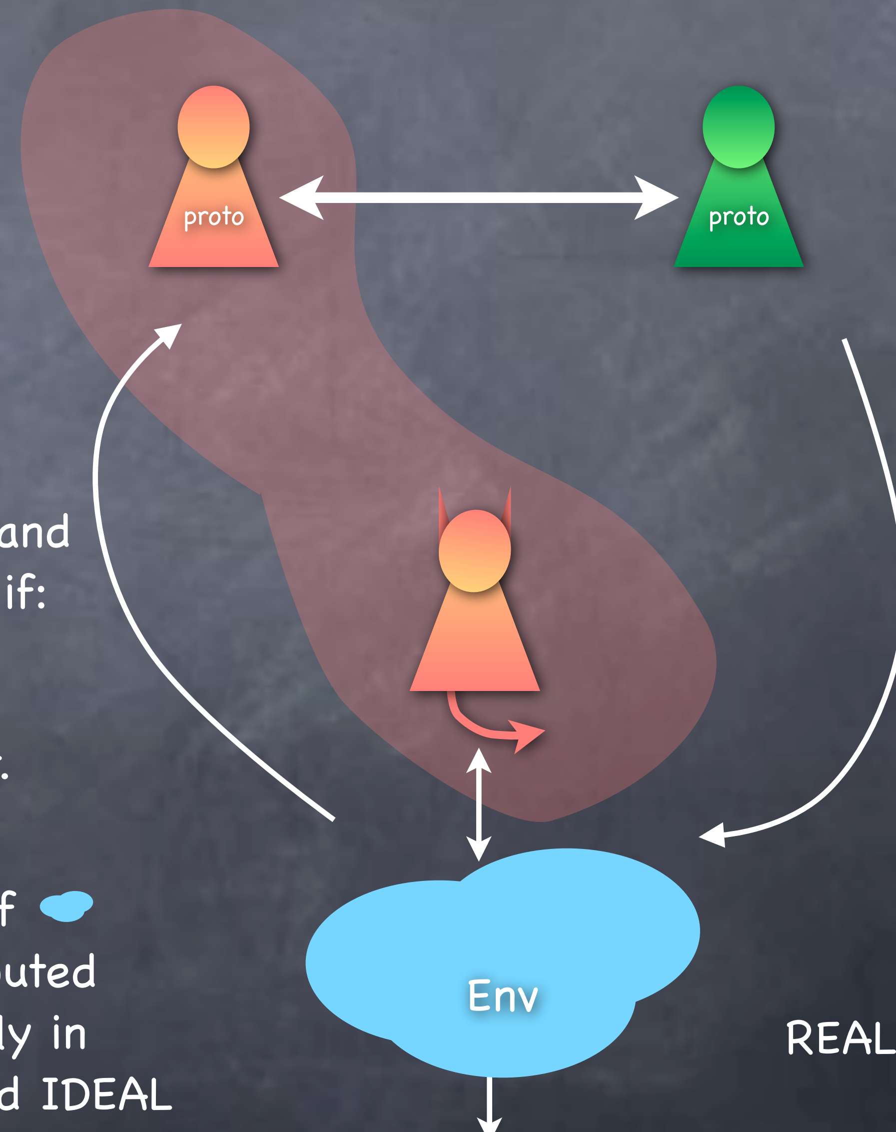

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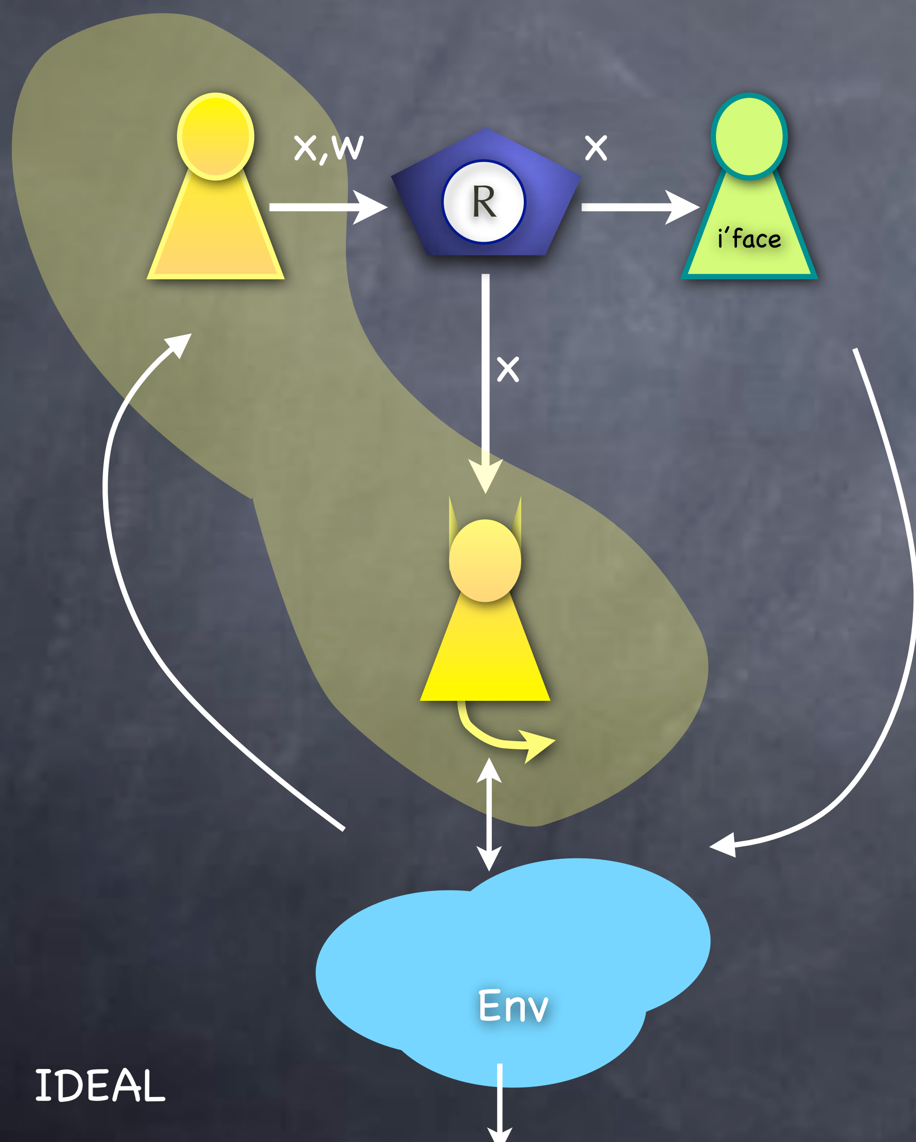

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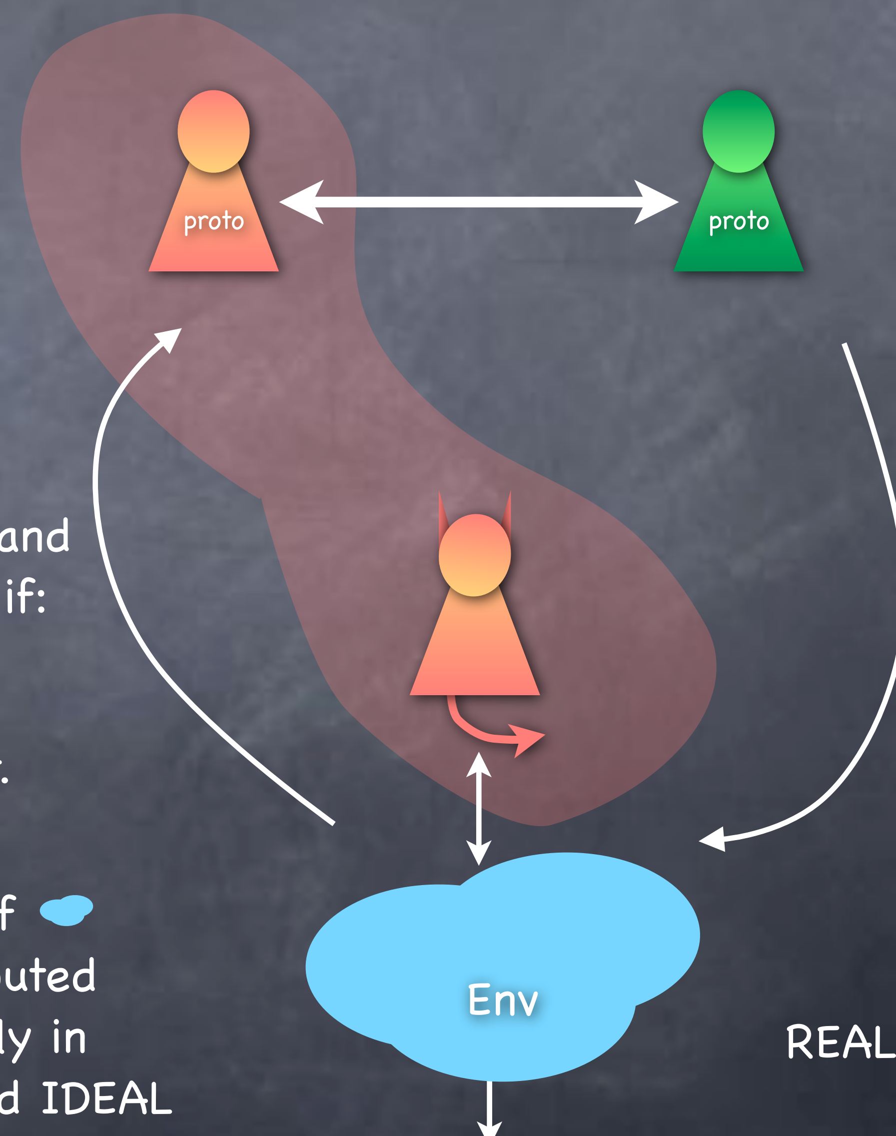

