Homomorphic Encryption

Lecture 18

And some applications

Homomorphic Encryption

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- Not covered today: Fully Homomorphic Encryption, which supports ring homomorphism (addition <u>and</u> multiplication of messages)

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  - Rerandomization useful even without homomorphism

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- Functionality gives "handles" to messages posted; accepts requests for posting fresh messages, or derived messages
- Unlinkability: Above, receiver gets only the message m_1+m_2 in IDEAL; is not told if it is a fresh message or derived from other messages

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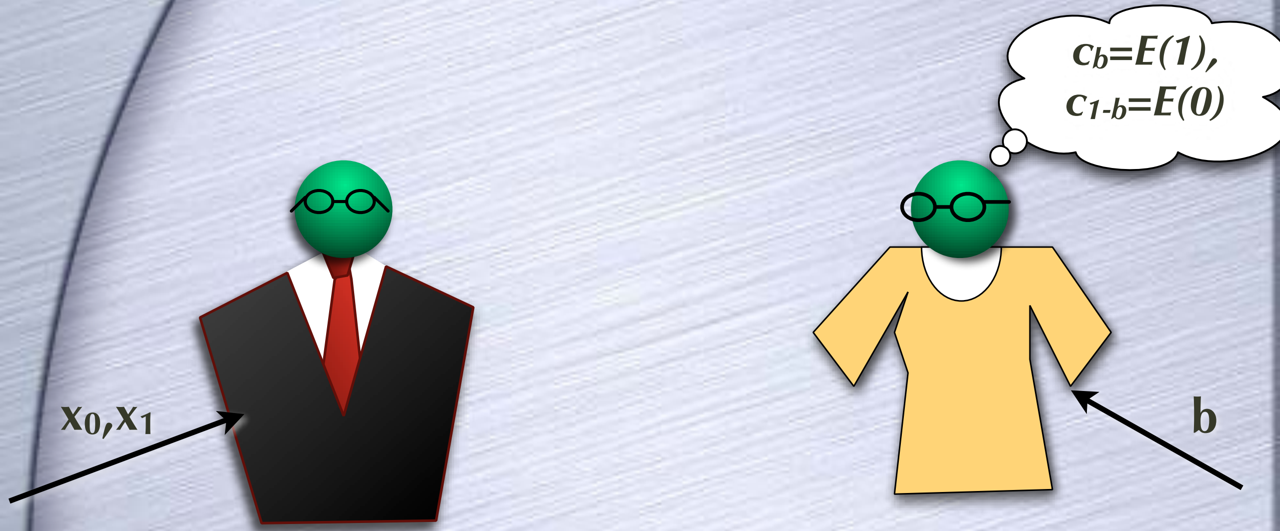

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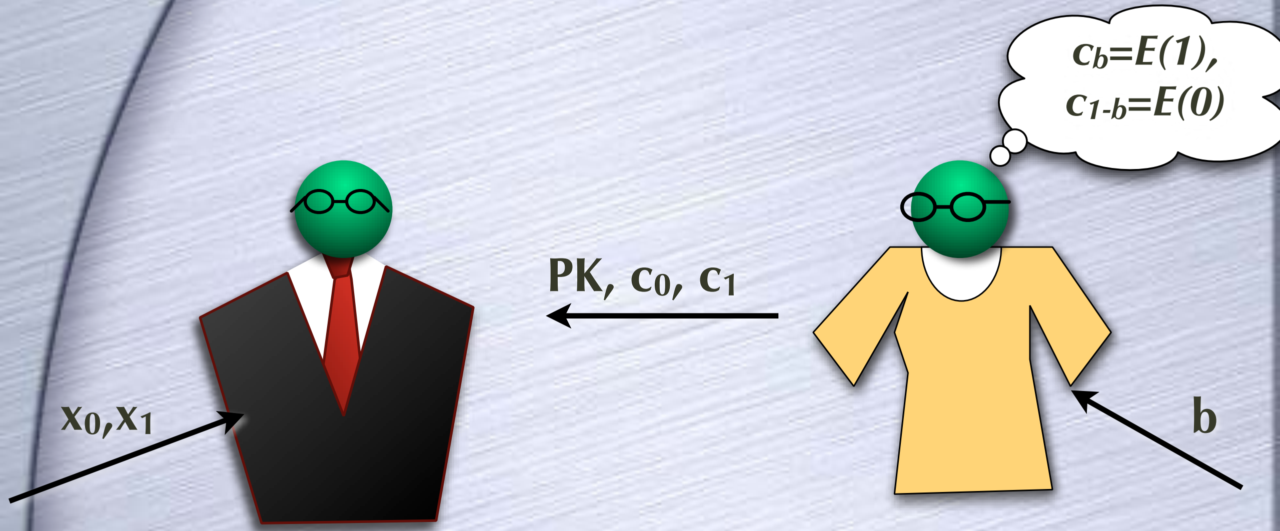

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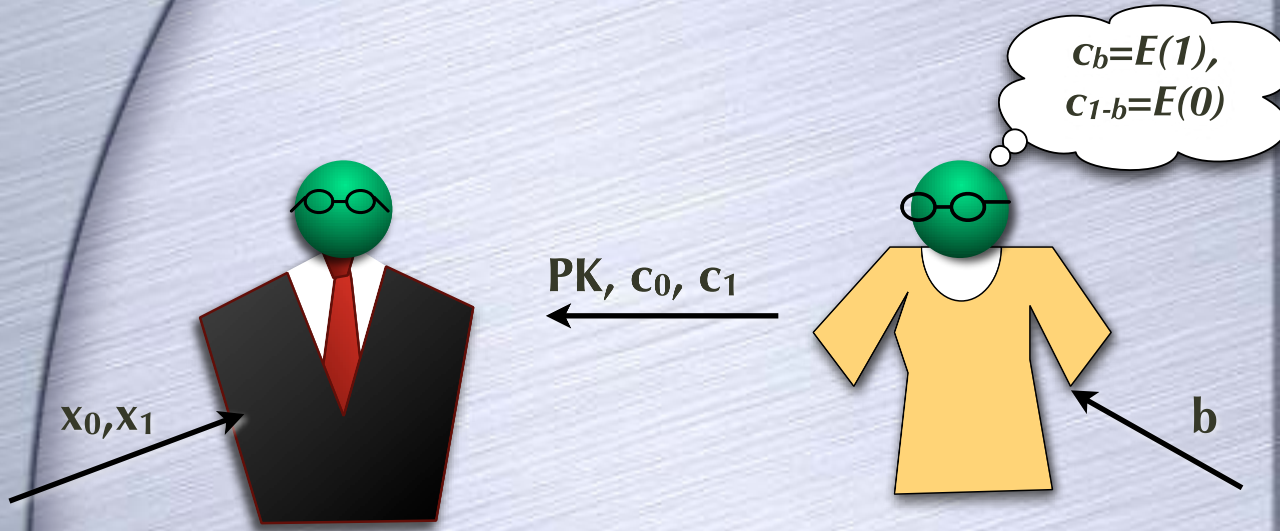

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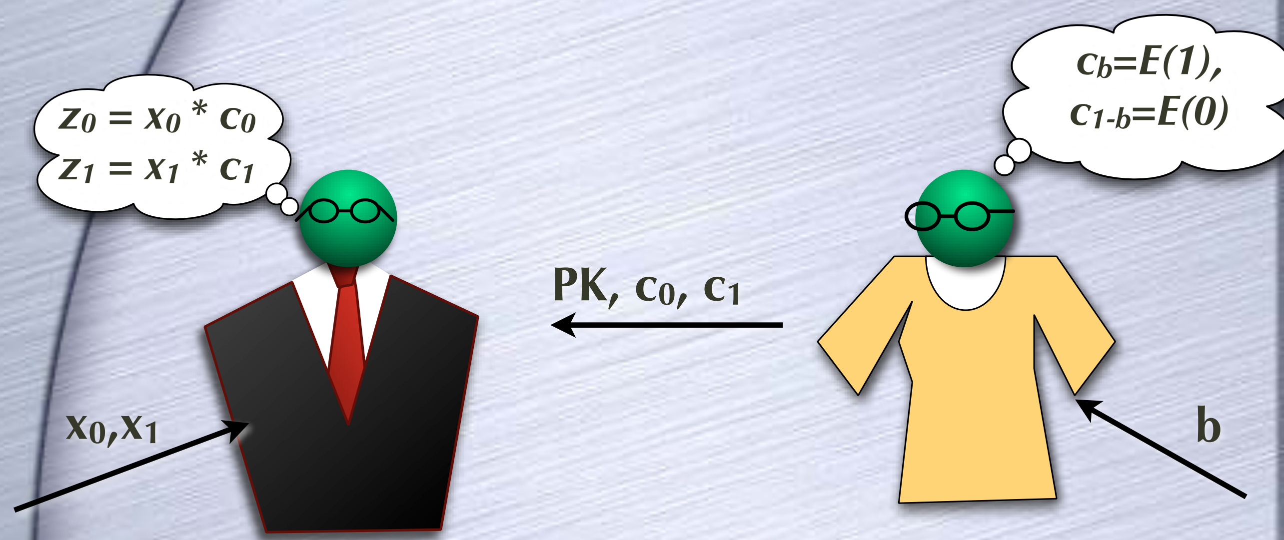

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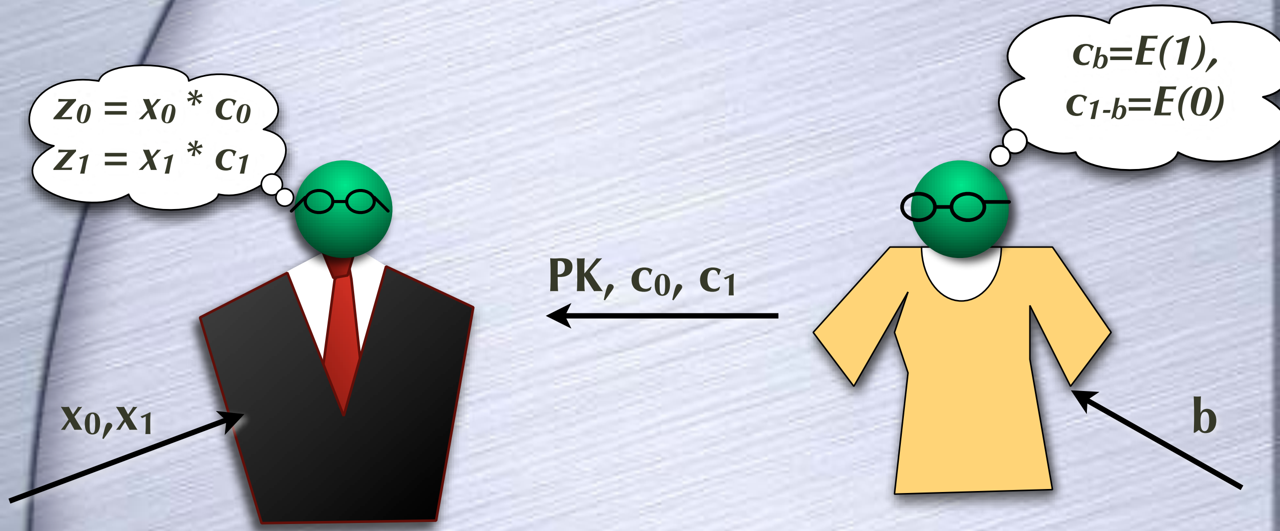

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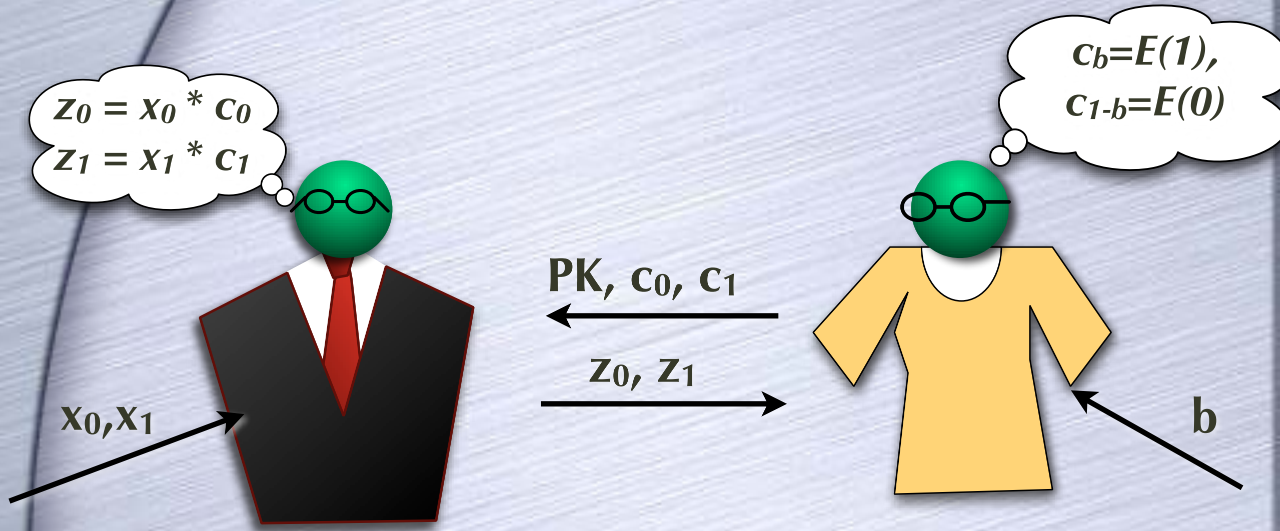

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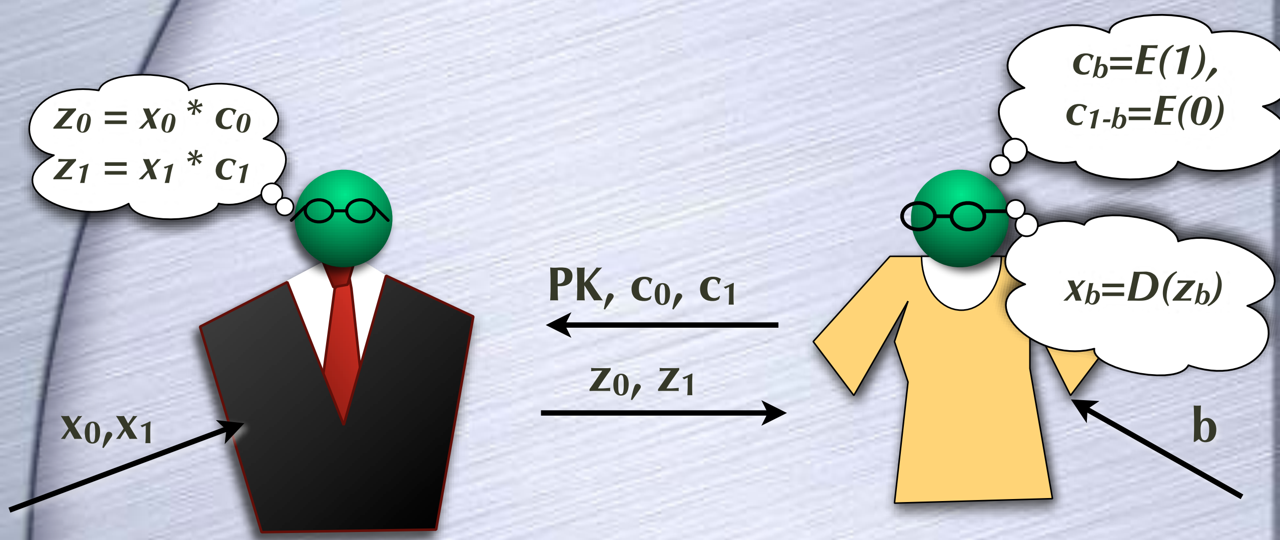

