Hash Functions

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Lecture 10

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Before we talk about digital signatures...

A Tale of Two Boxes

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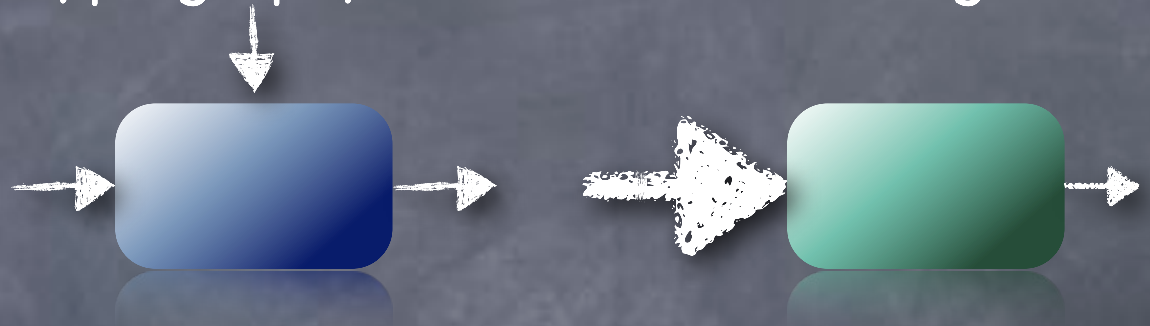

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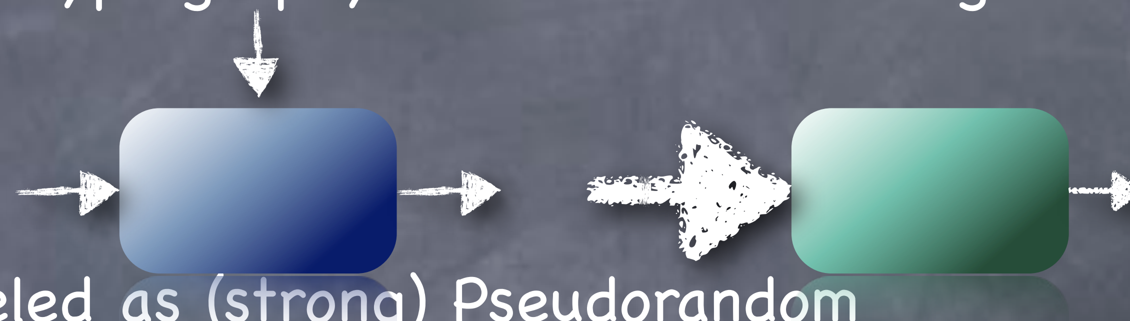

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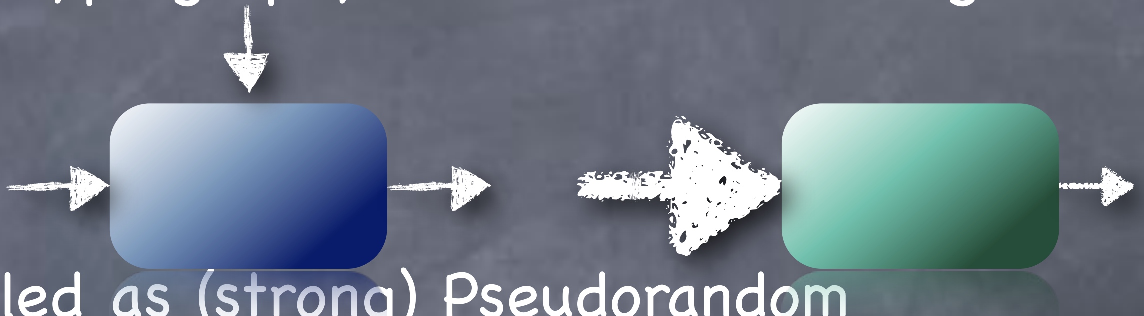

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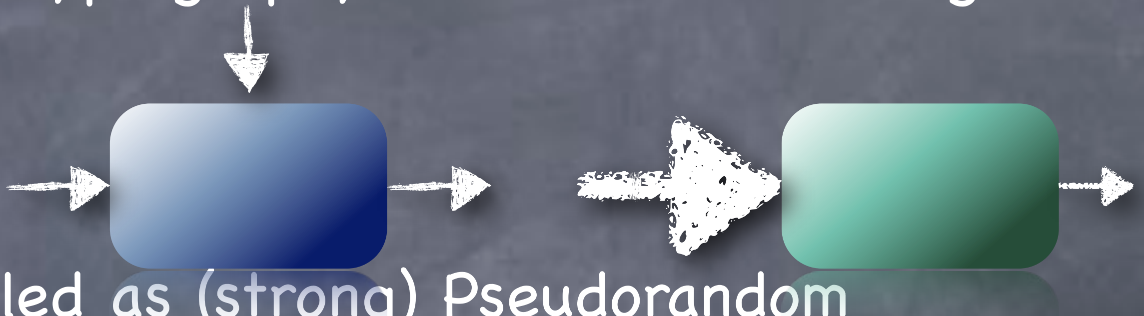

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  - Today: understanding security requirements on hash functions

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  - Typical security requirement: “collision resistance”
  - Also sometimes: some kind of unpredictability

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  - Main goal: to "avoid collisions". Will see several variants of the problem

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- Not collision-resistant under any of the following definitions
- Alternately, could be considered as have already been randomly chosen from a family (and security parameter fixed too)
  - Usually involves hand-picked values (e.g. "I.V." or "round constants") built into the standard

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- CRHF the strongest; UOWHF still powerful (will be enough for digital signatures)

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- CRHF implies second pre-image collision resistance and, if sufficiently compressing, then pre-image collision resistance [Exercise]

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  - Birthday attack effectively halves the hash length (say security parameter) over "naïve attack"

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  - ∀x≠y,w,z $Pr_{h\leftarrow\mathcal{H}}$ [h(x)=w, h(y)=z] = $1/|Z|^2$
    - ∀x≠y $Pr_{h\leftarrow\mathcal{H}}$ [h(x)=h(y)] = 1/|Z|

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- k-Universal:
  - $\forall x_1..x_k$ (distinct), $z_1..z_k$, $\Pr_{h\leftarrow\mathcal{H}}[\forall i\ h(x_i)=z_i\ ] = 1/|Z|^k$

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    - $\Pr_h$ [Chop(h(x)) = w, Chop(h(y)) = z]
      $= \Pr_h$ [h(x) = w0 or w1, h(y) = z0 or z1] $= 4/|Z|^2 = 1/|Z'|^2$

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    - Will see shortly, how to extend the domain to arbitrarily long strings (without increasing output size)

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