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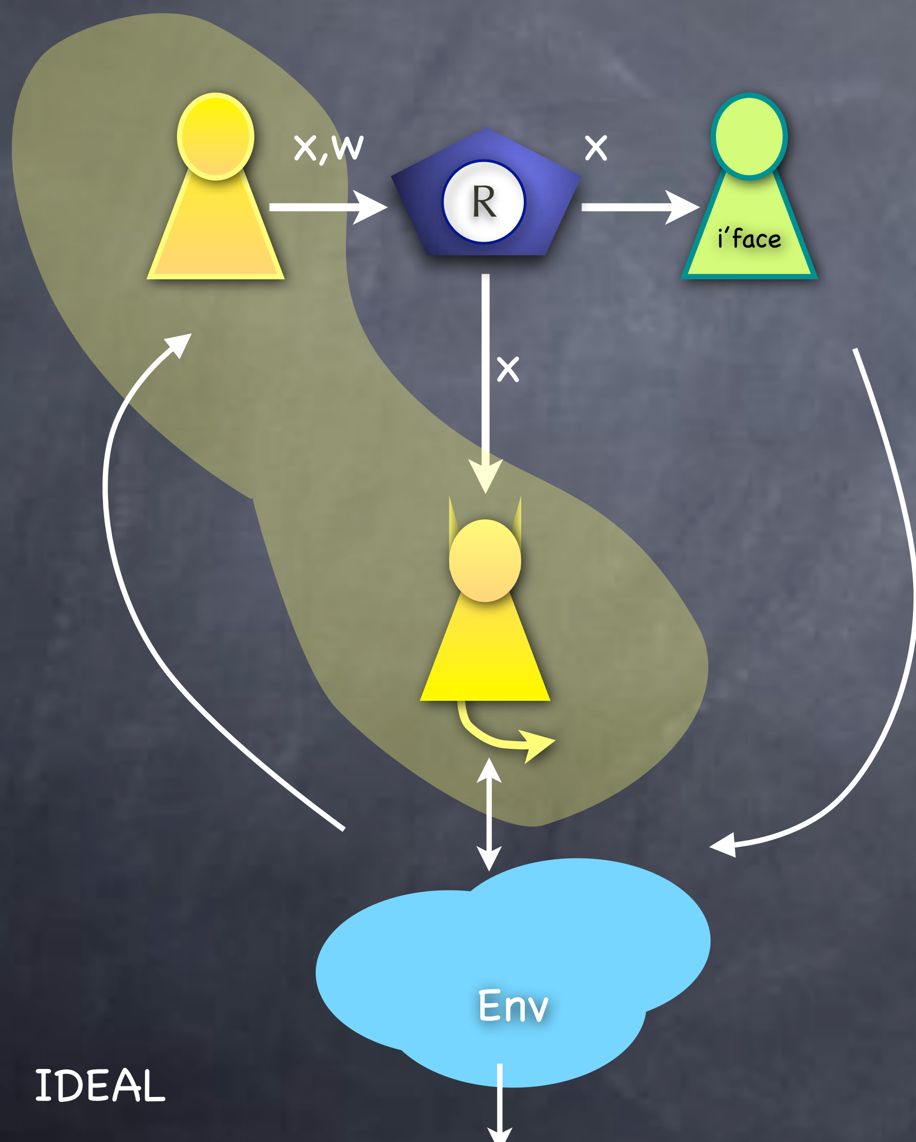

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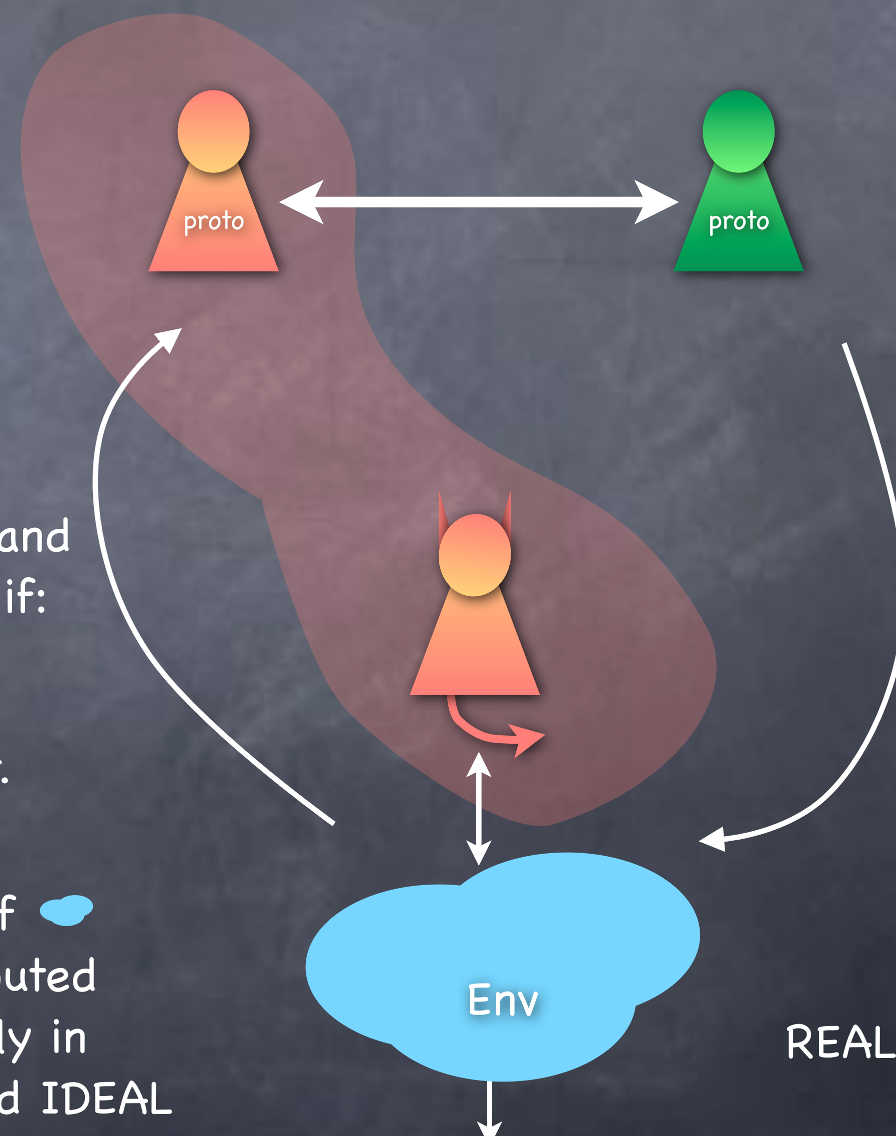

SIM ZK

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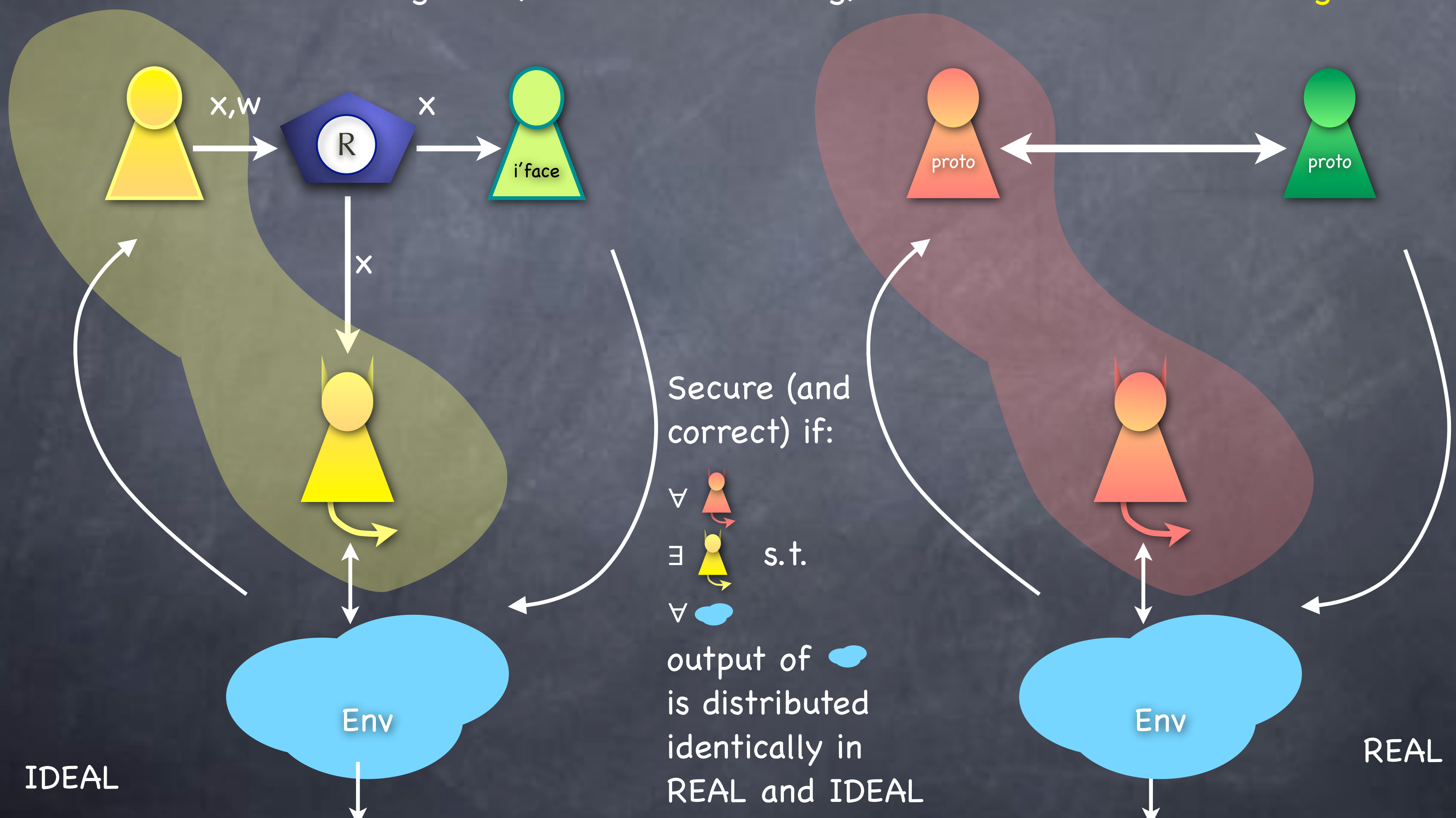
  - Then simulator is a witness extractor
- Adding this (in standalone setting) makes it a Proof of Knowledge



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  - ZKPoK, Statistical ZK Arguments, O(1)-round ZK, ...

A ZK Proof for Graph Colorability

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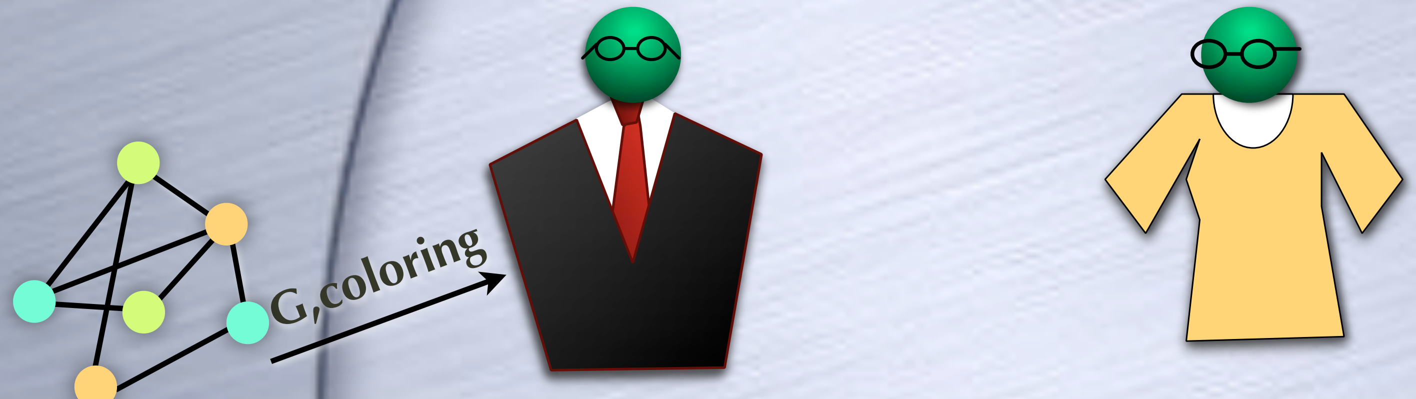