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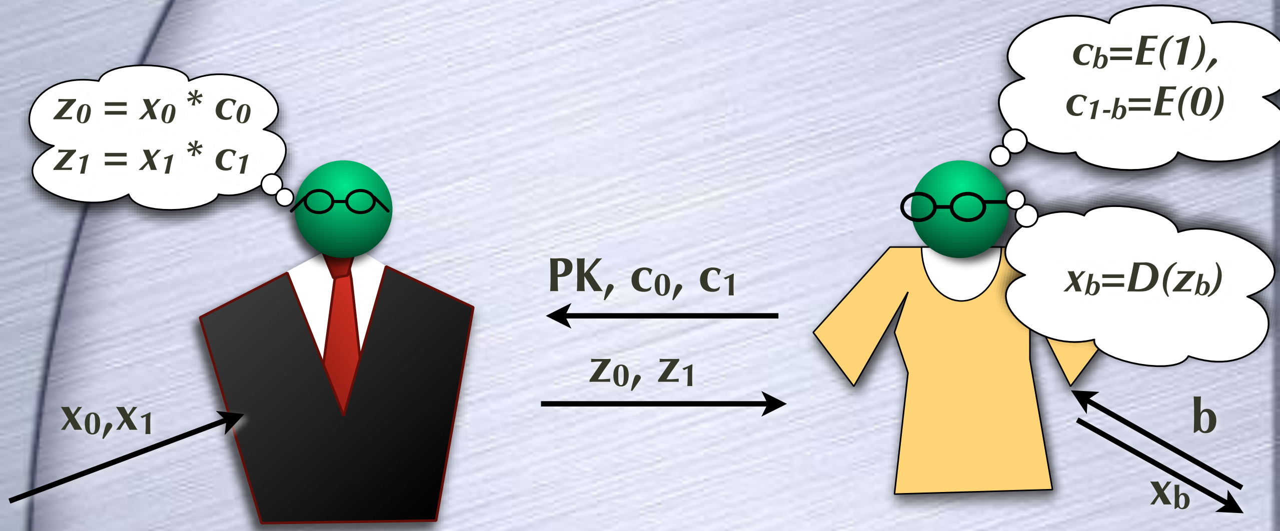

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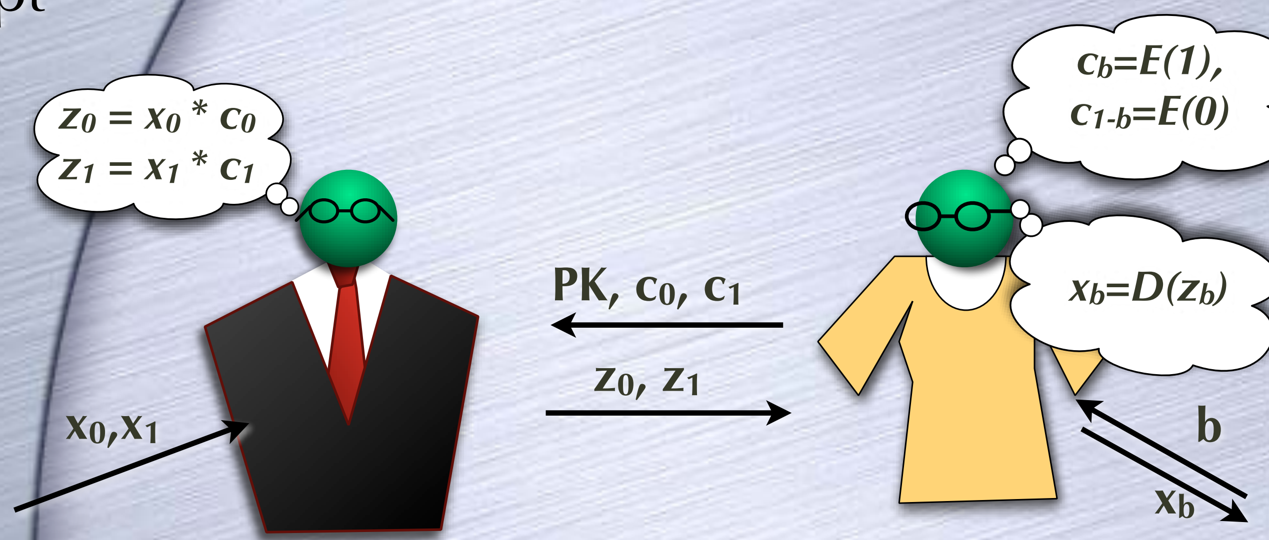

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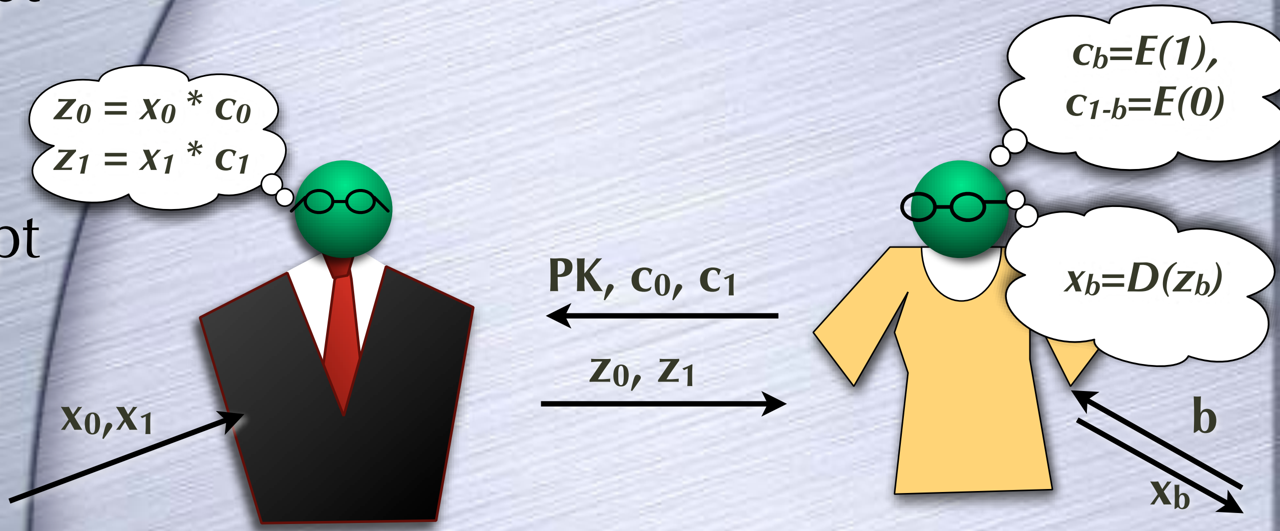
- Simulation for passive-corrupt sender: Extract x_0,x_1 from input; set c_0,c_1 to be say E(1)



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  - When message space is $\mathbb{Z}_n$: additively homomorphic encryption

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- Unlinkability: ReRand(c) = c.Enc(0)

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For Paillier, can use exponentiation

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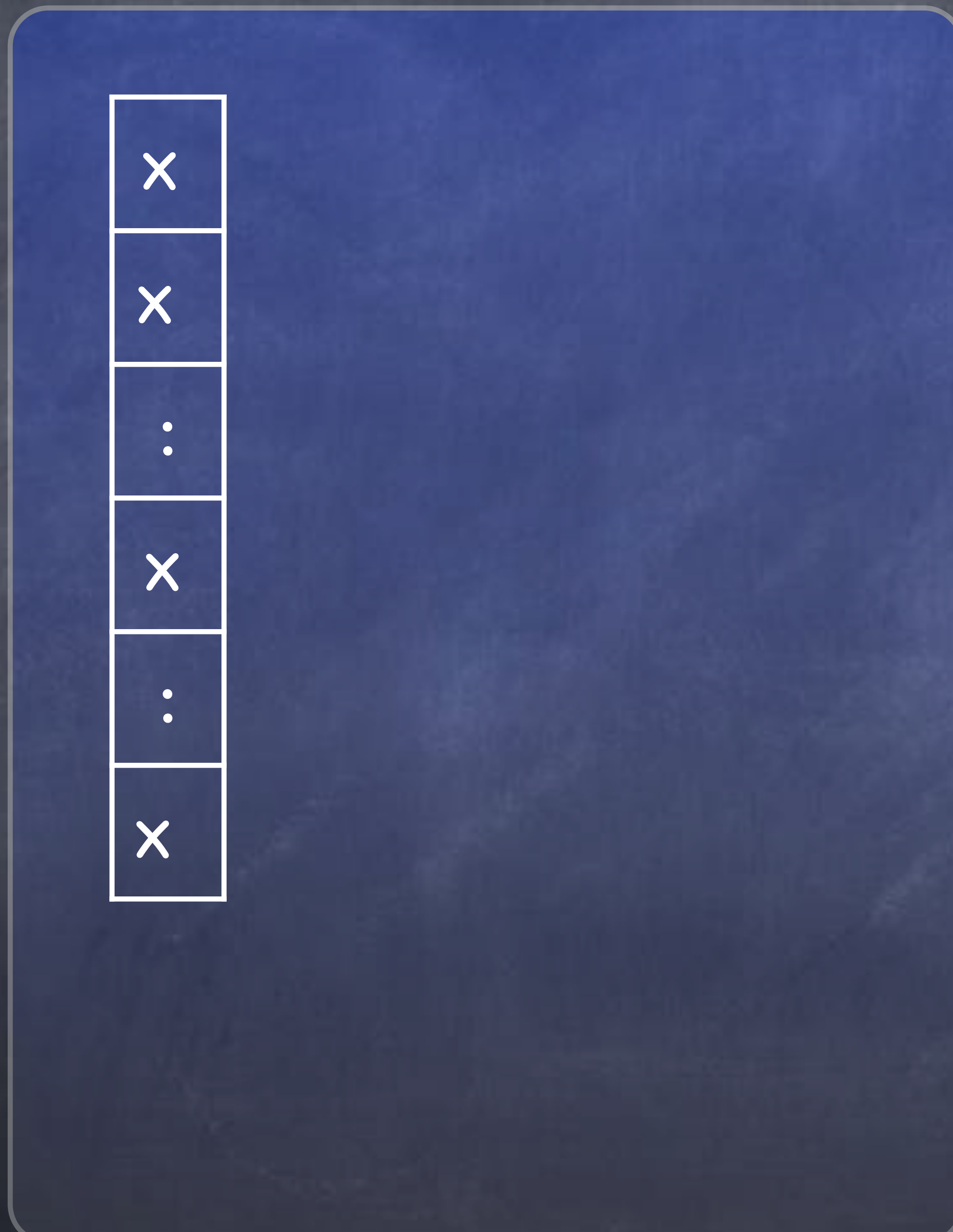
i



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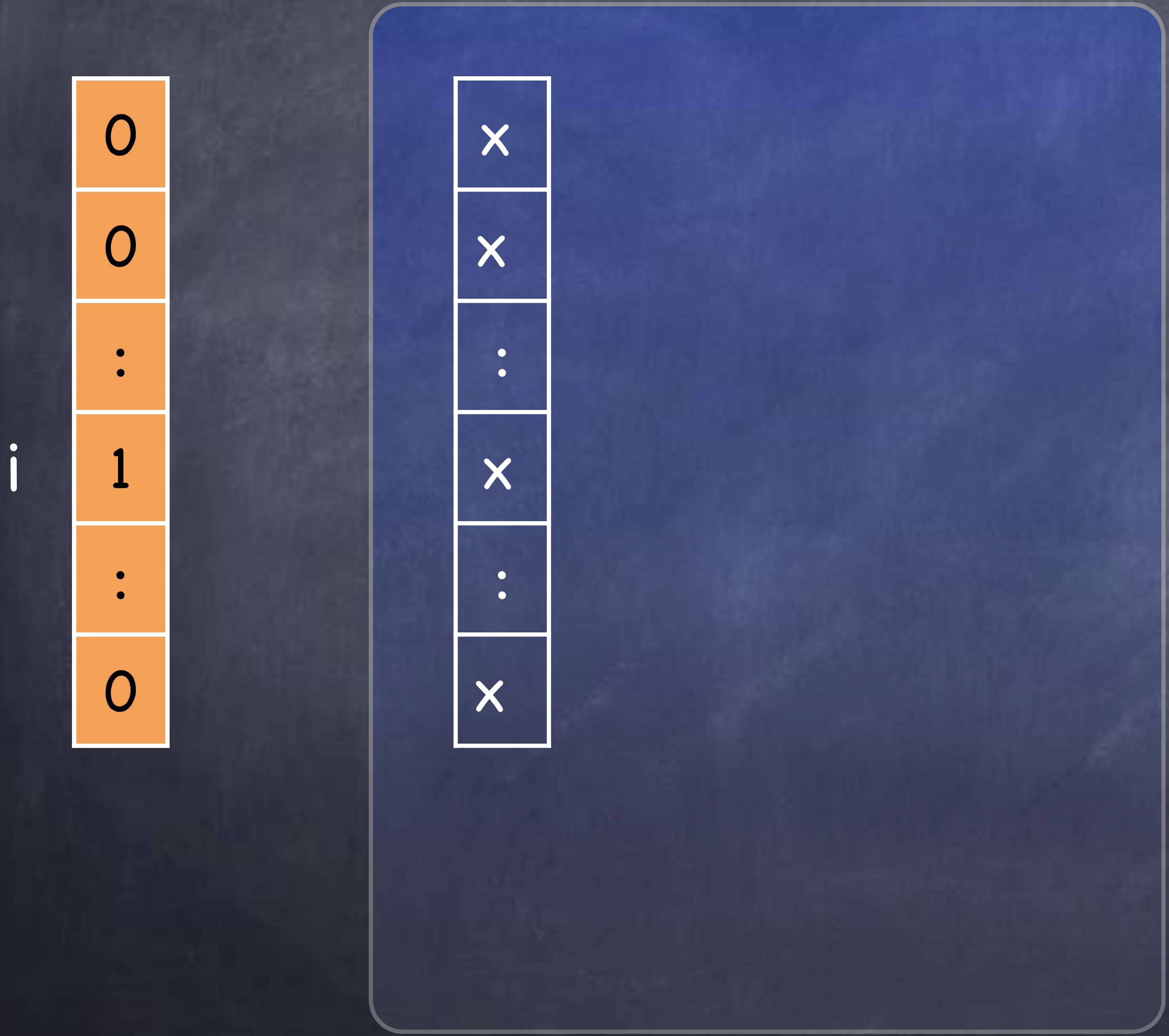

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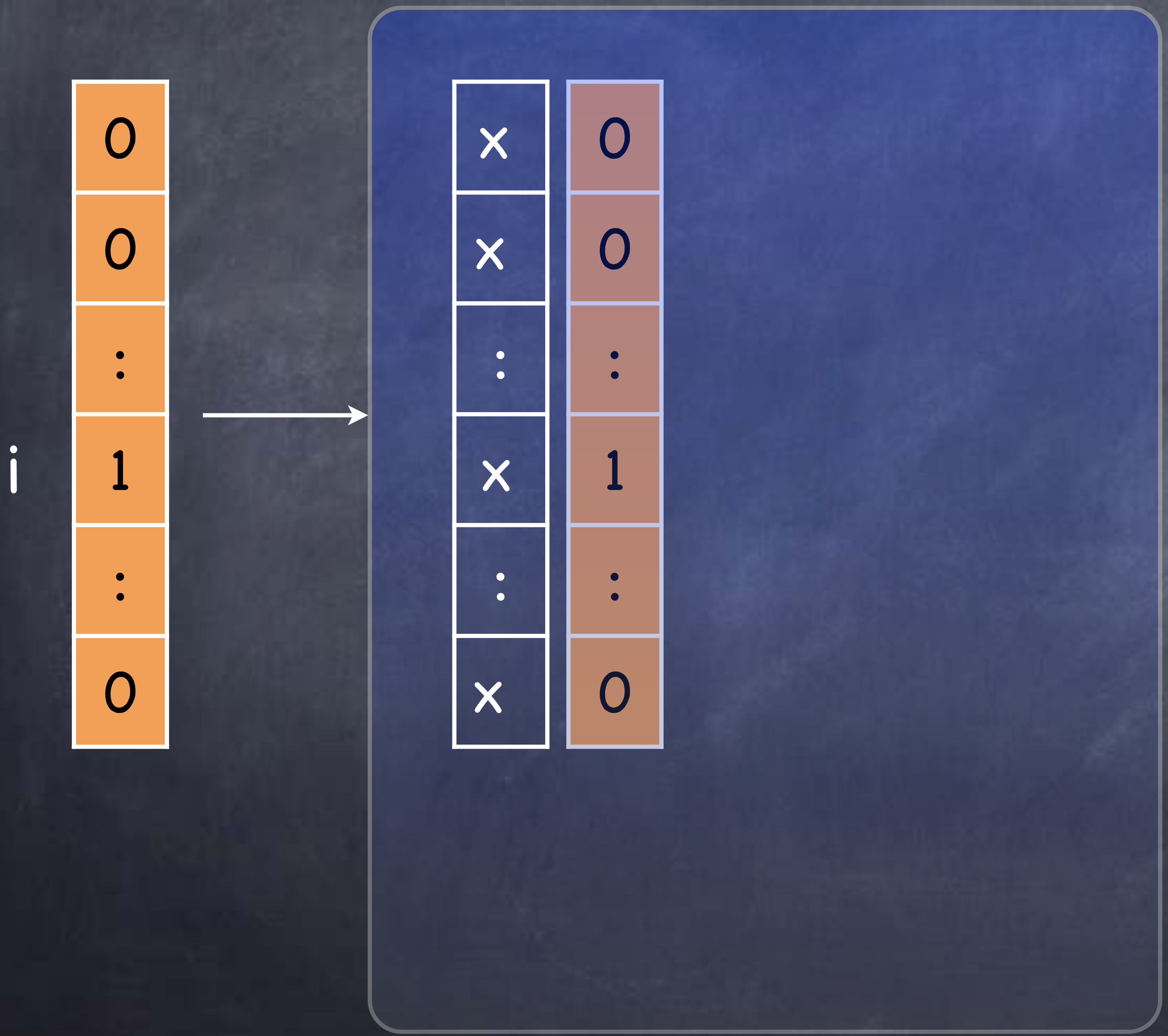

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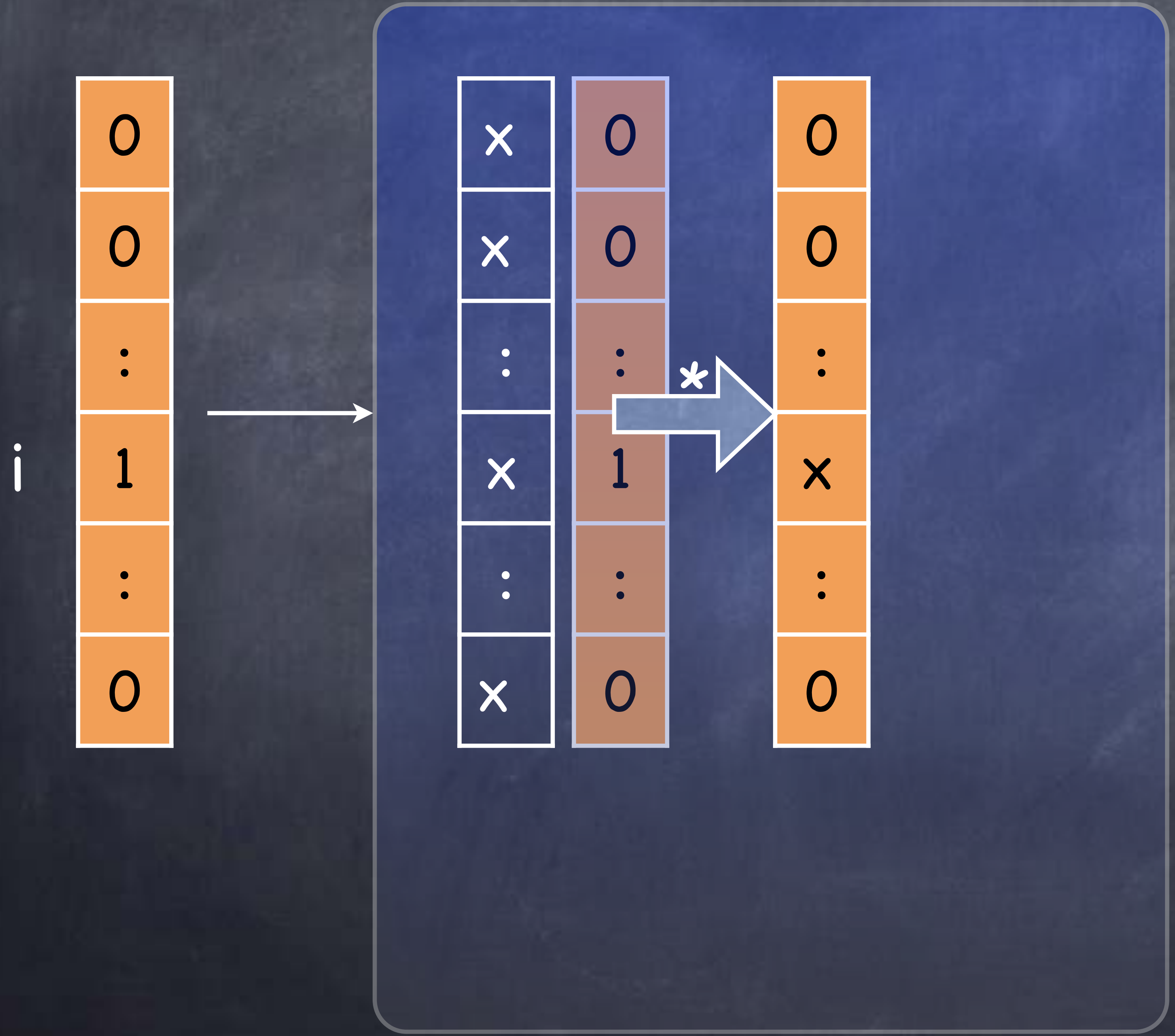

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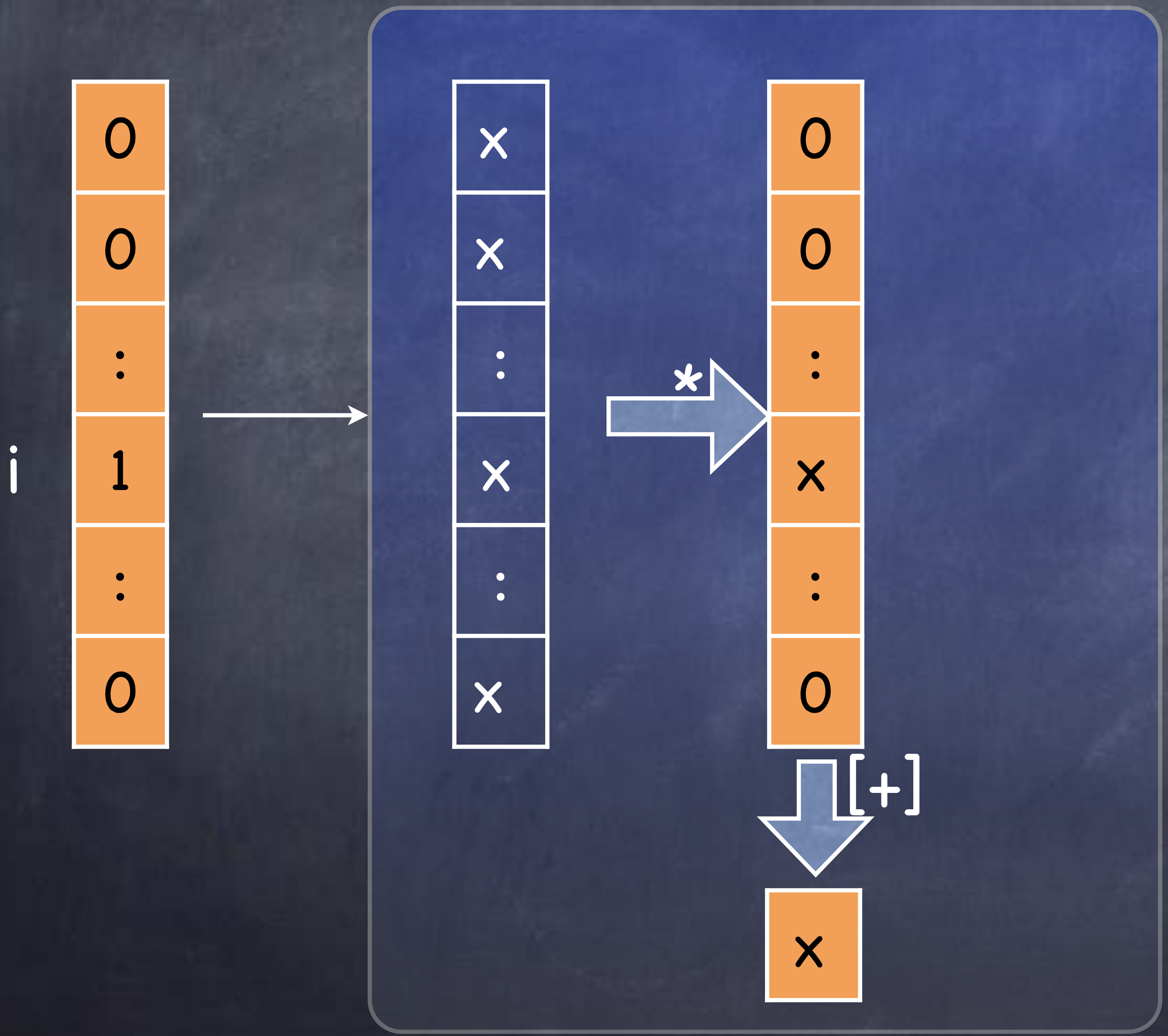

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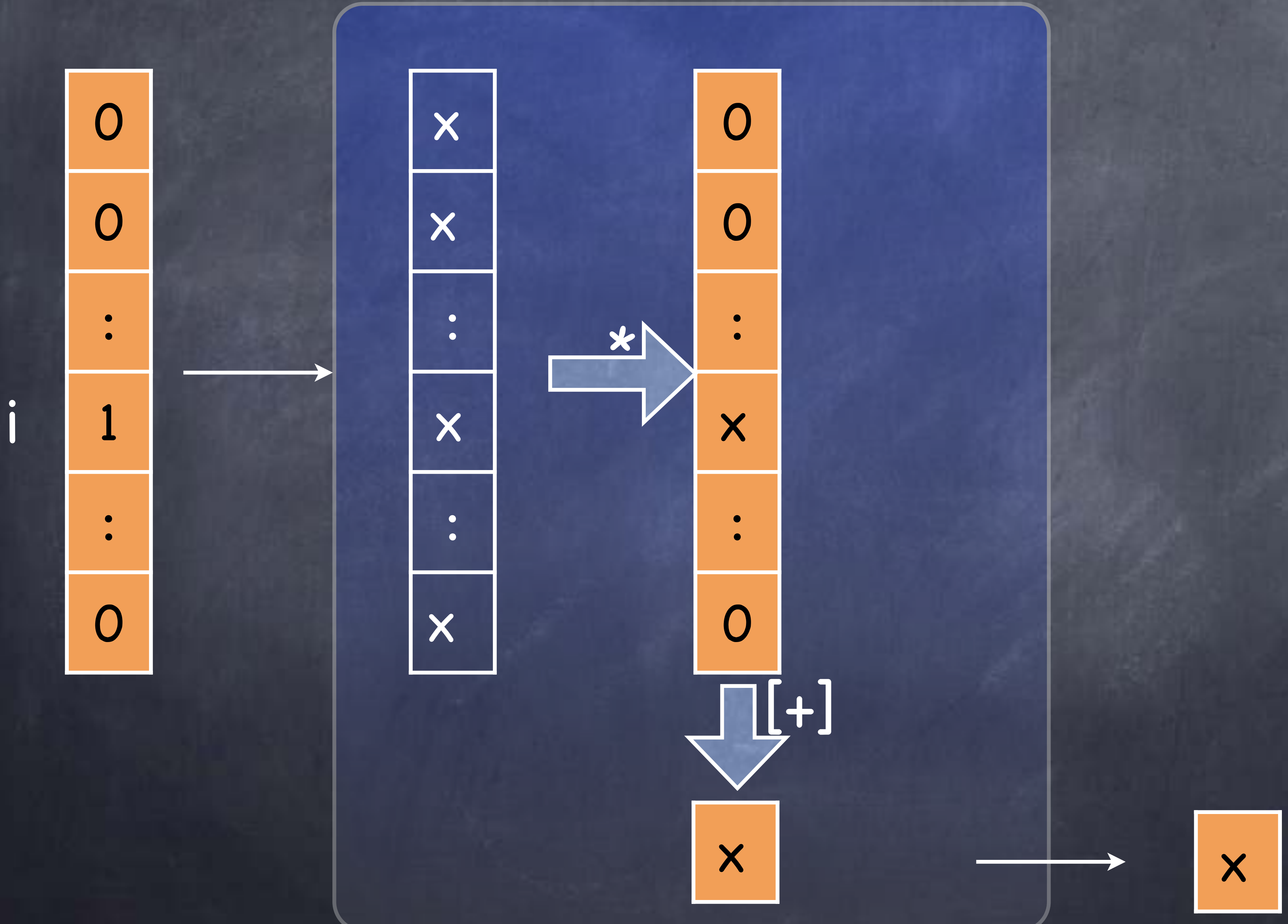

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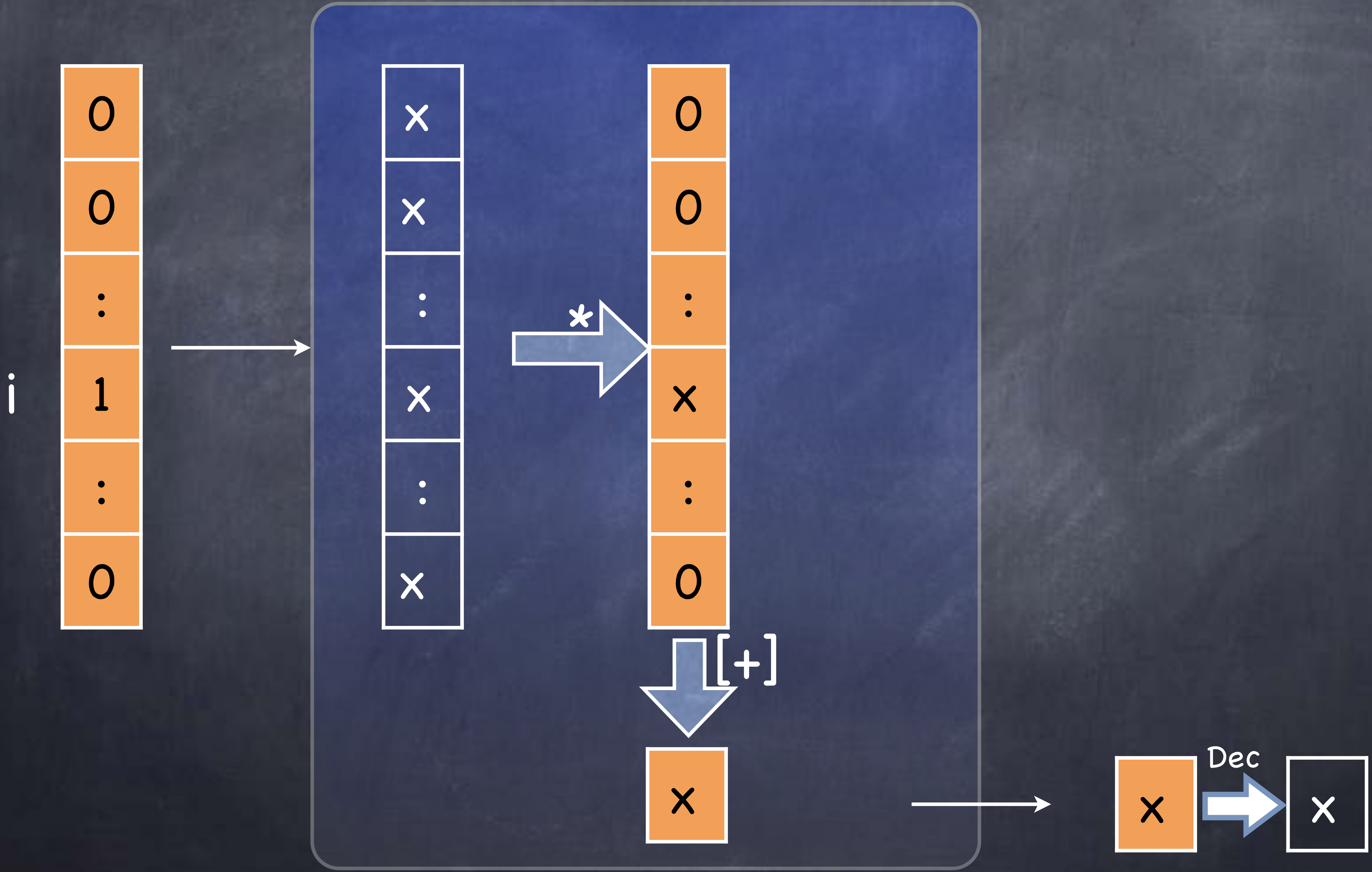