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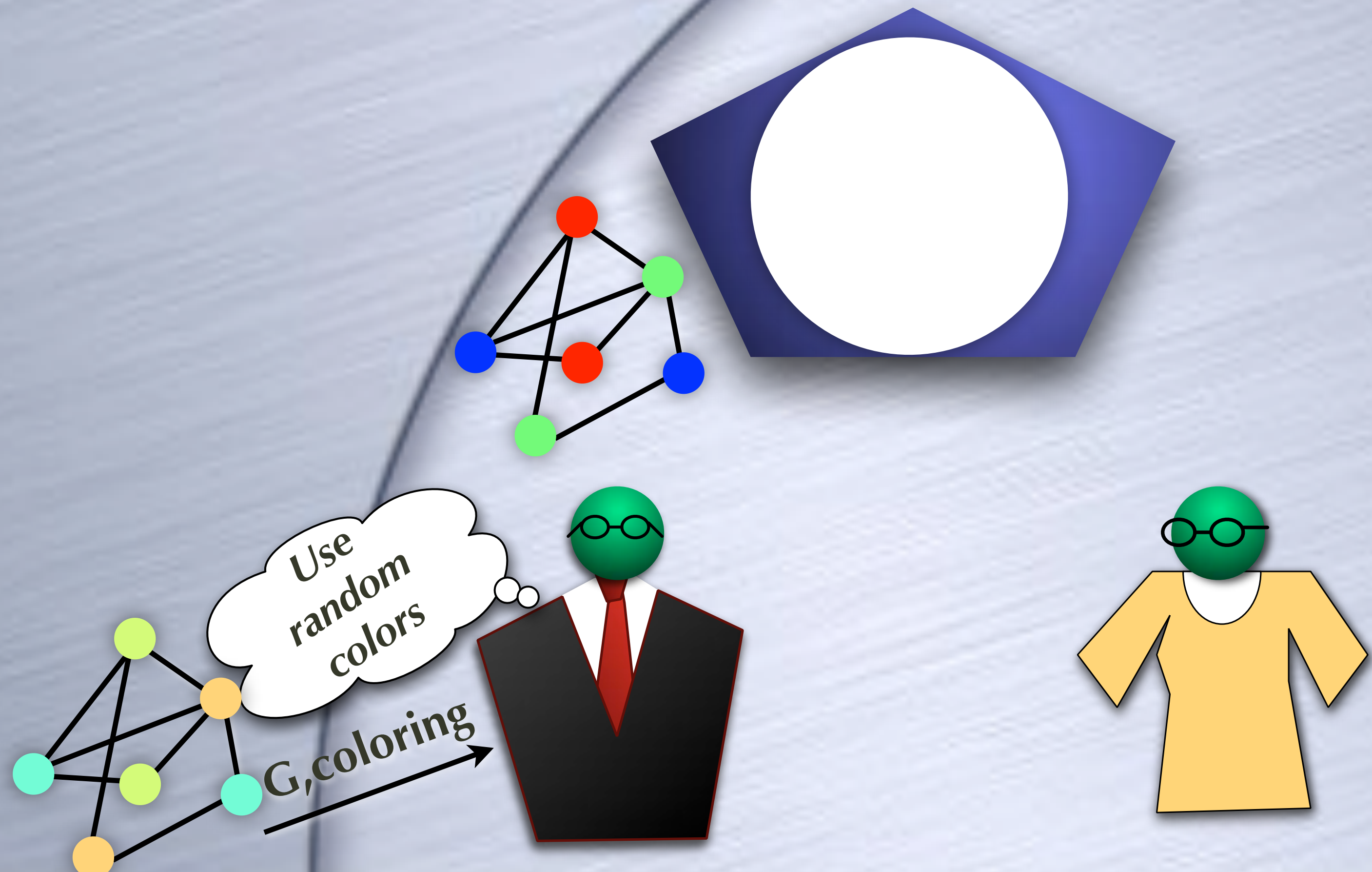

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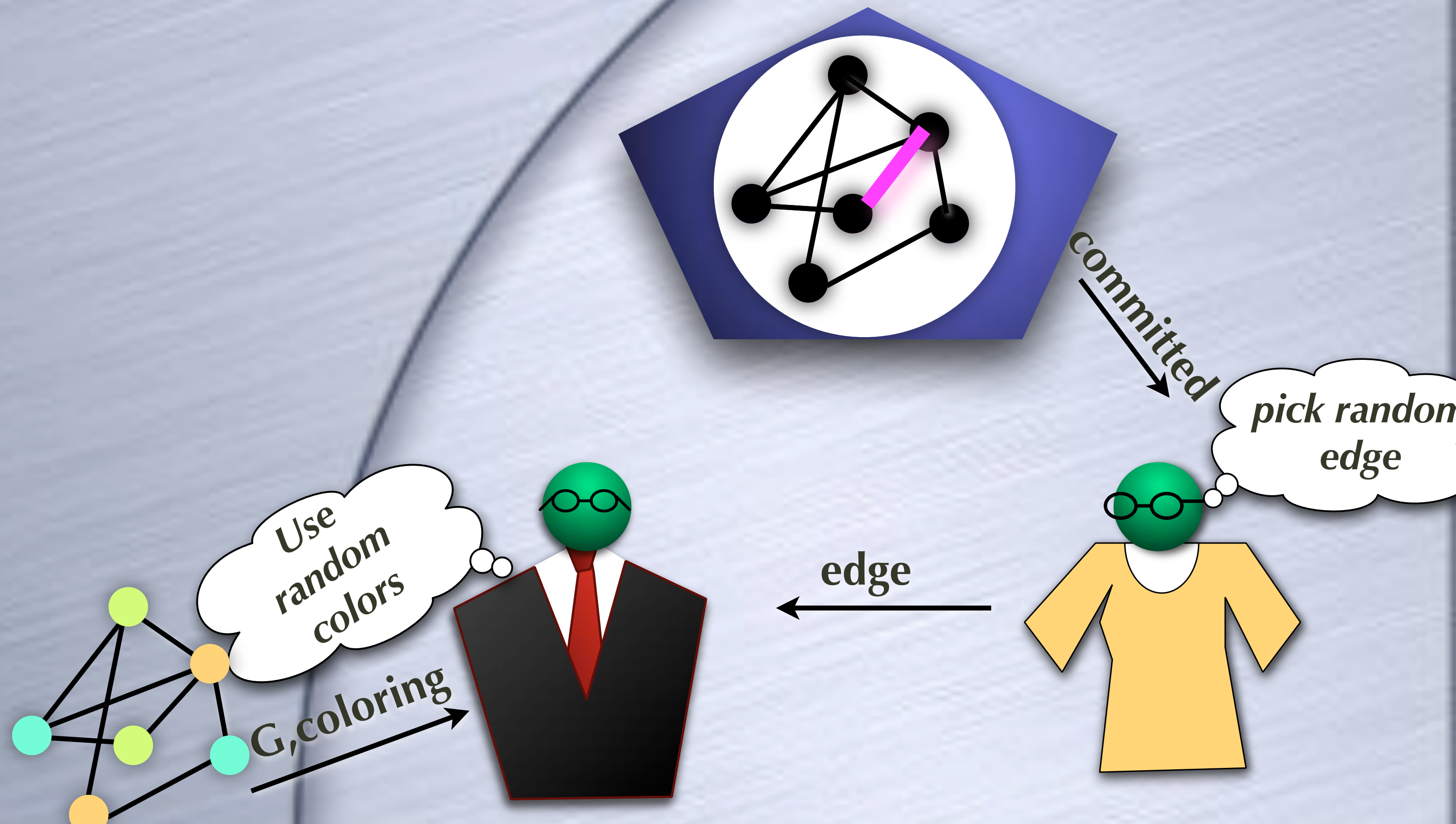

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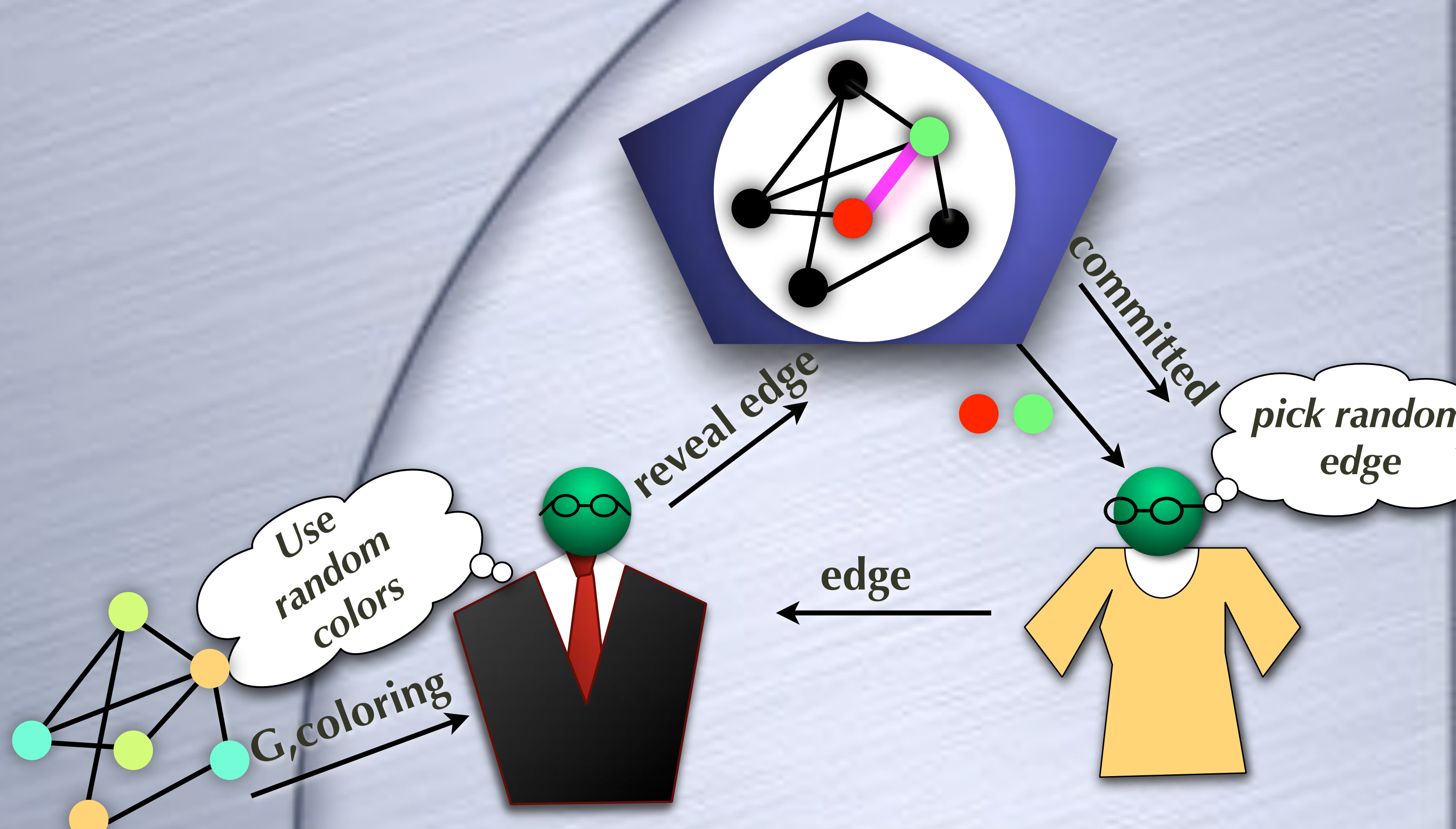

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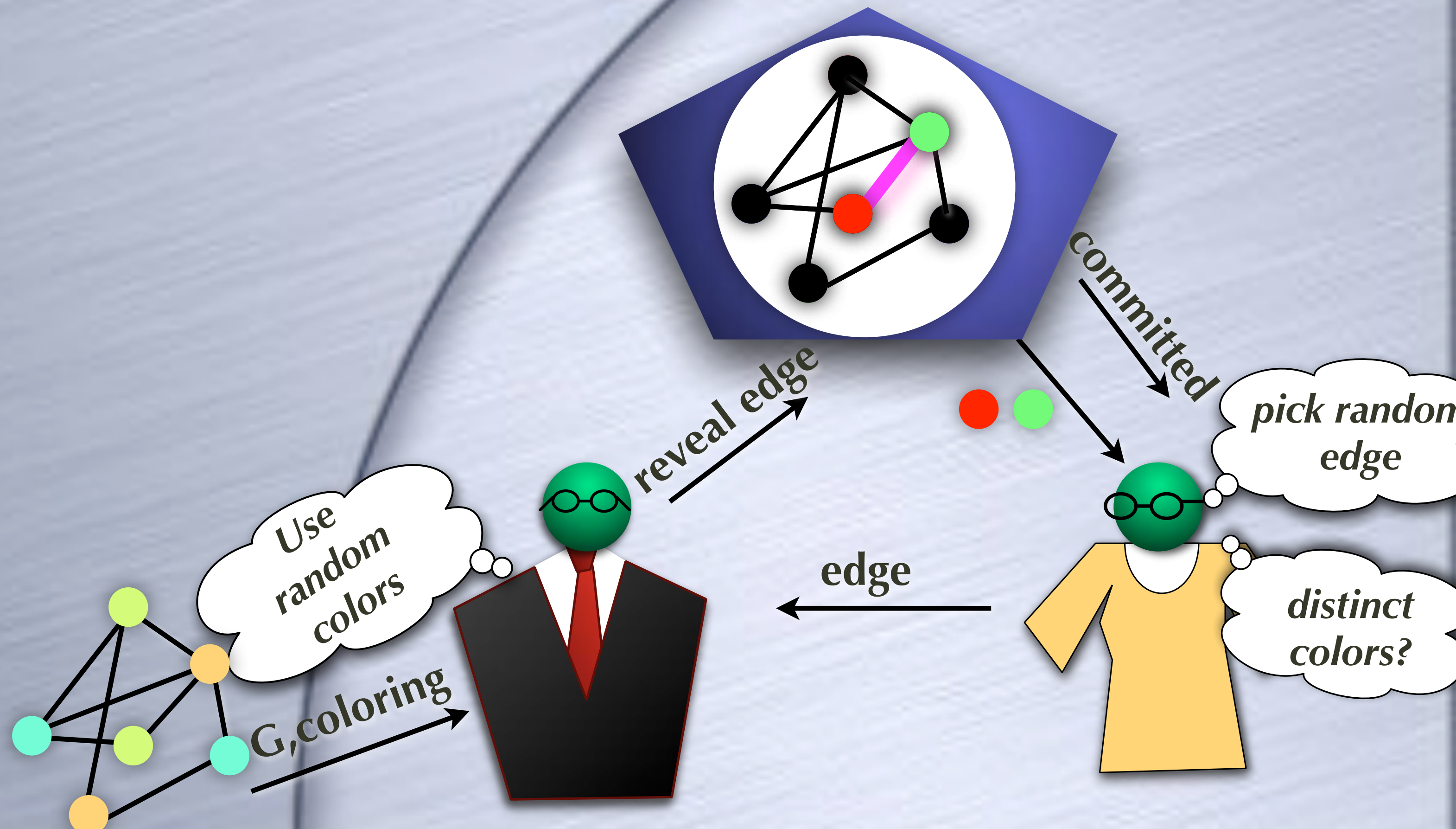

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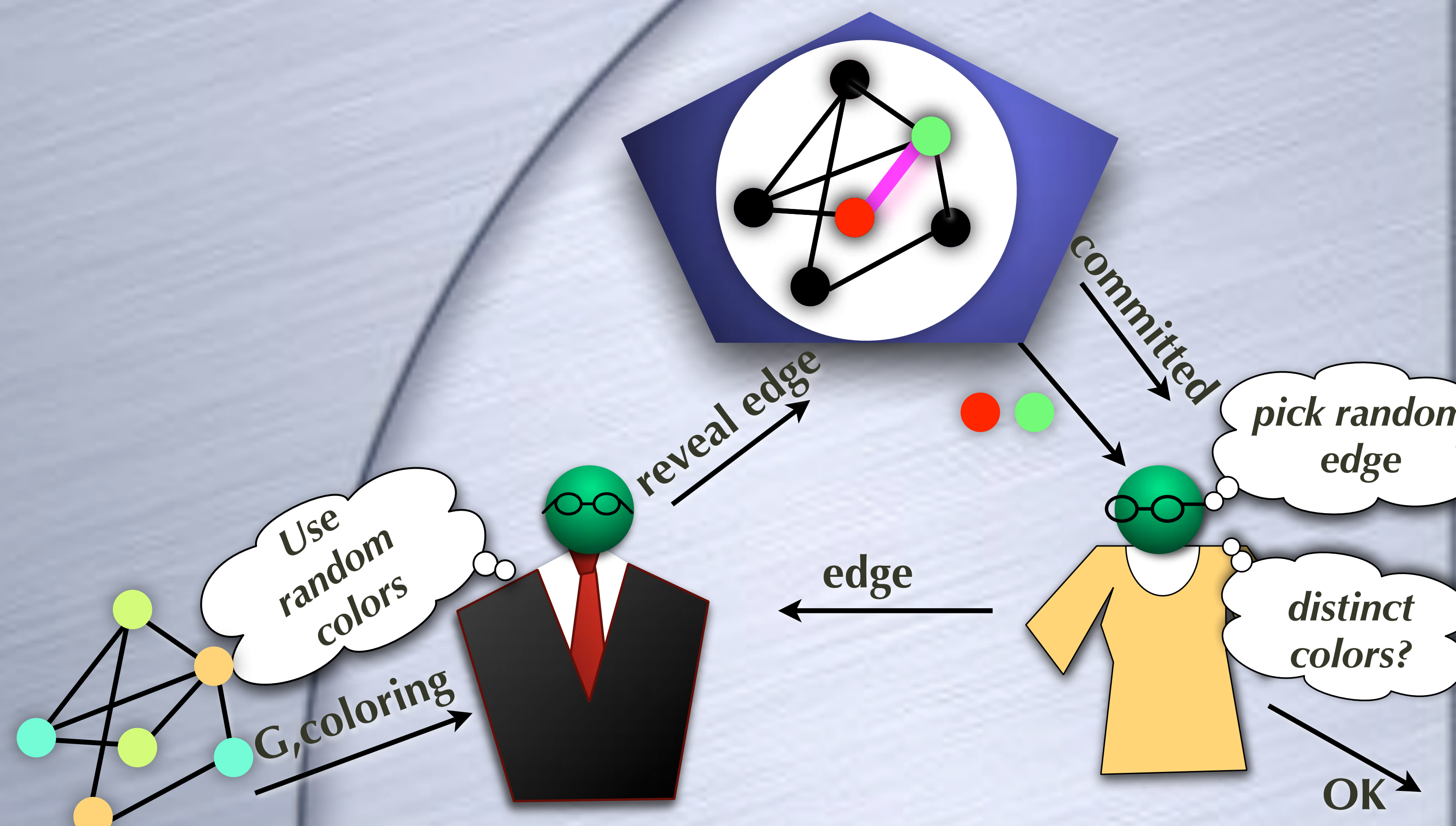

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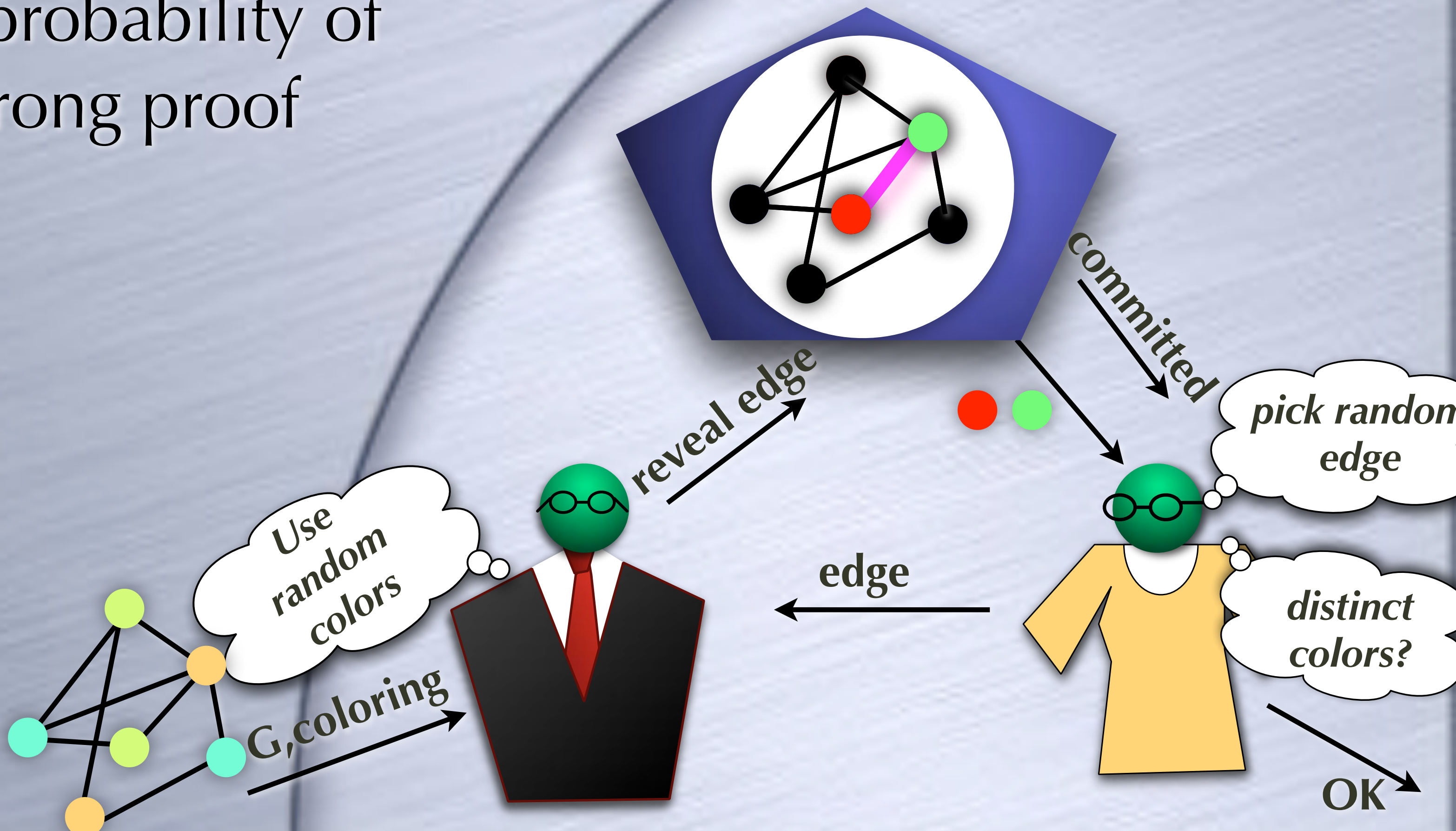
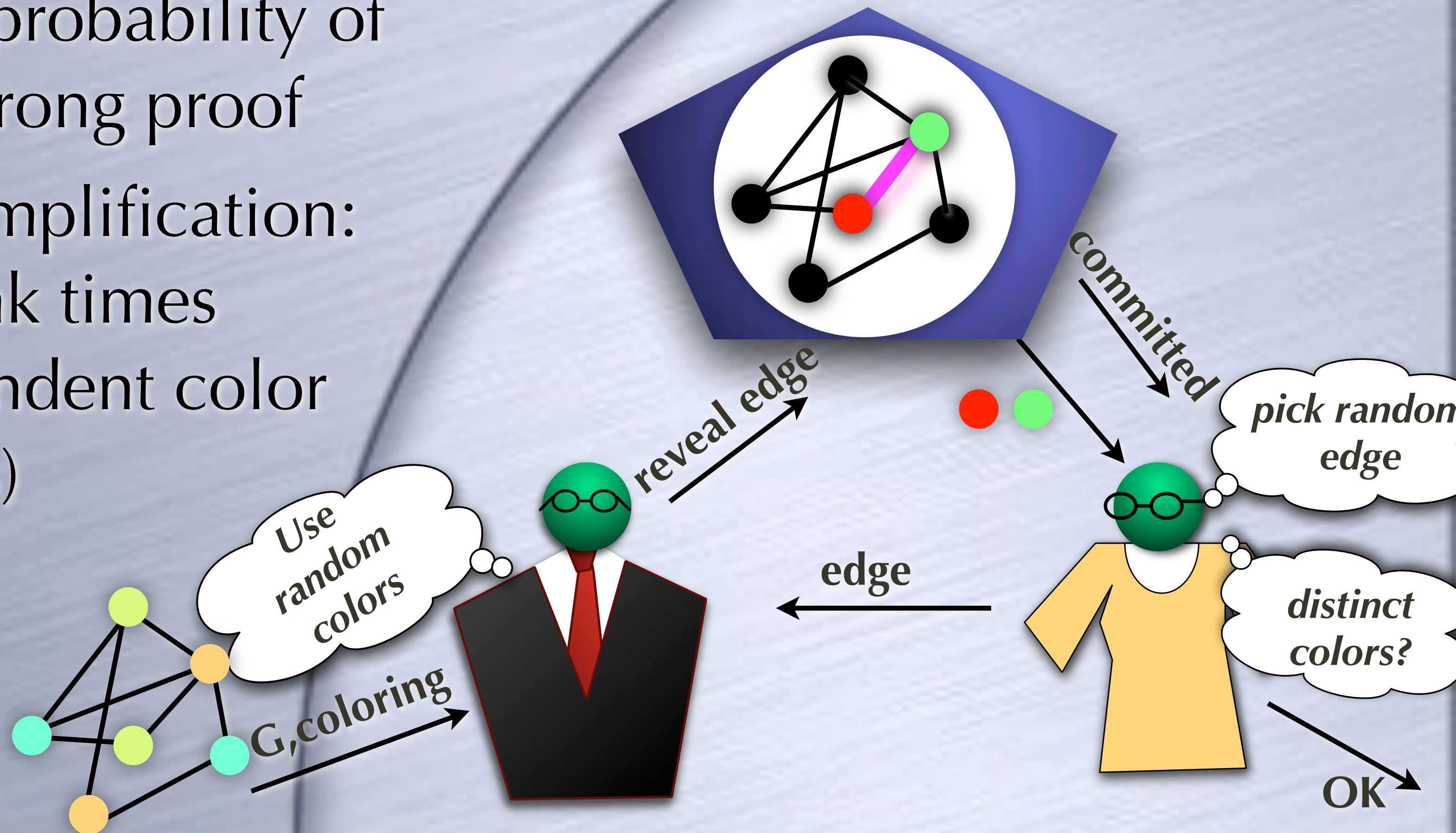