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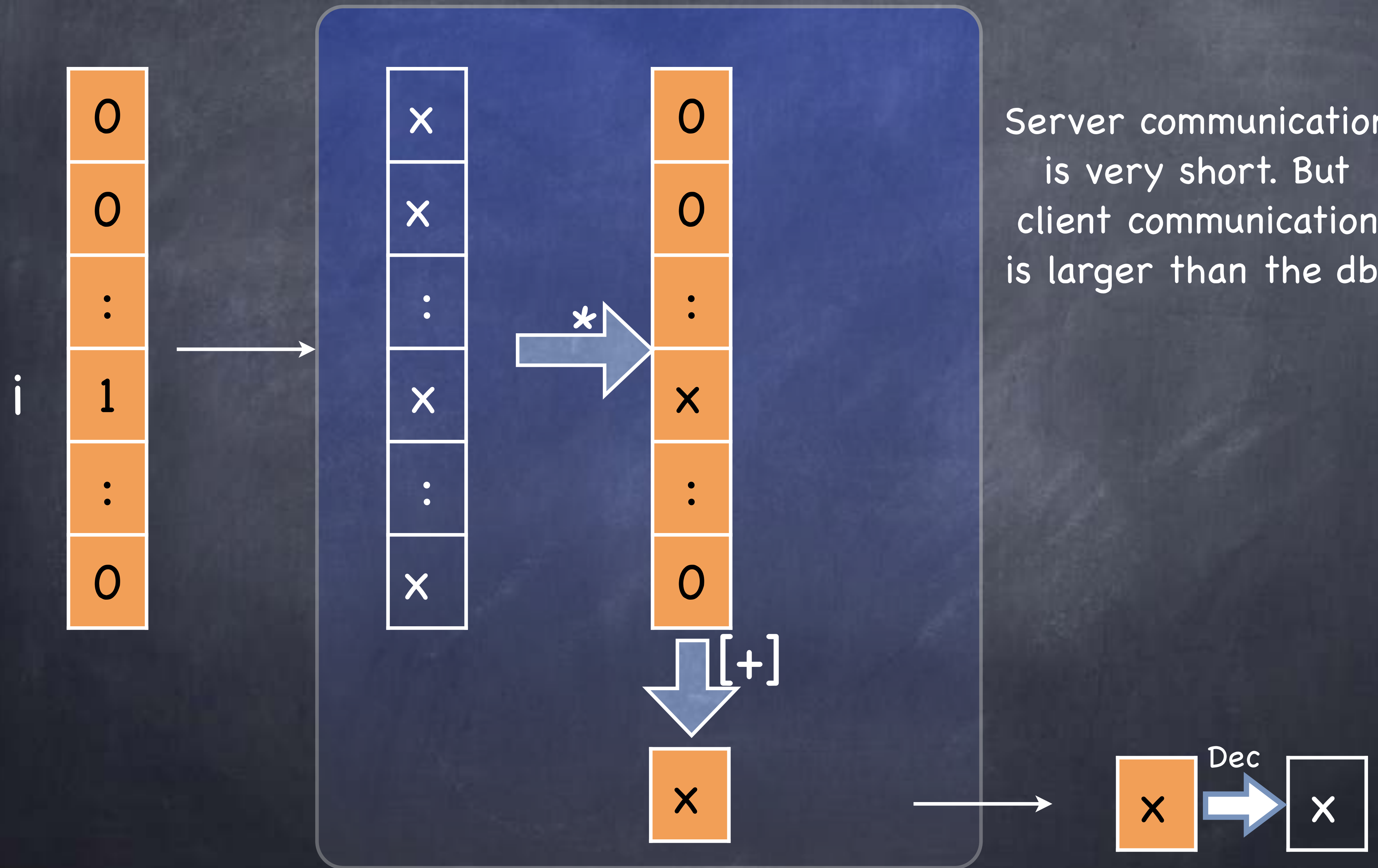

Server communication is very short. But client communication is larger than the db!

Private Information Retrieval

x					x
x					x
:					:
x			x		x
:					:
x					x

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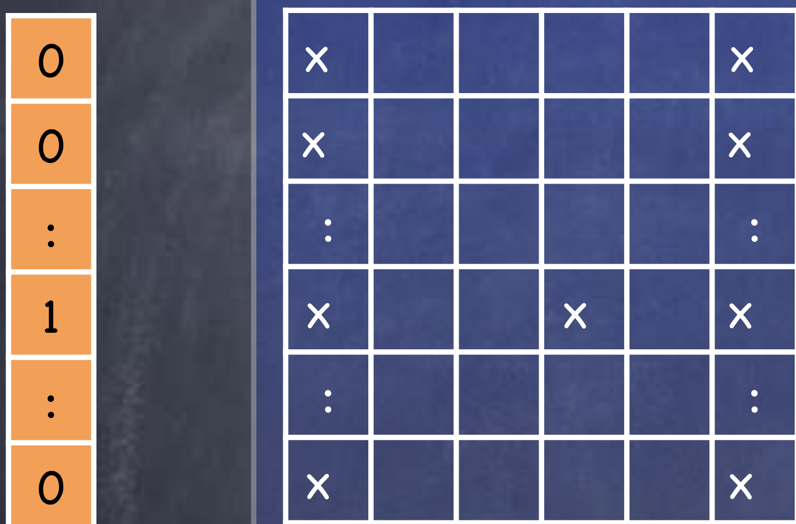

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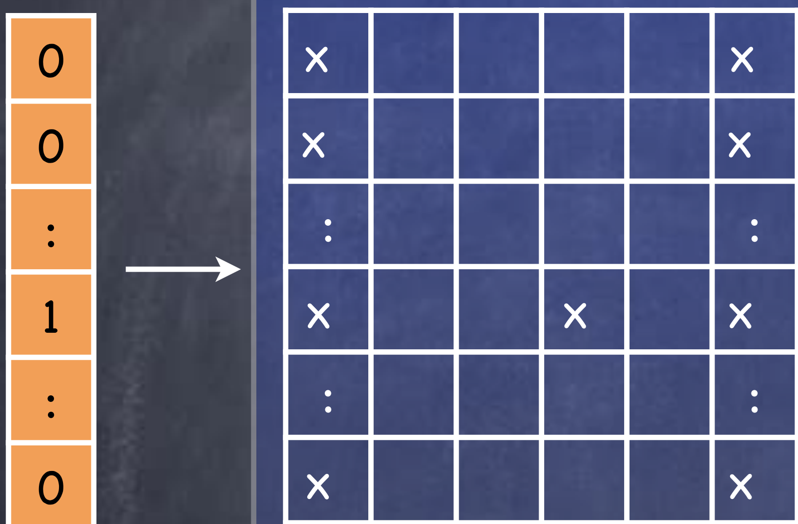

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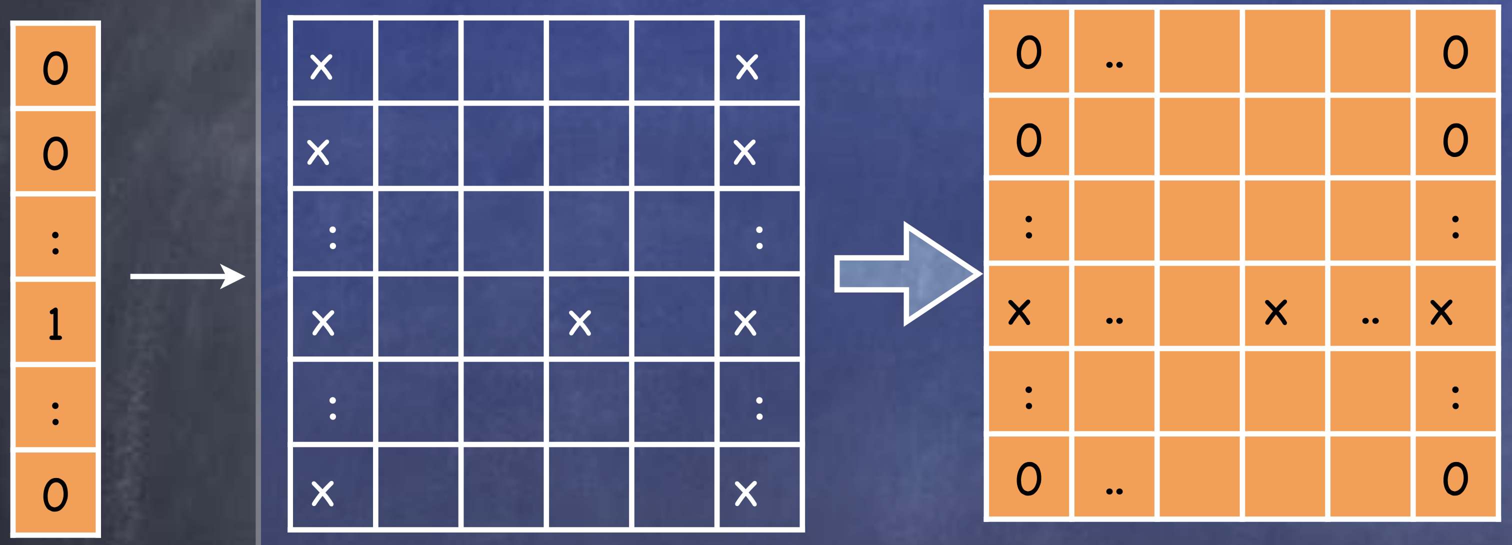

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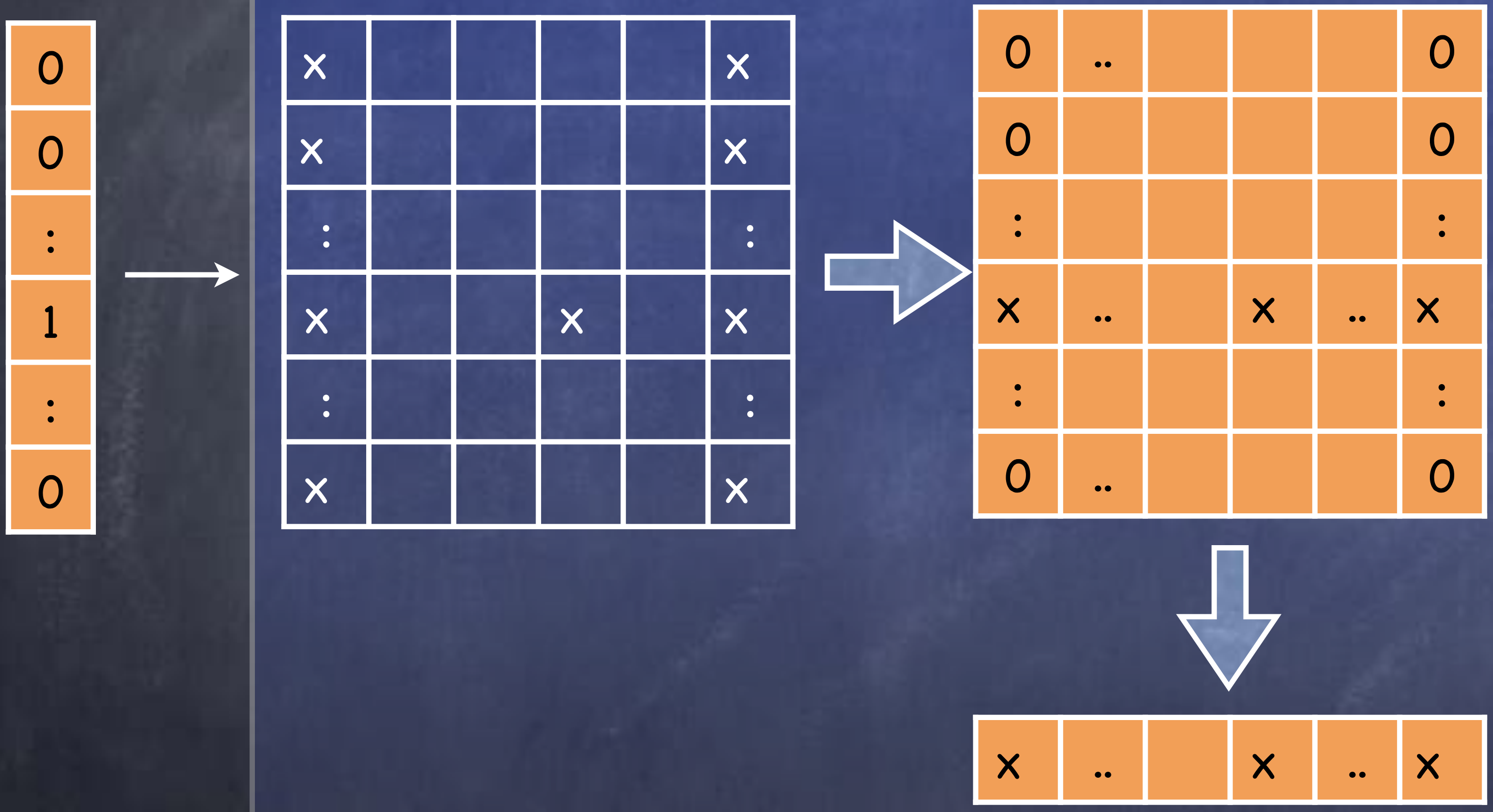

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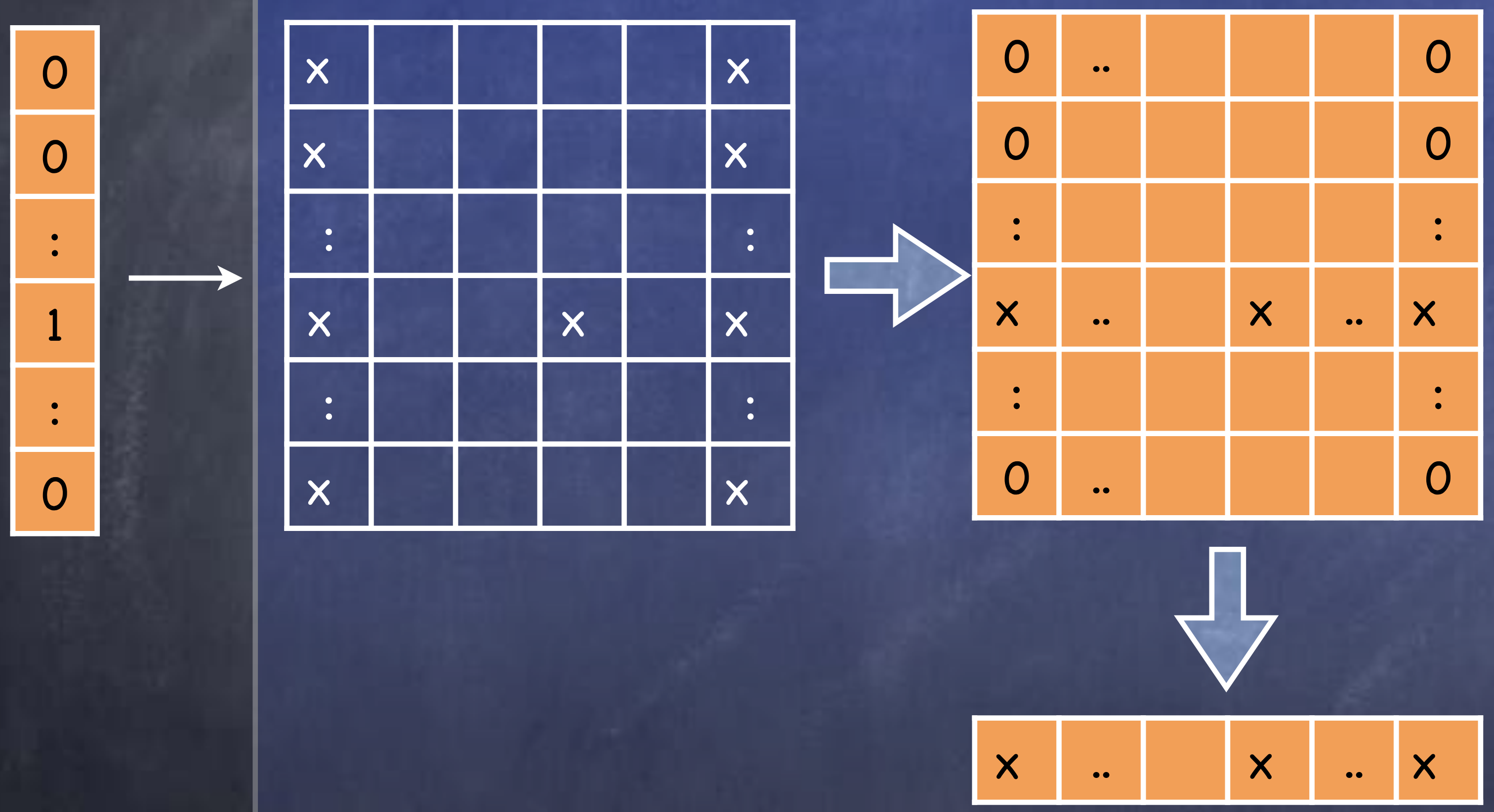

Use PIR again!

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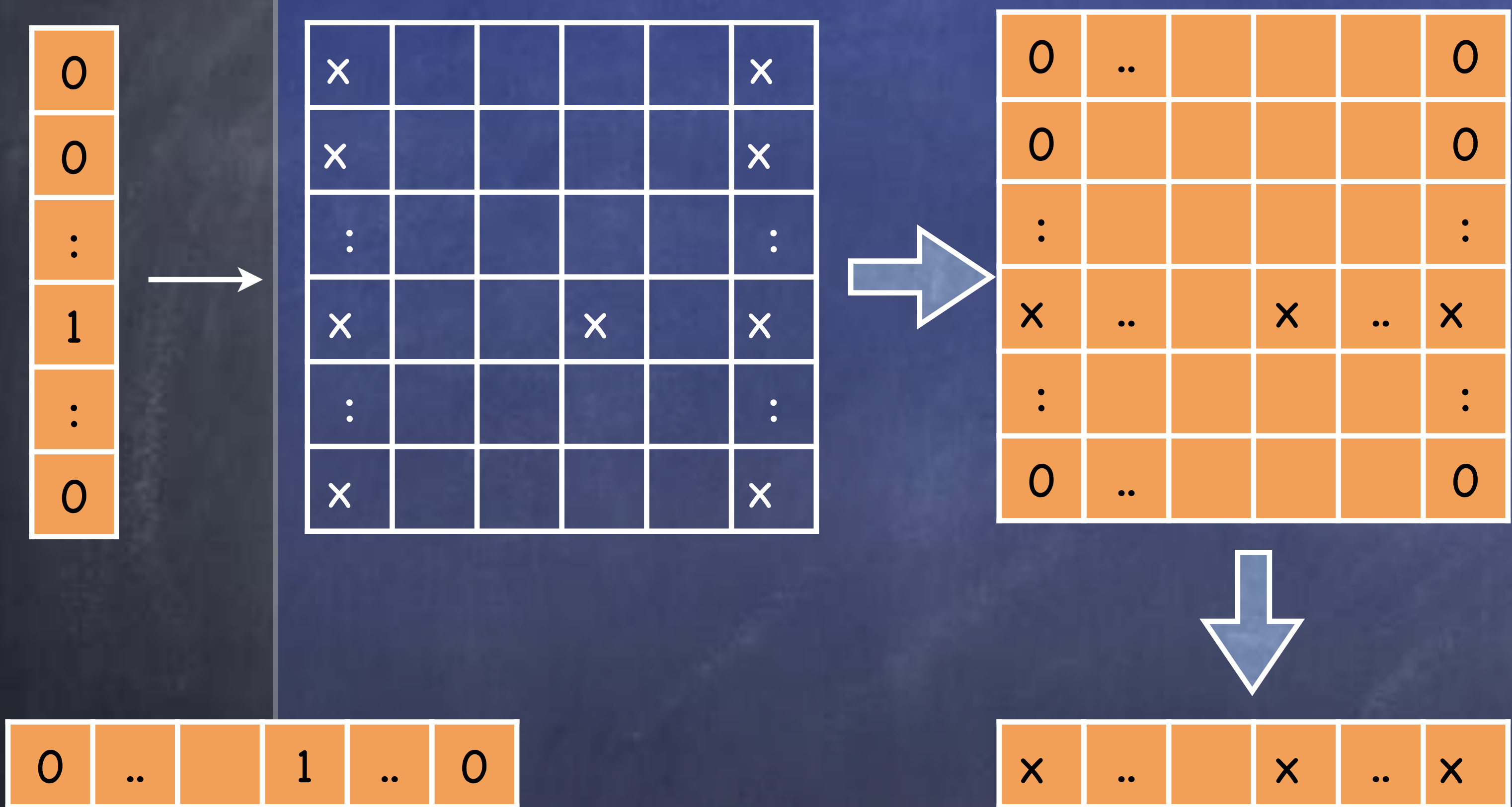

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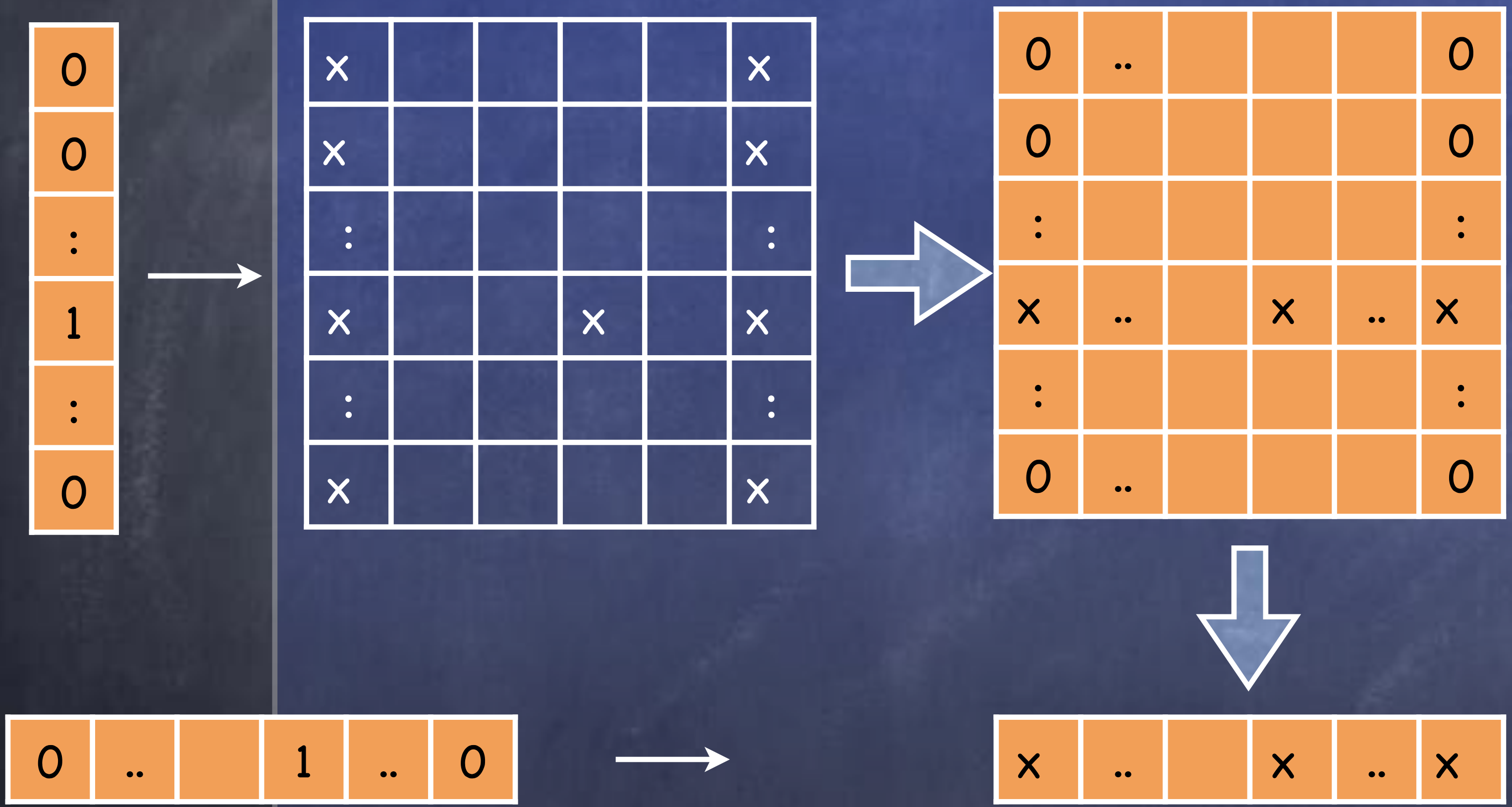

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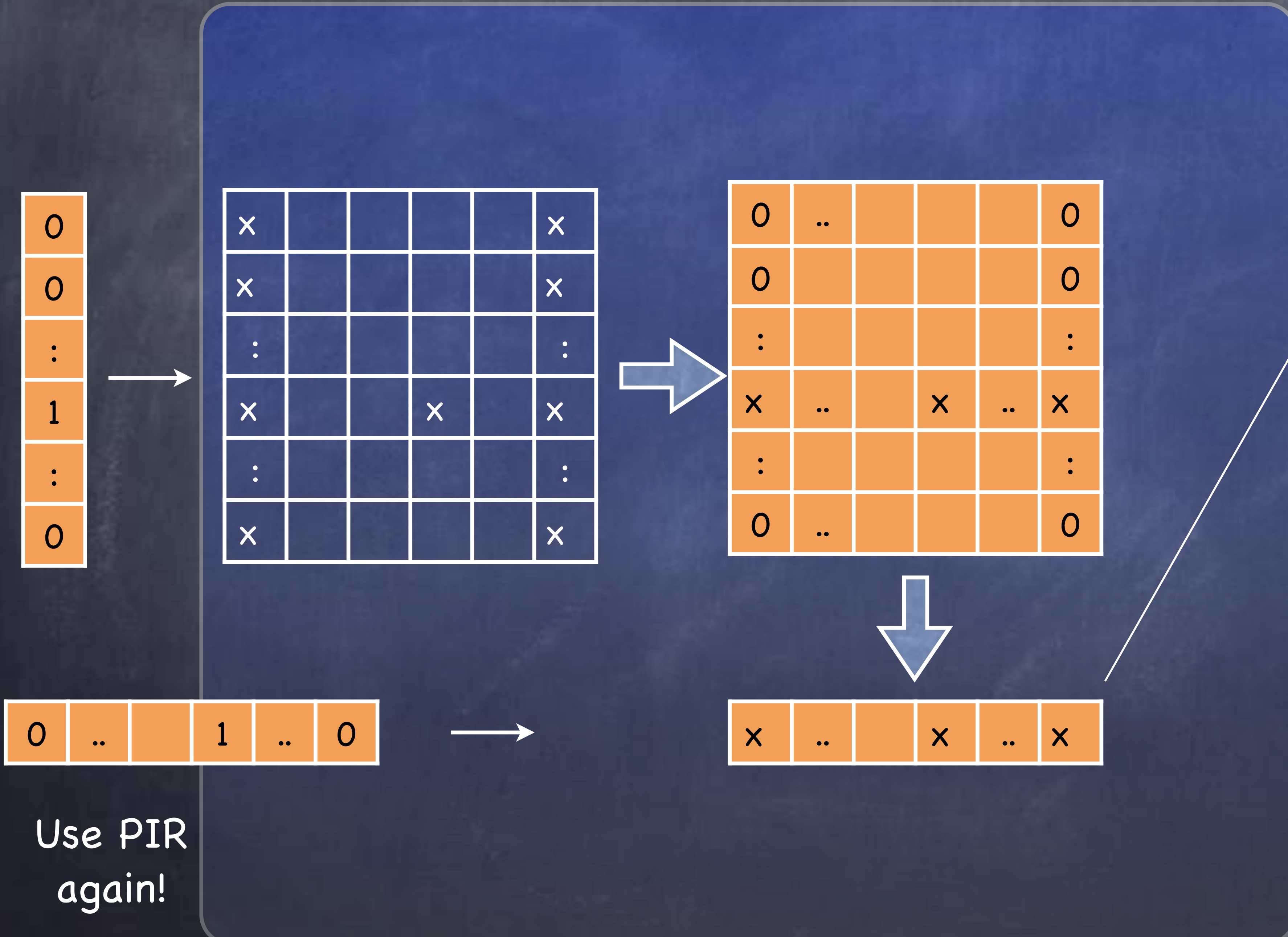

Considering ciphertext as plaintext for the sub-PIR

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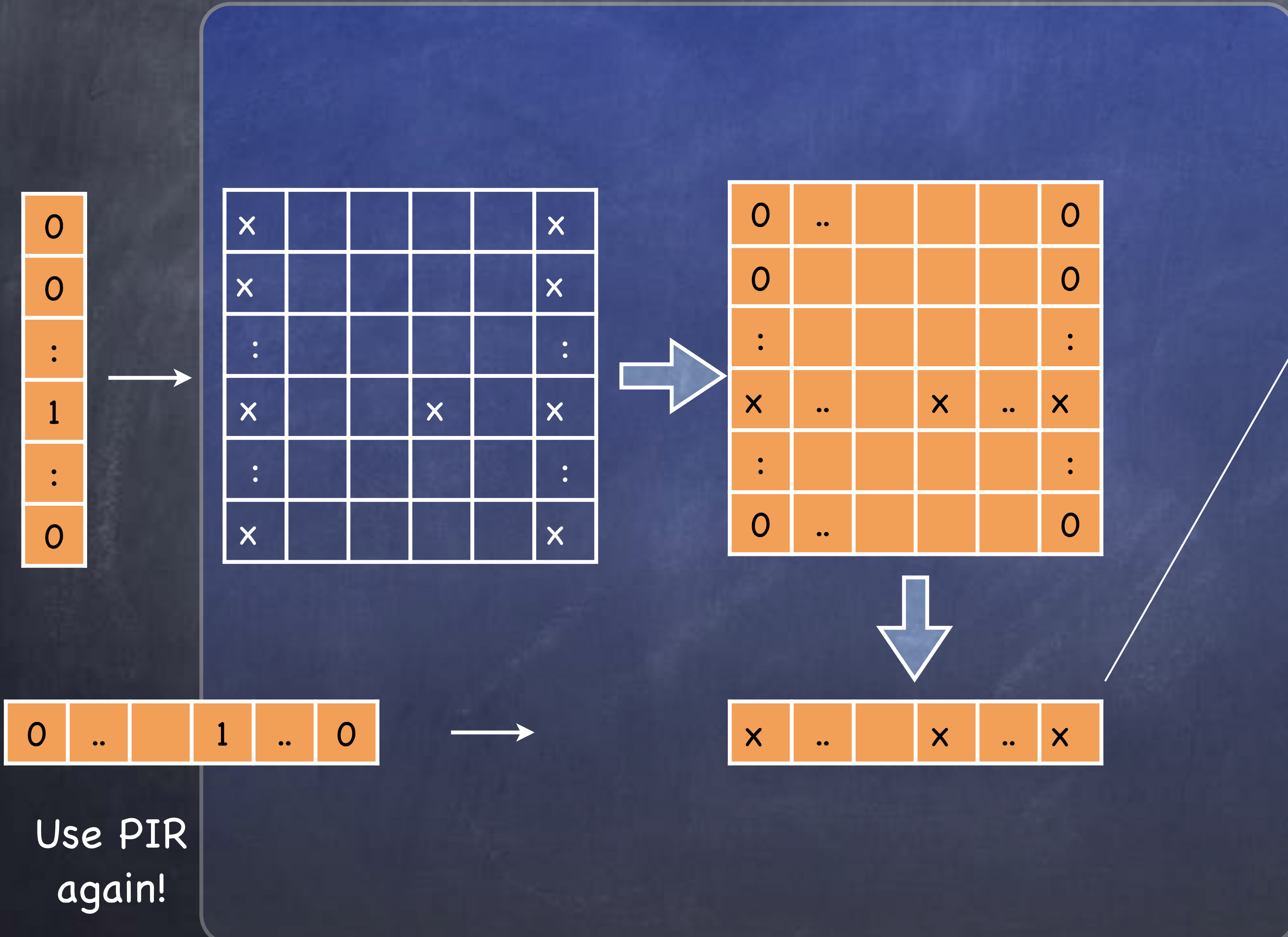