A ZK Proof for Graph Colorability

- Uses a commitment protocol as a subroutine
- At least 1/m probability of catching a wrong proof
- Soundness amplification: Repeat say mk times (with independent color permutations)



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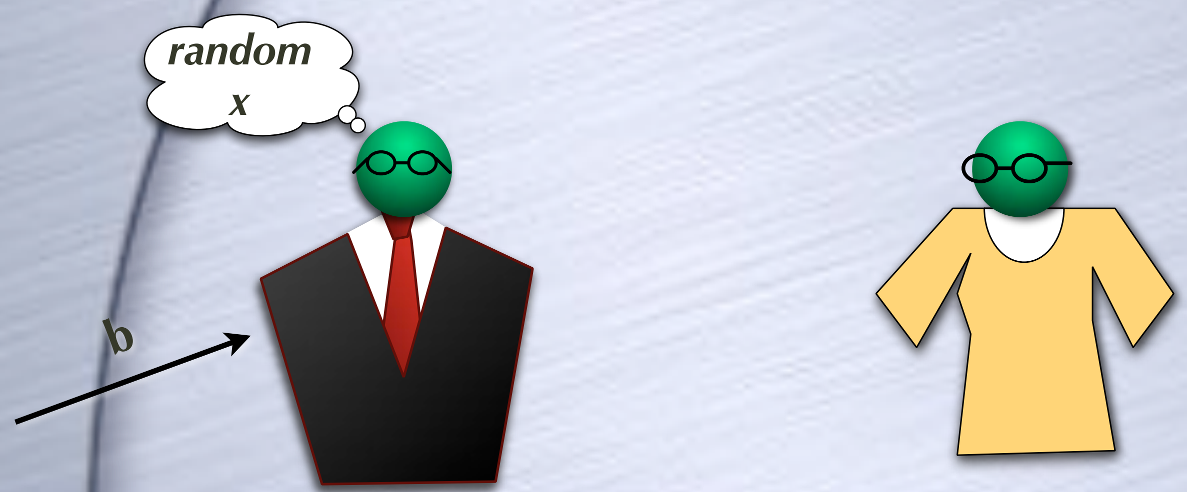

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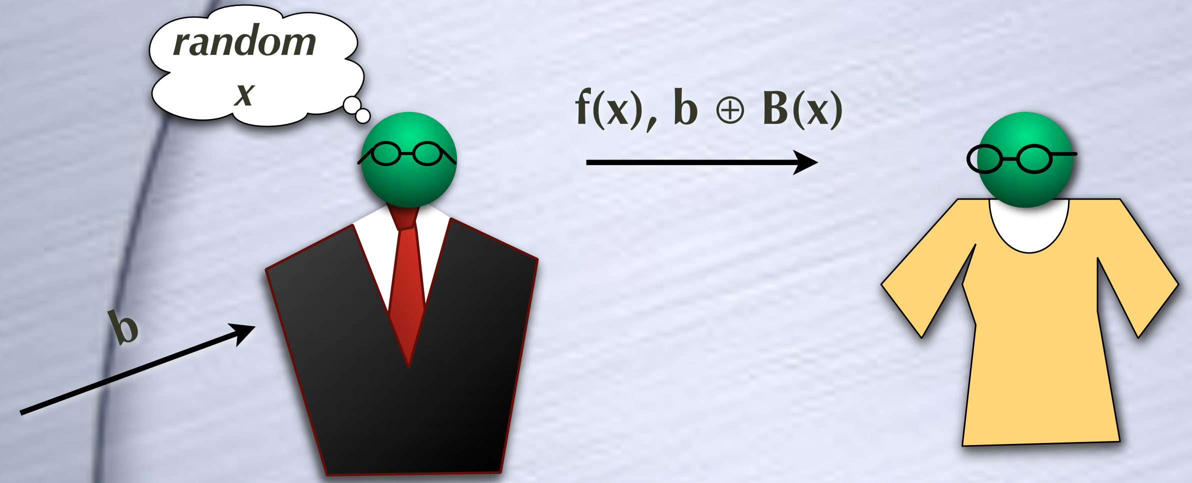

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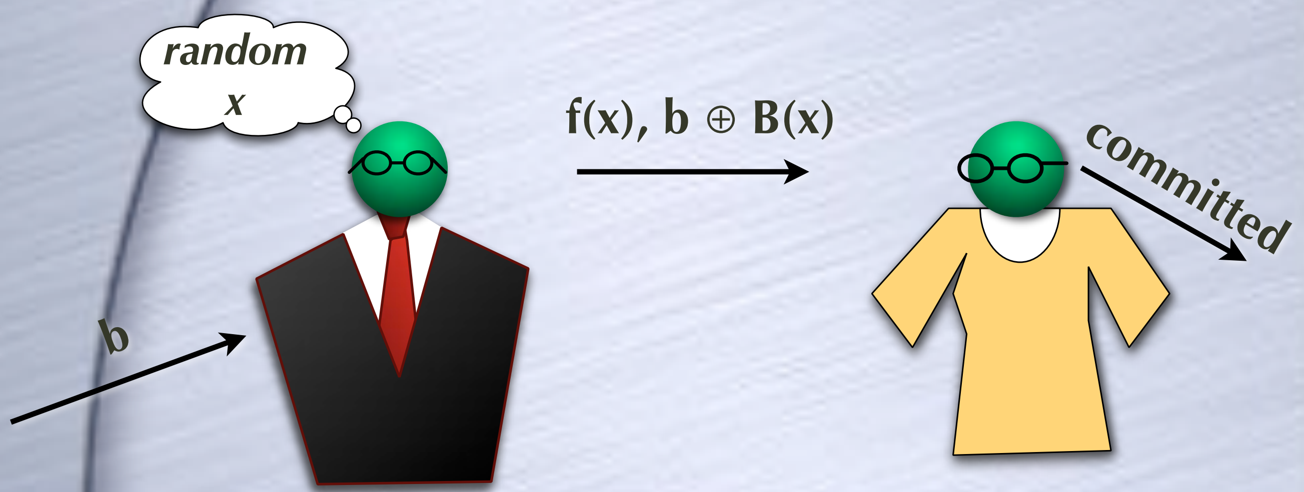

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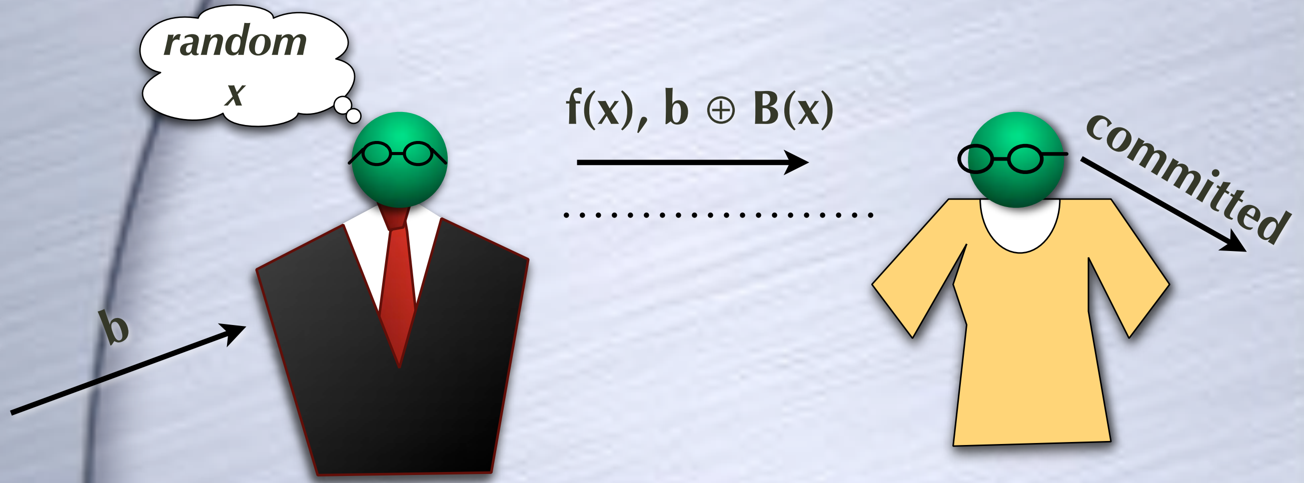