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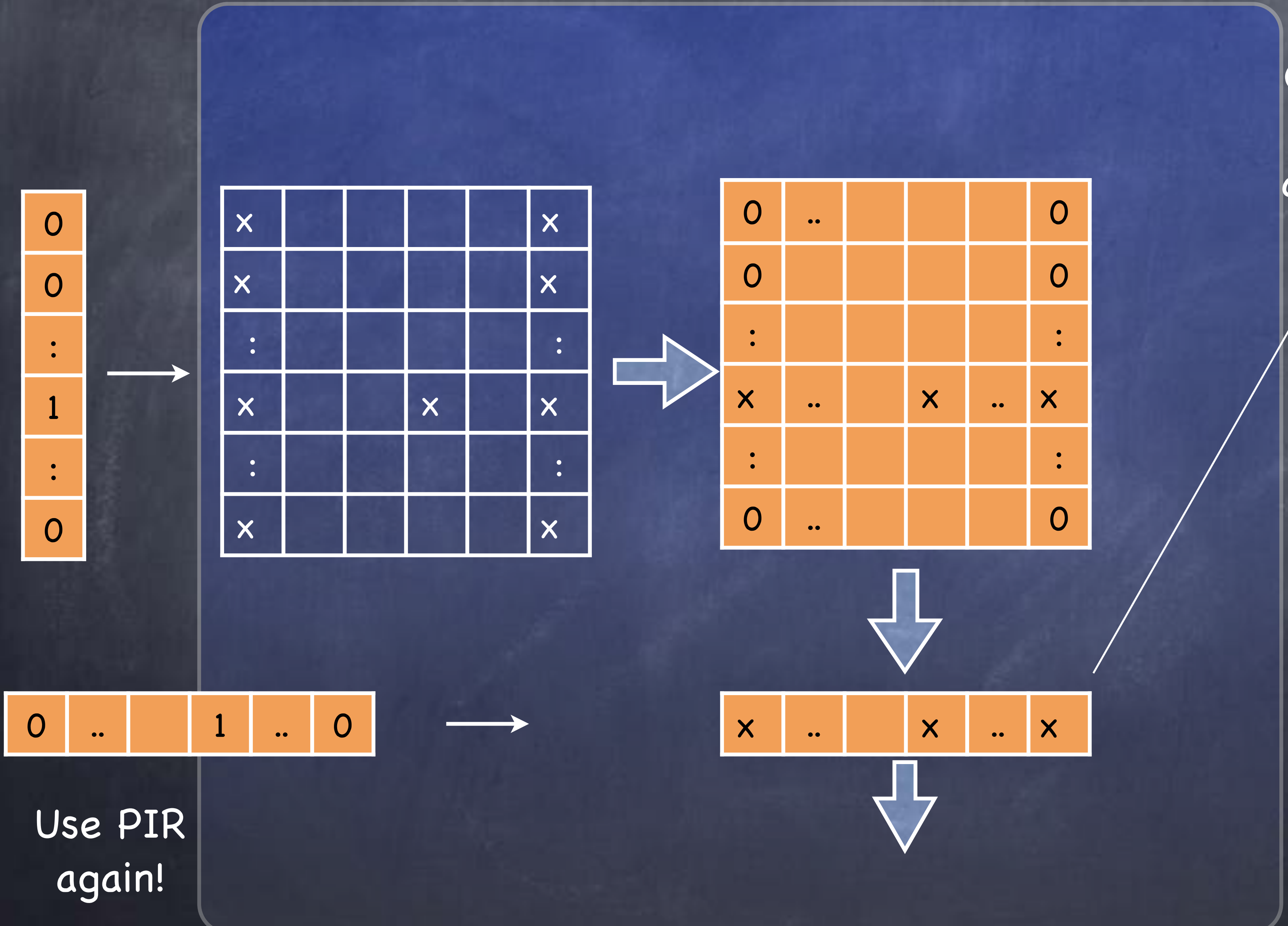

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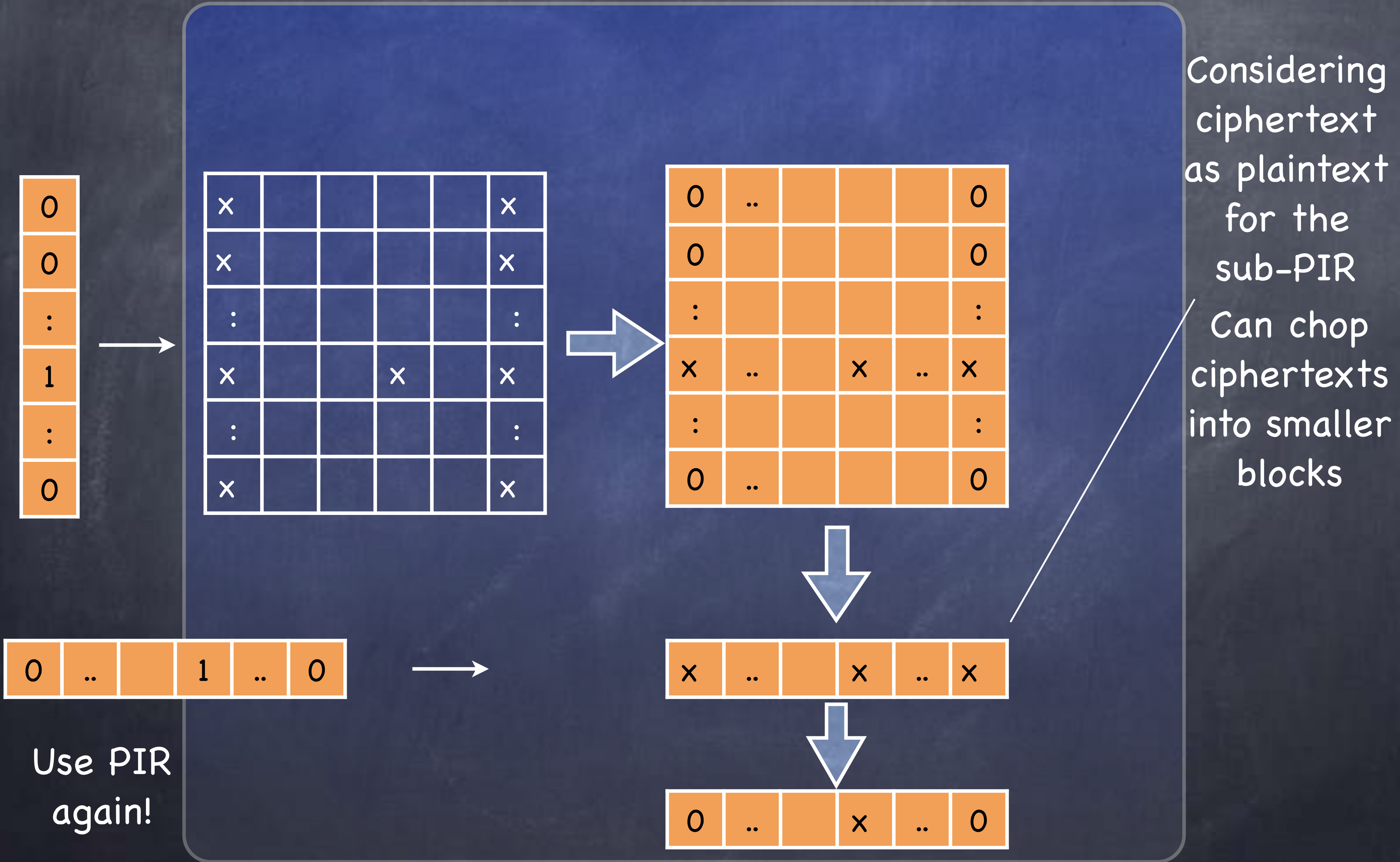

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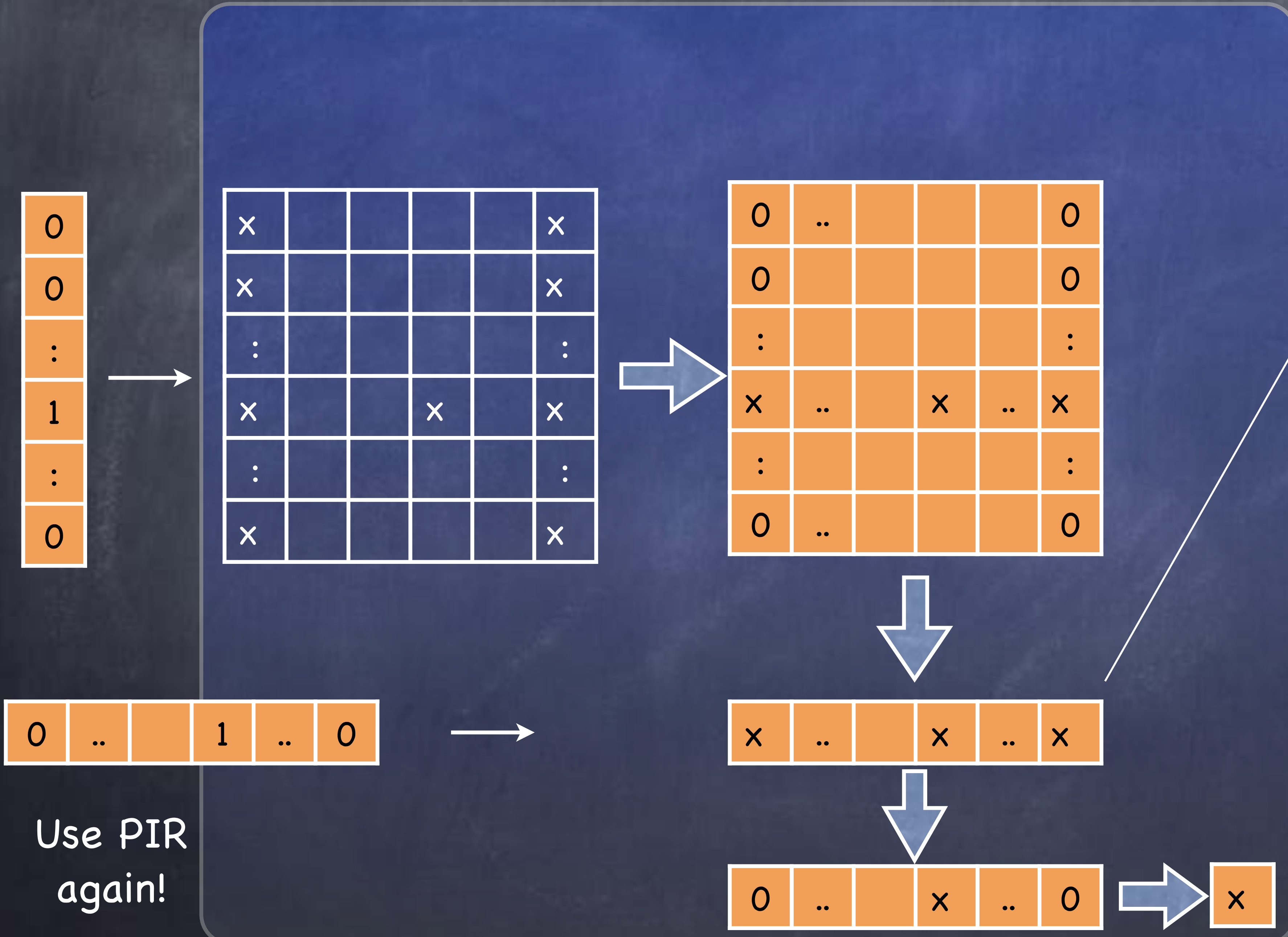

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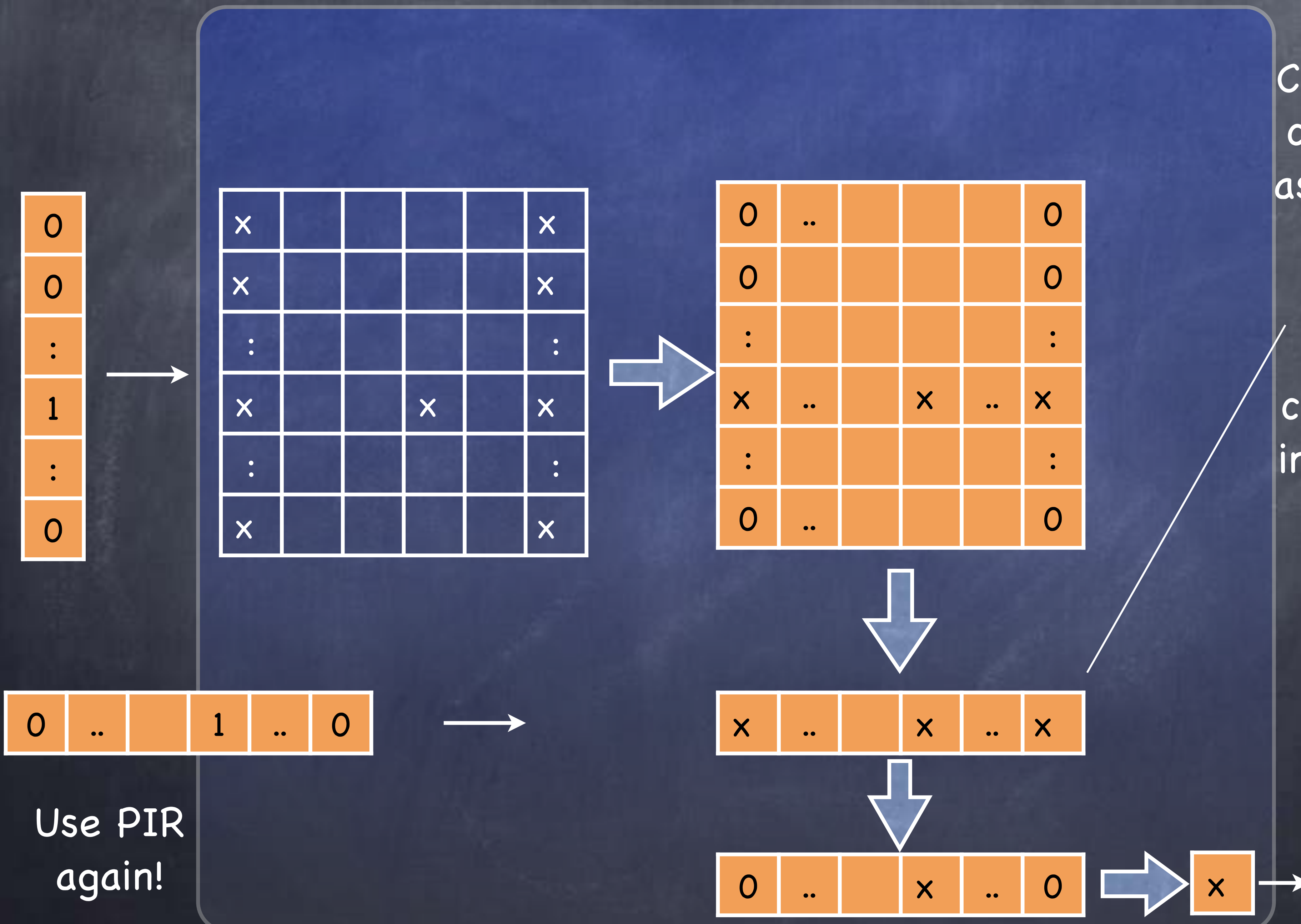

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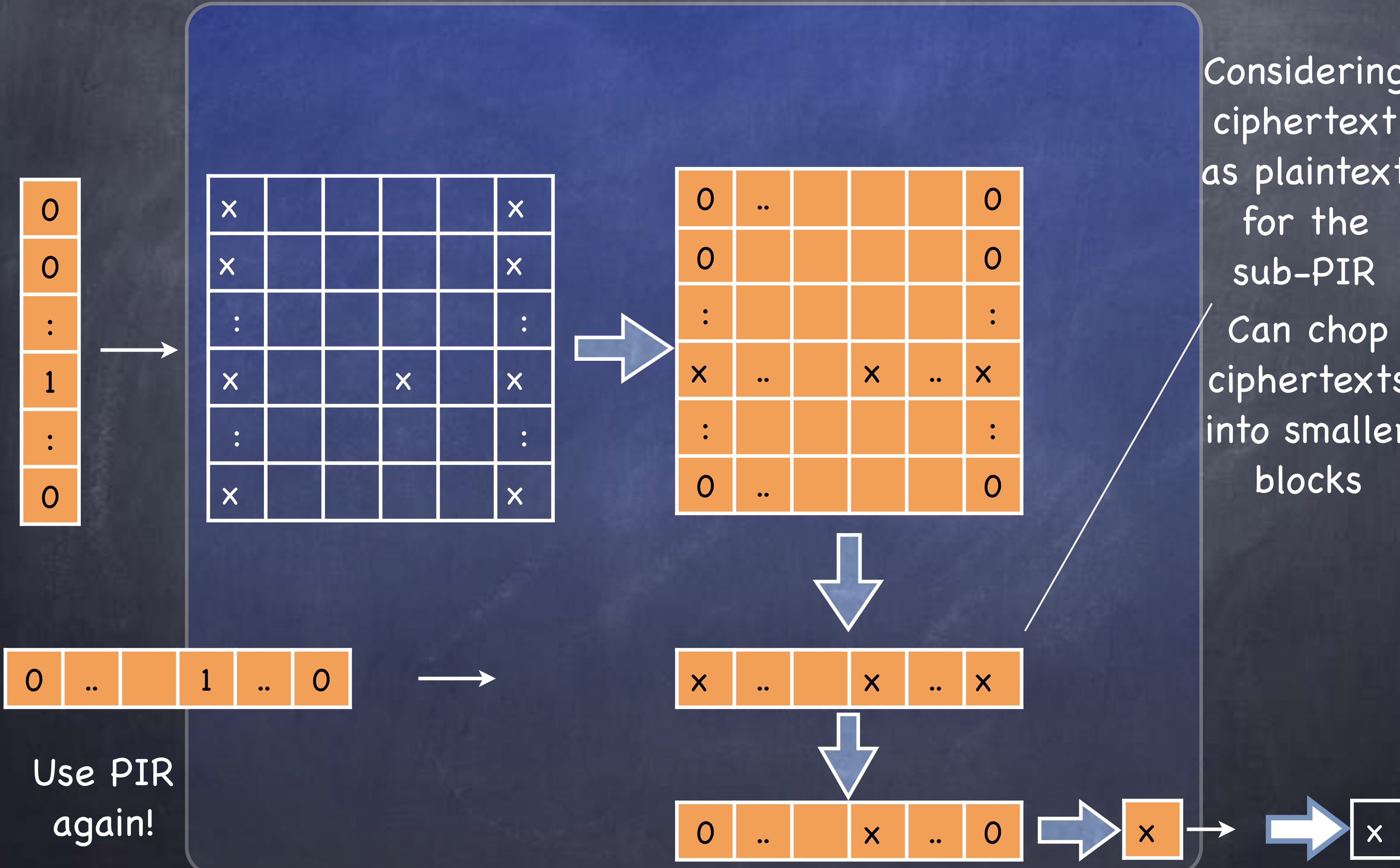

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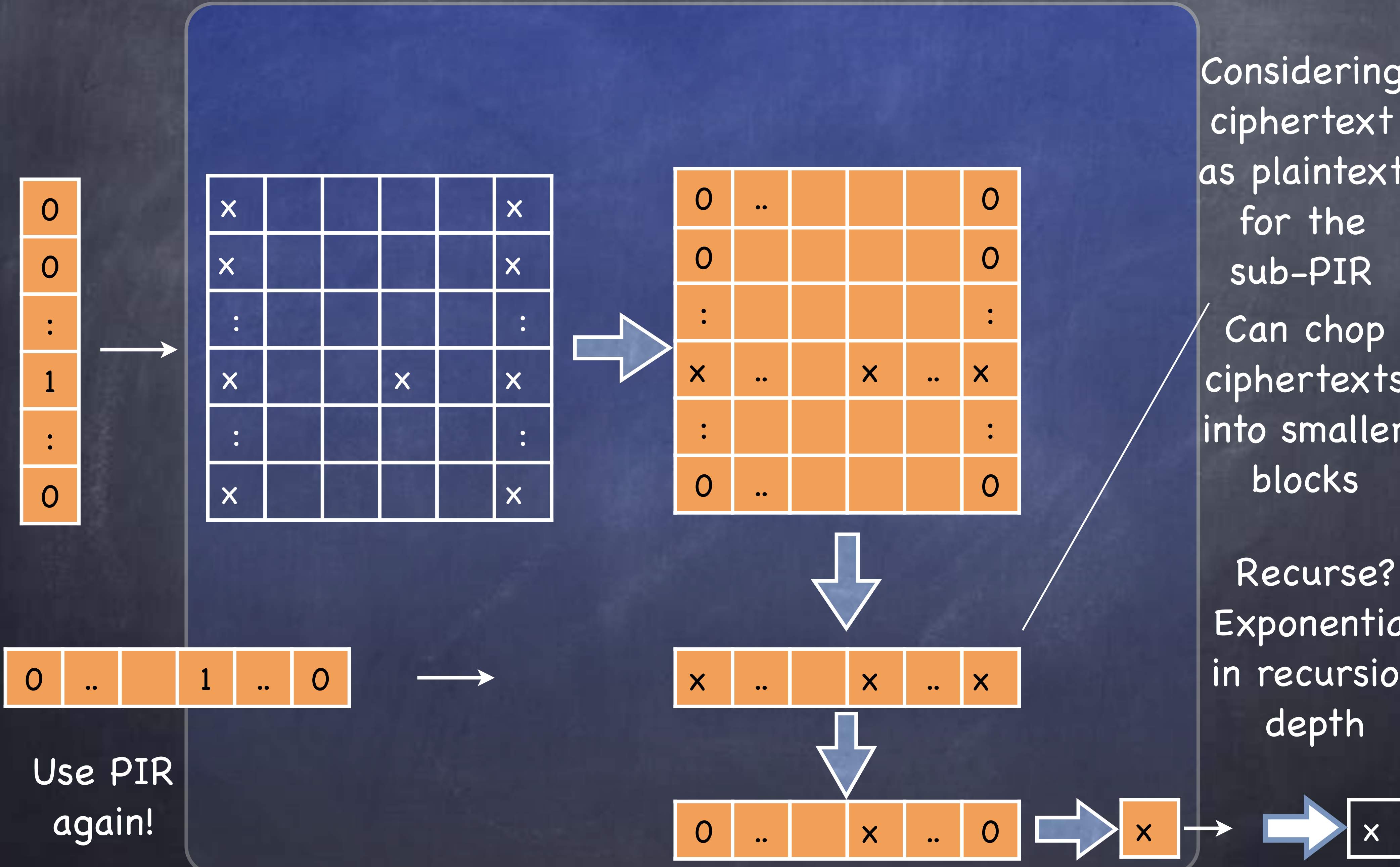

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Recurse? Exponential in recursion depth

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- Does such a family of encryption schemes exist?

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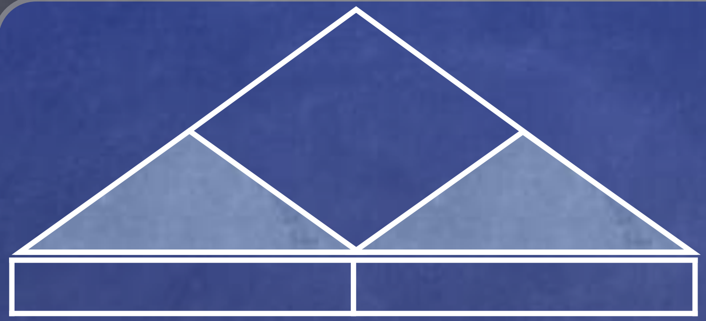
Final PIR protocol



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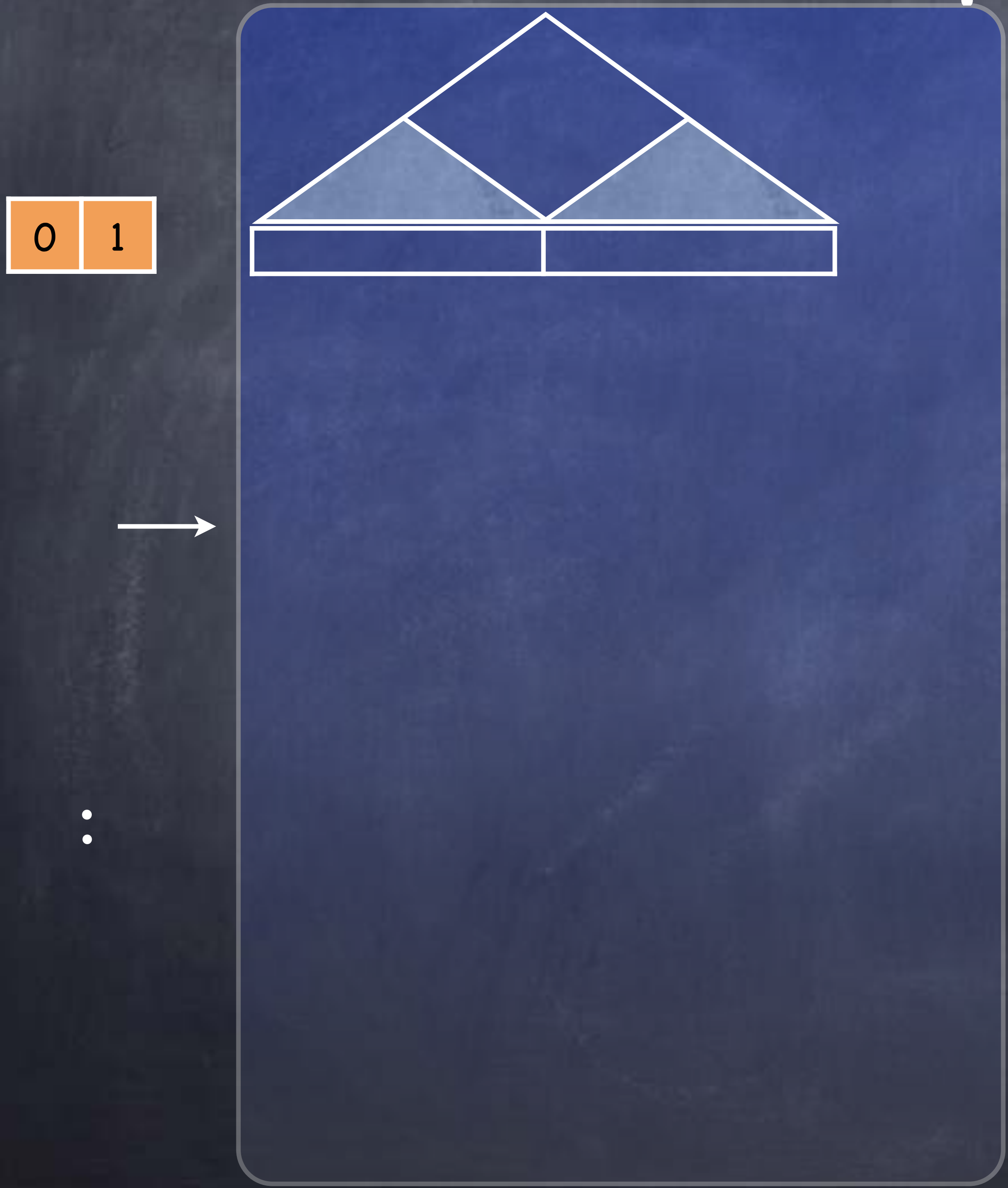

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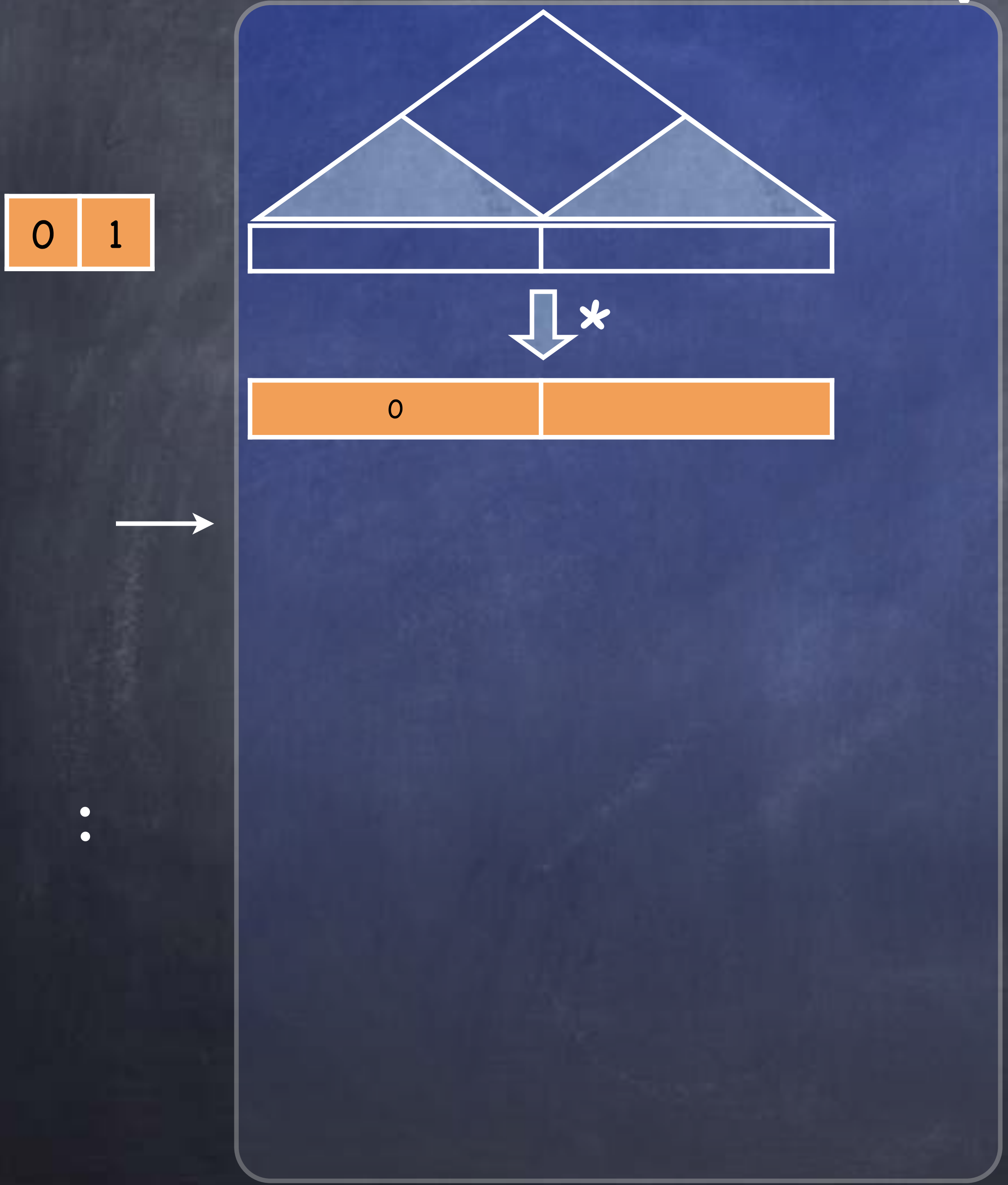

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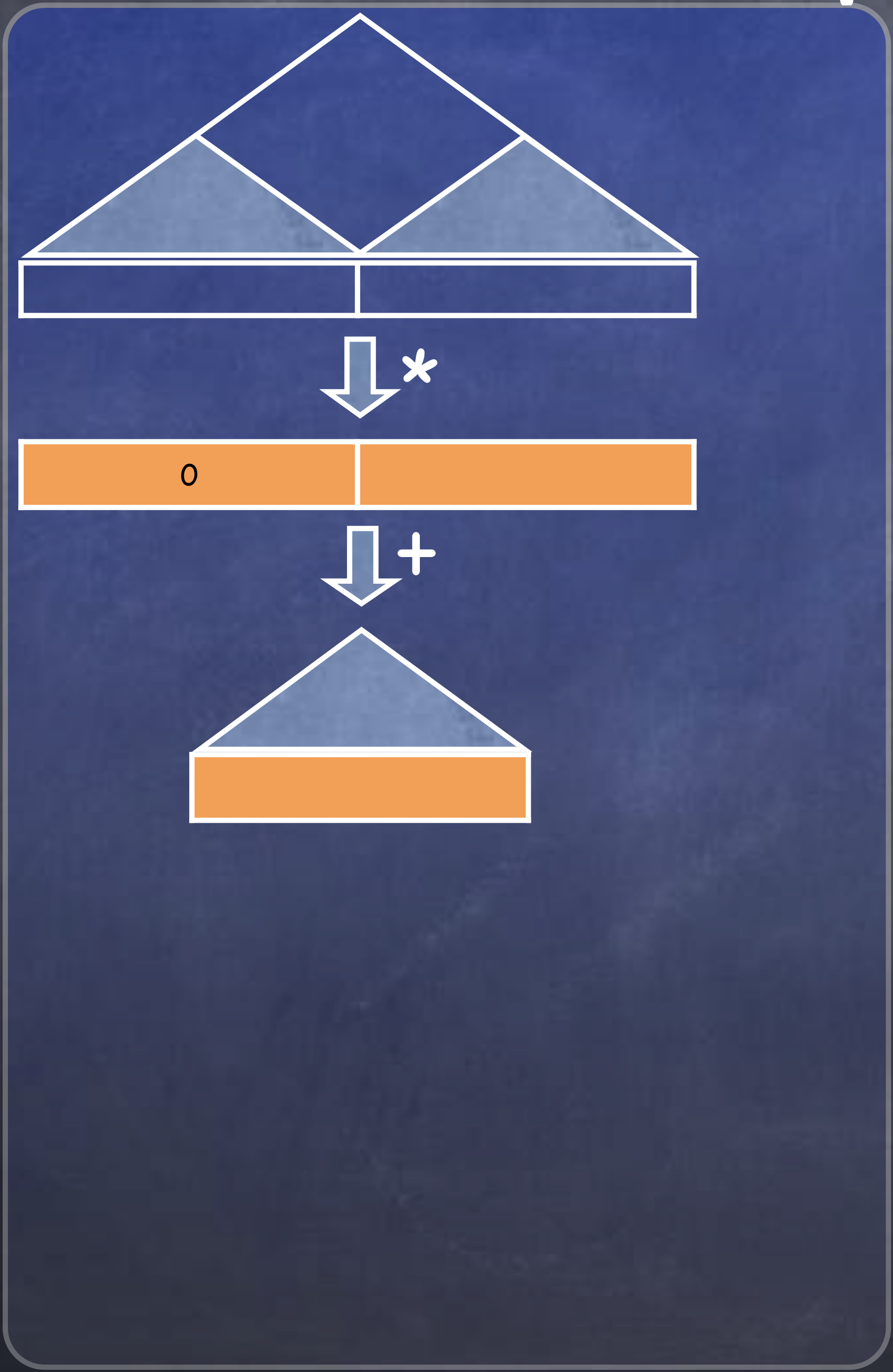