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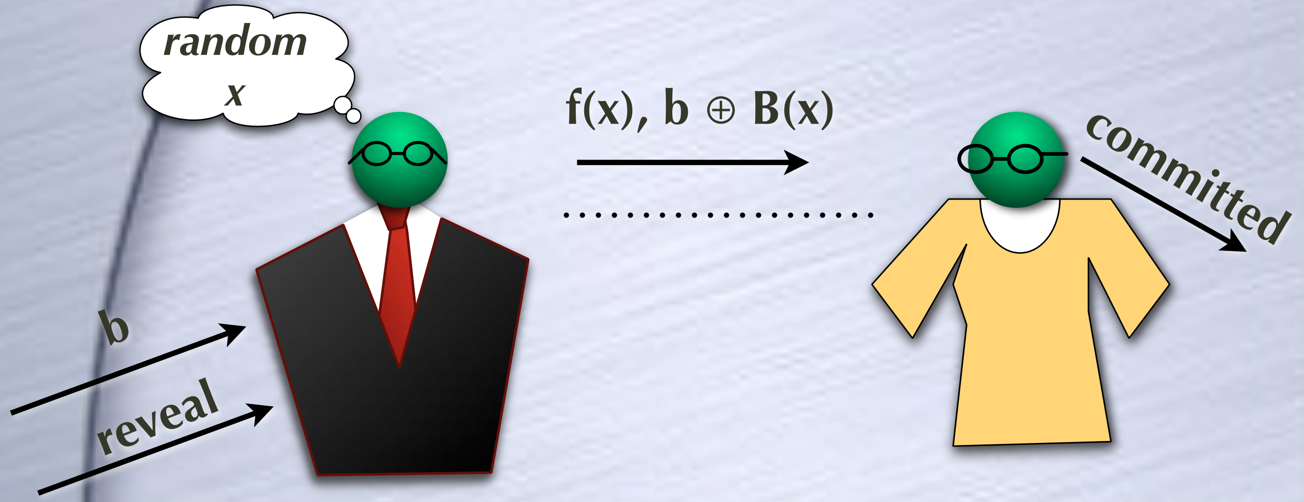

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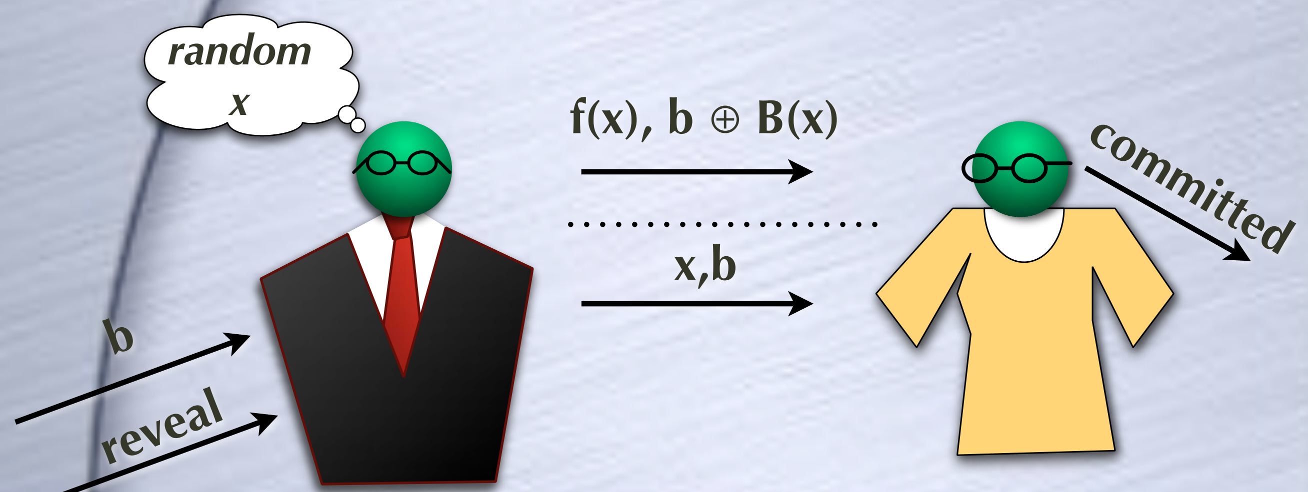

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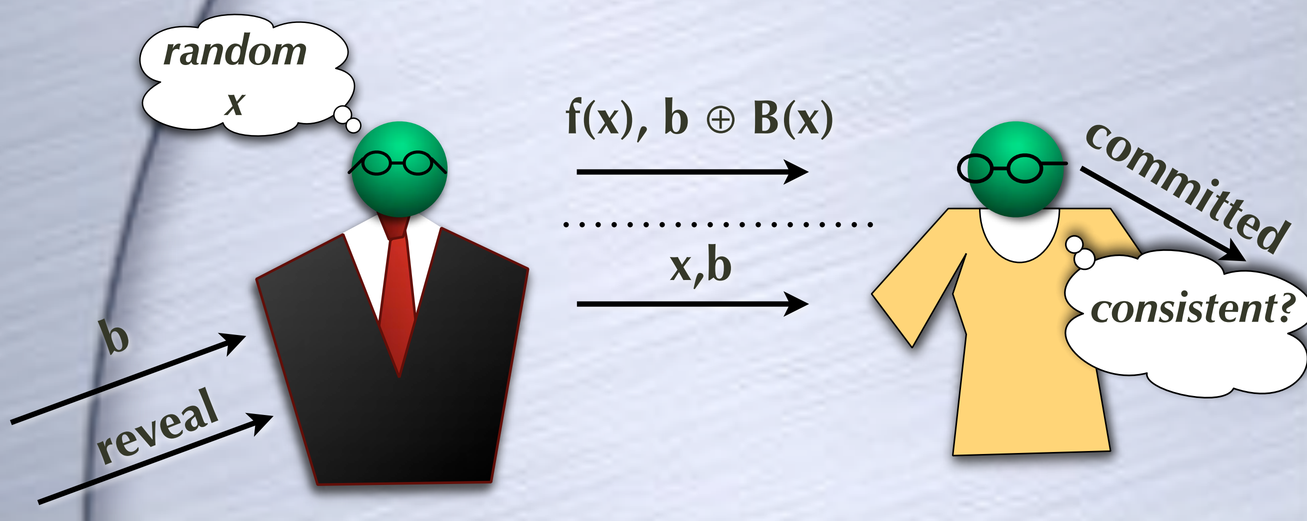

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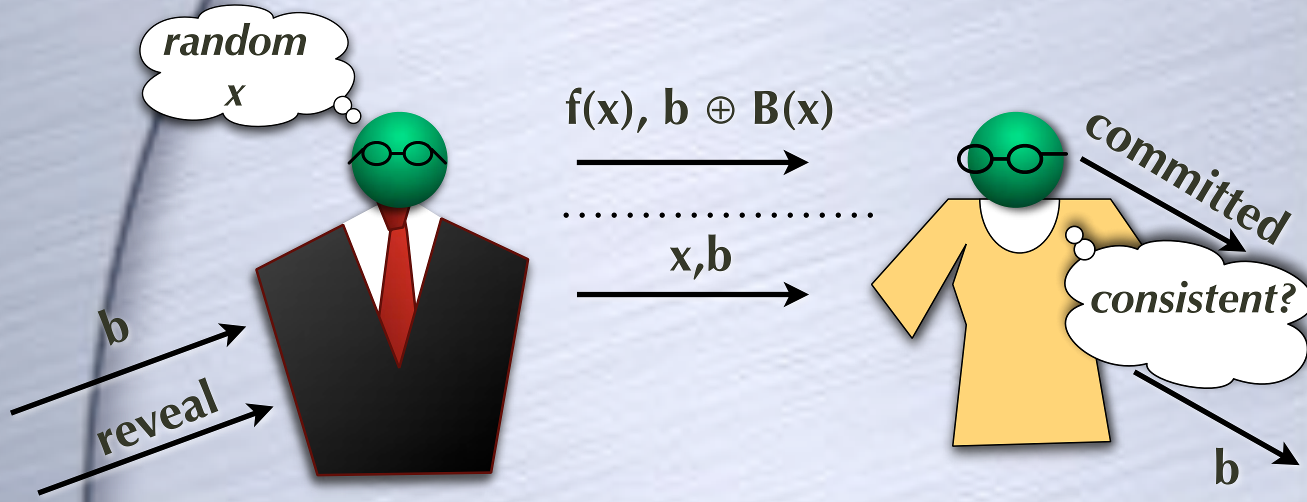

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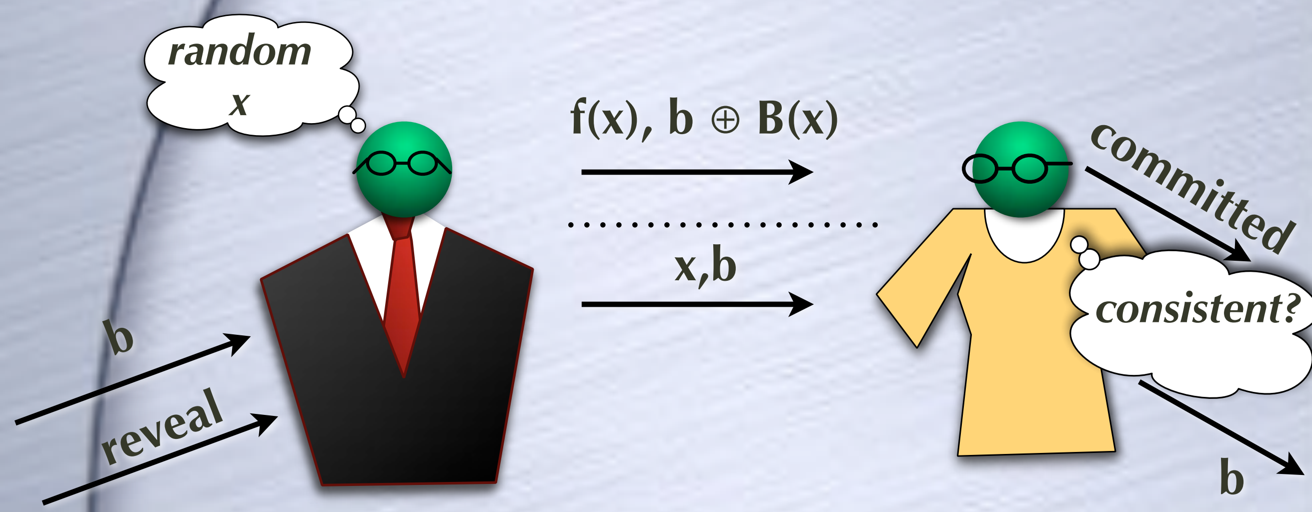
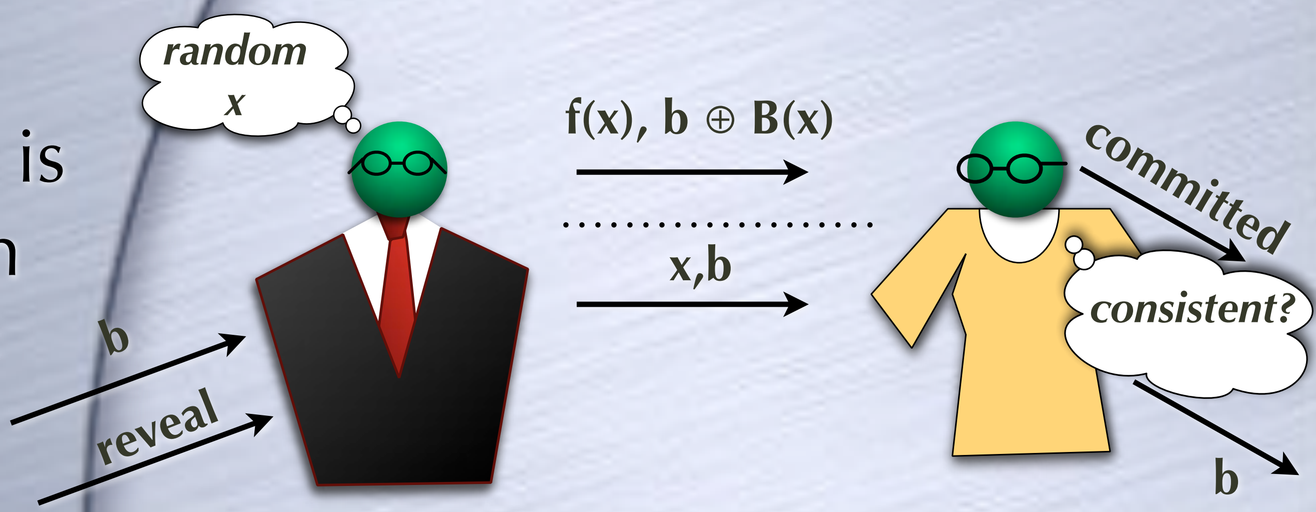


A Commitment Protocol

- Using a OWP f and a hardcore predicate for it B
- Satisfies only classical (IND) security, in terms of hiding and binding
- Perfectly binding because f is a permutation
- Hiding because B(x) is pseudorandom given f(x)



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$\xrightarrow{x_1}$

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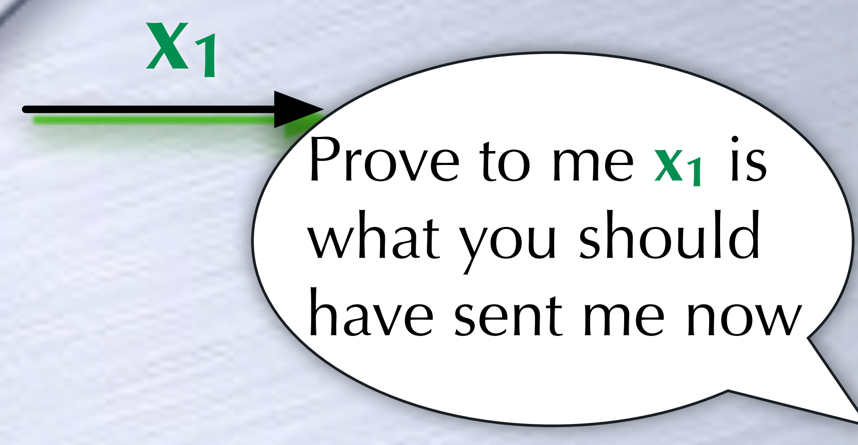

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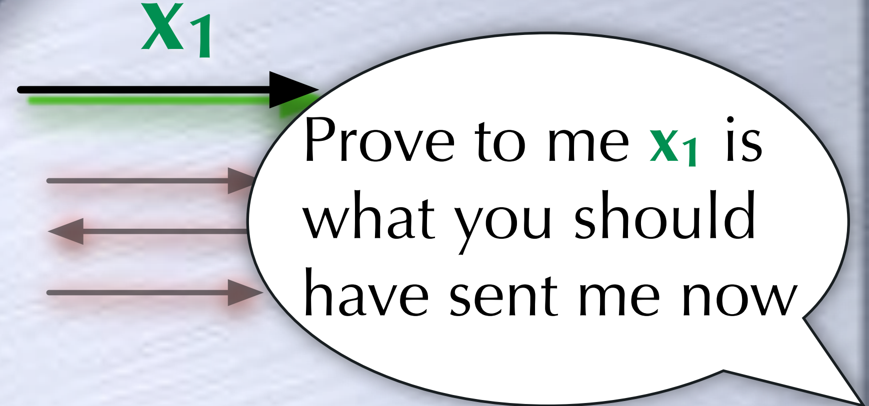

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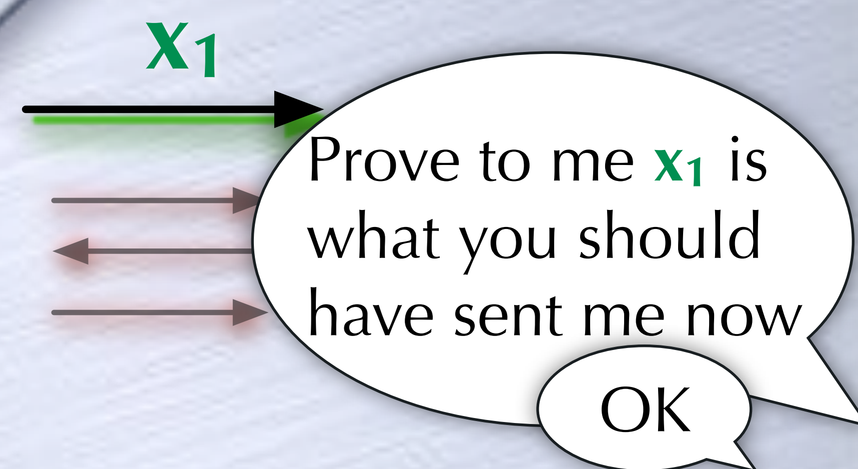

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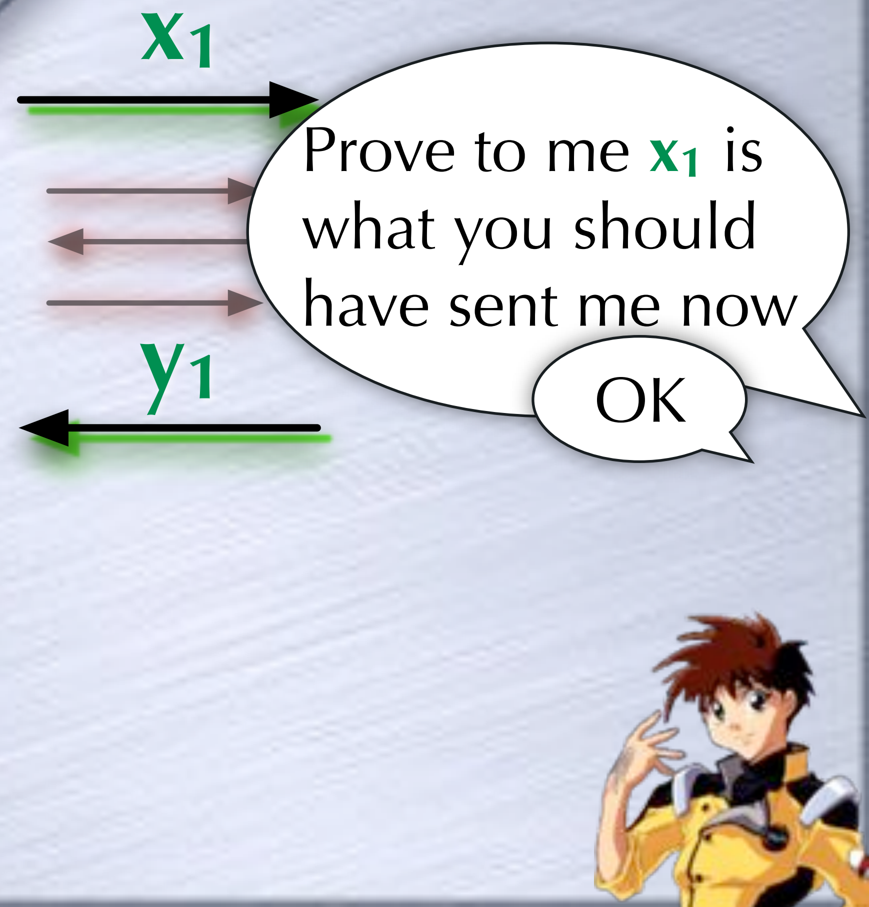

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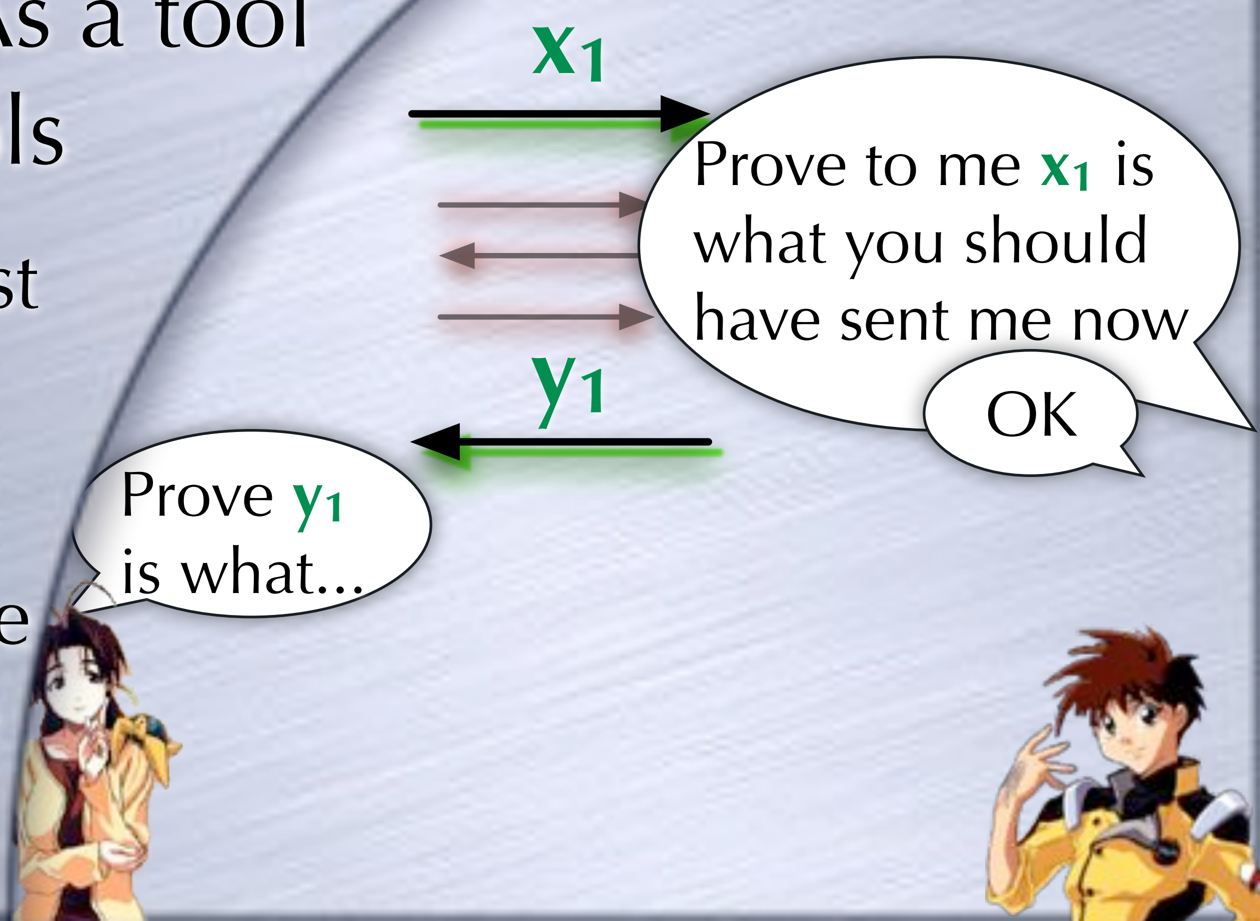

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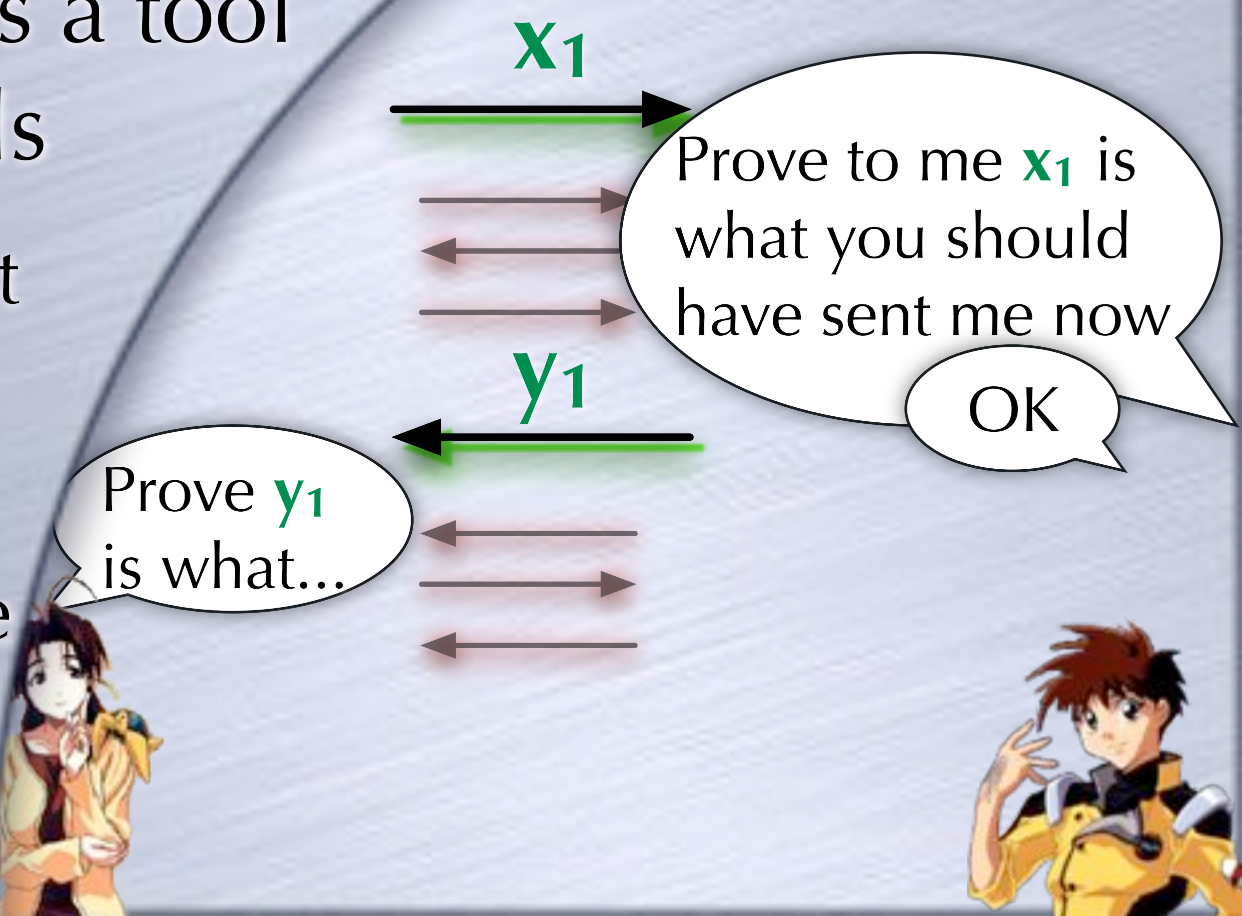

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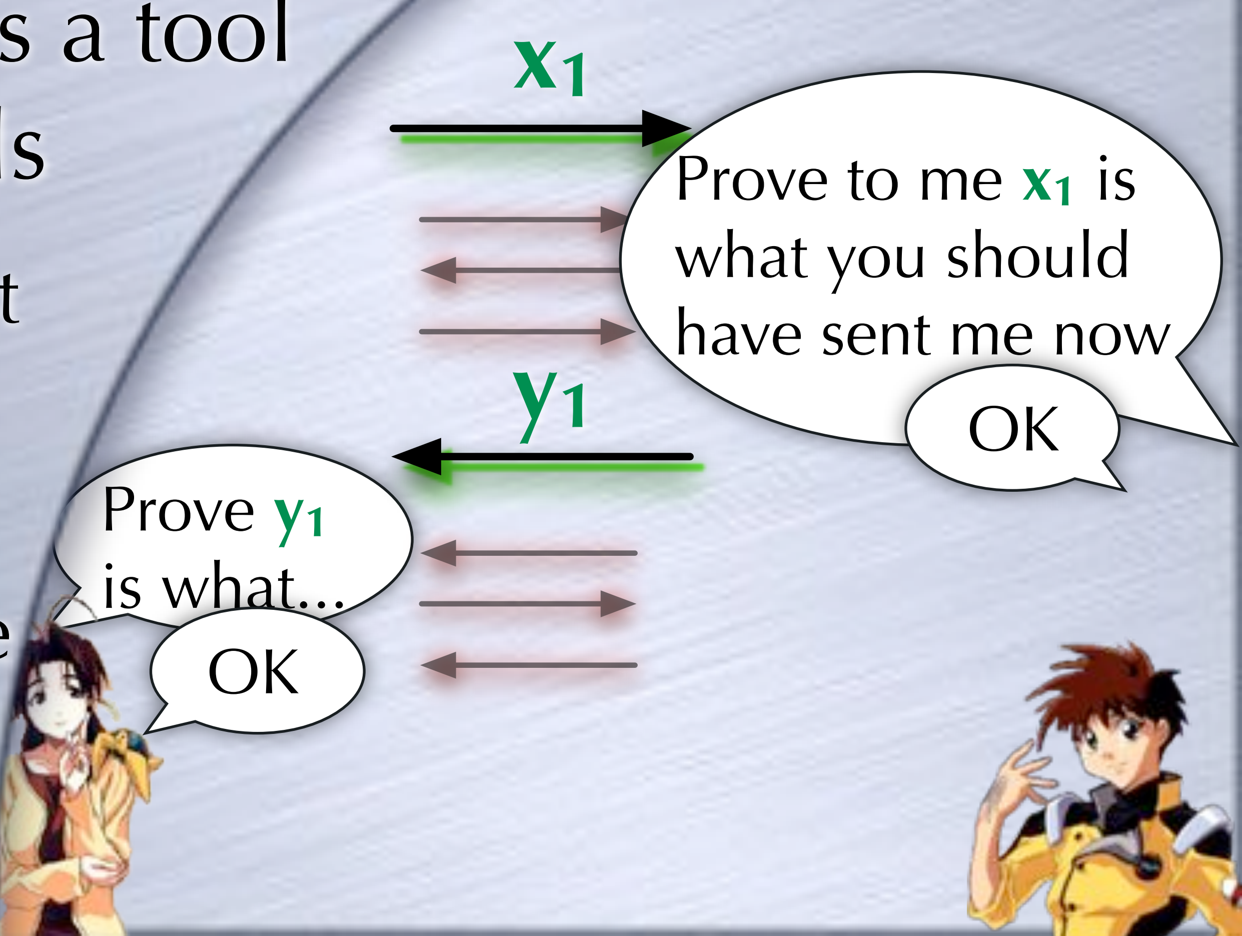

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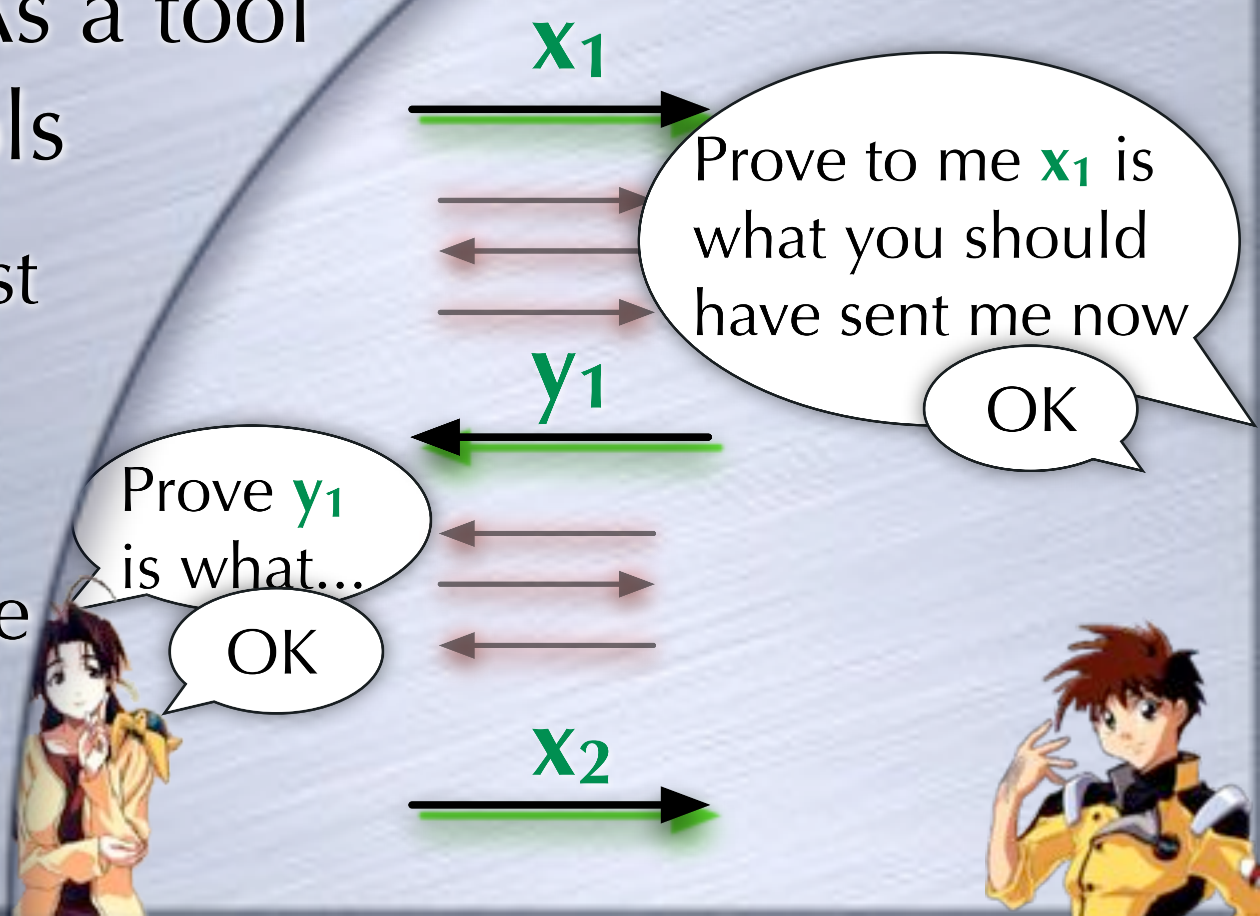

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