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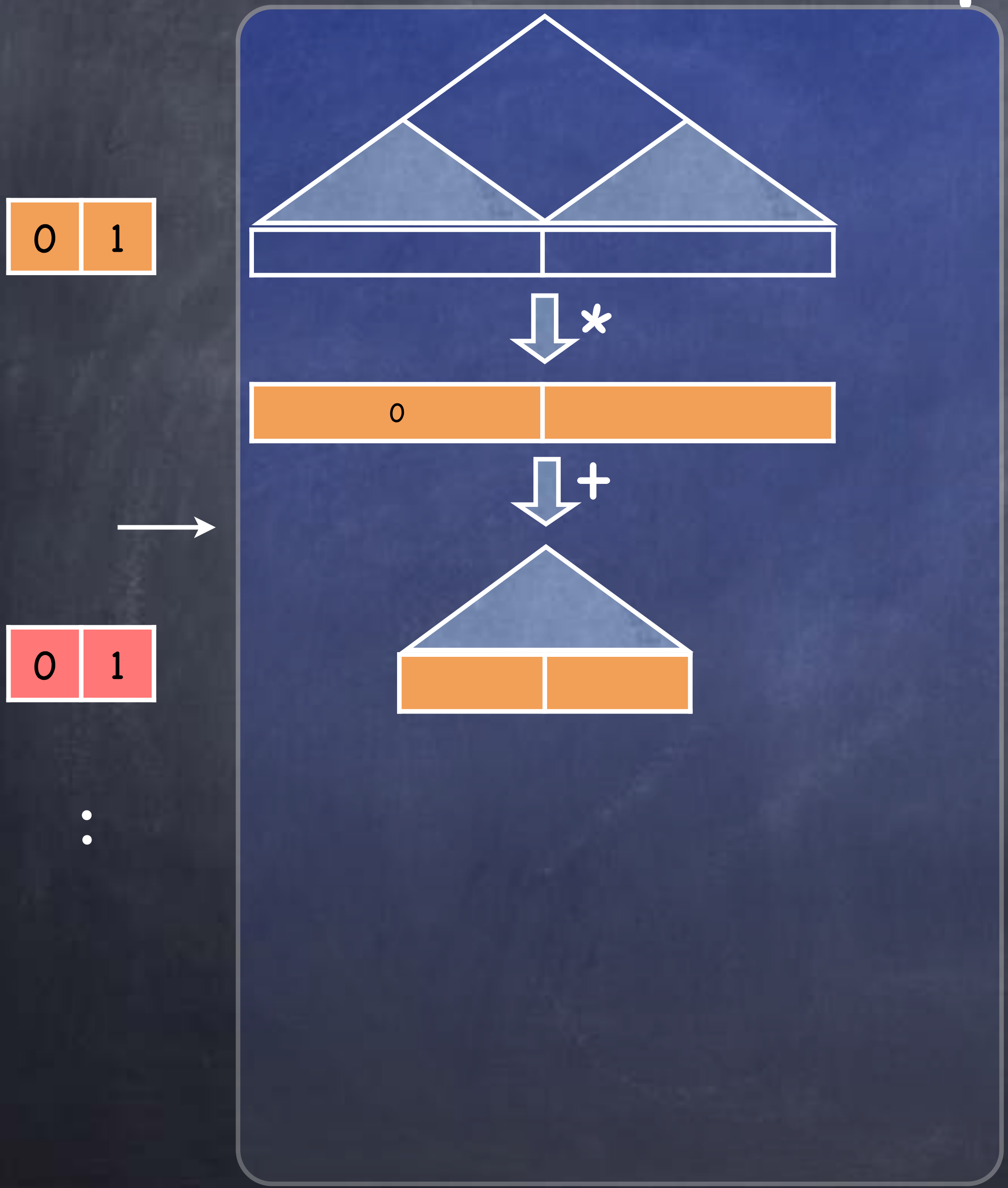

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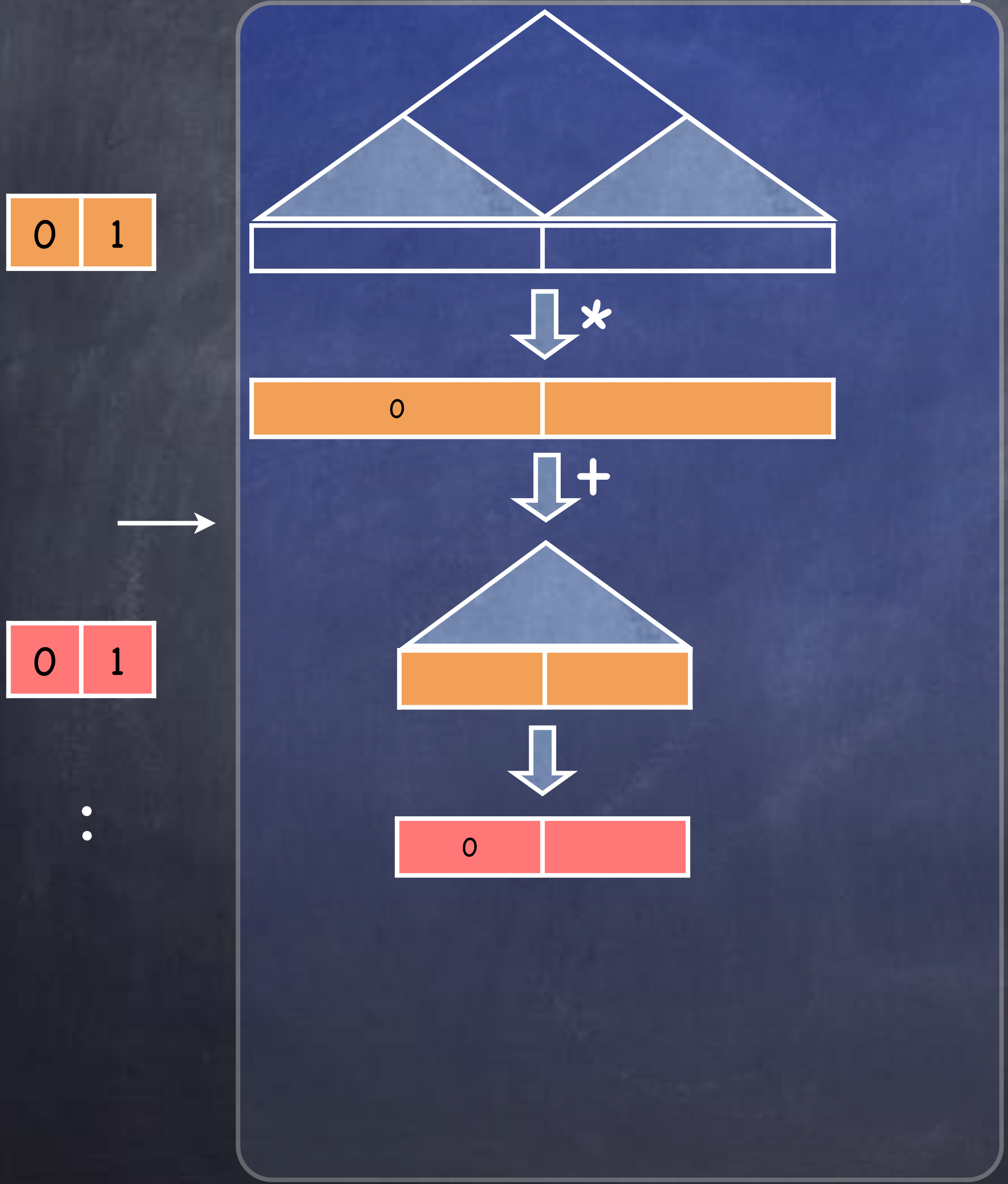

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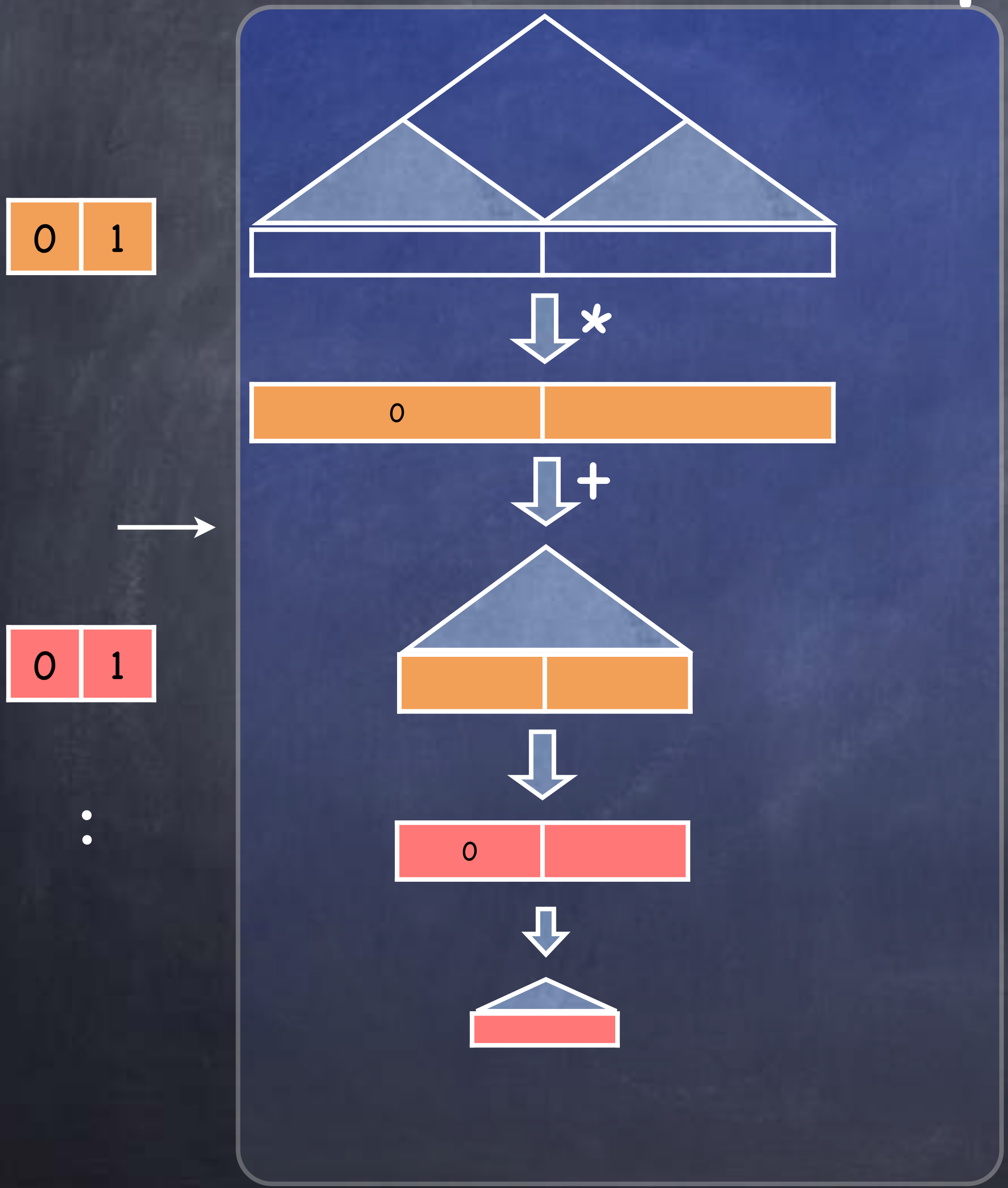

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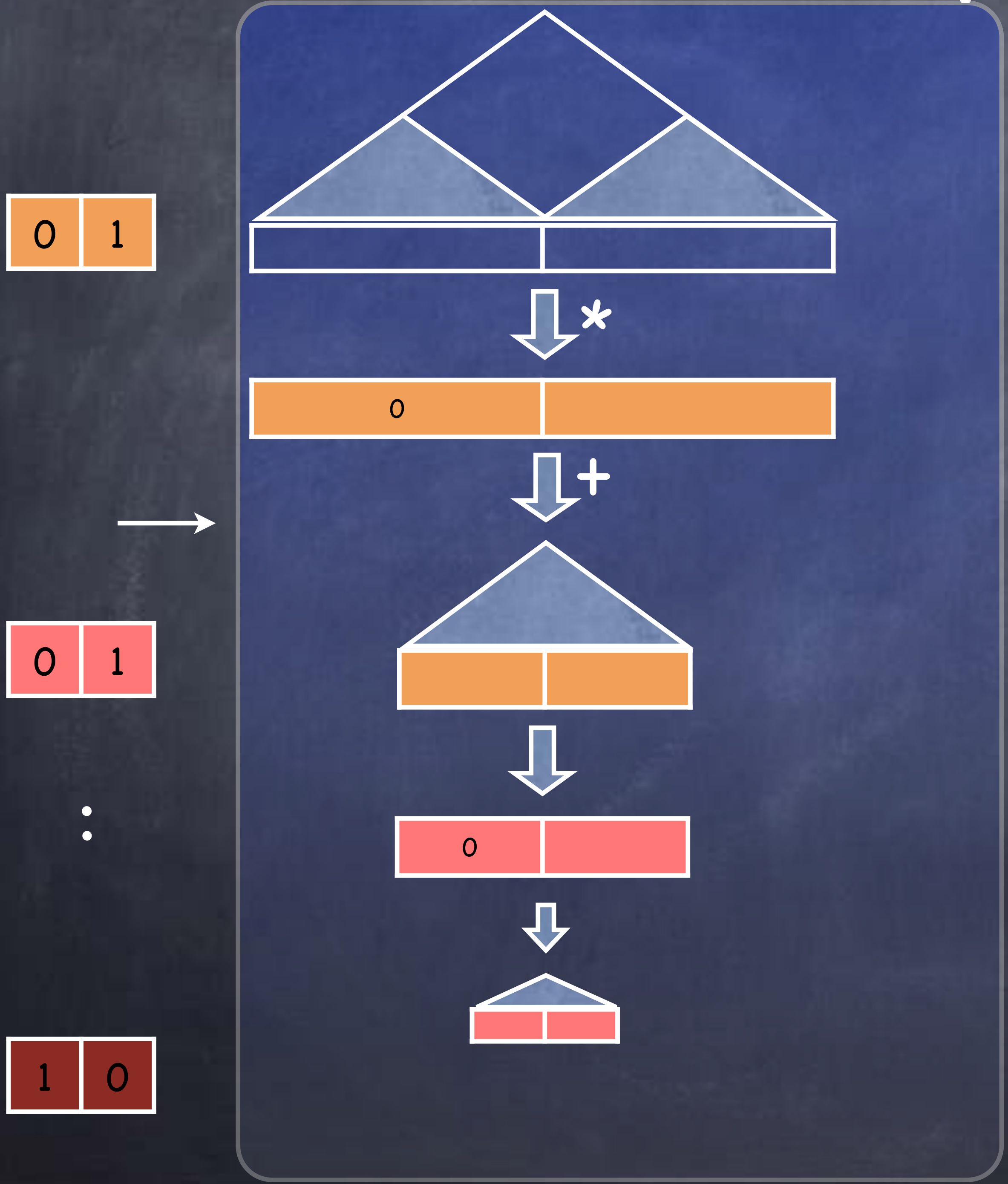

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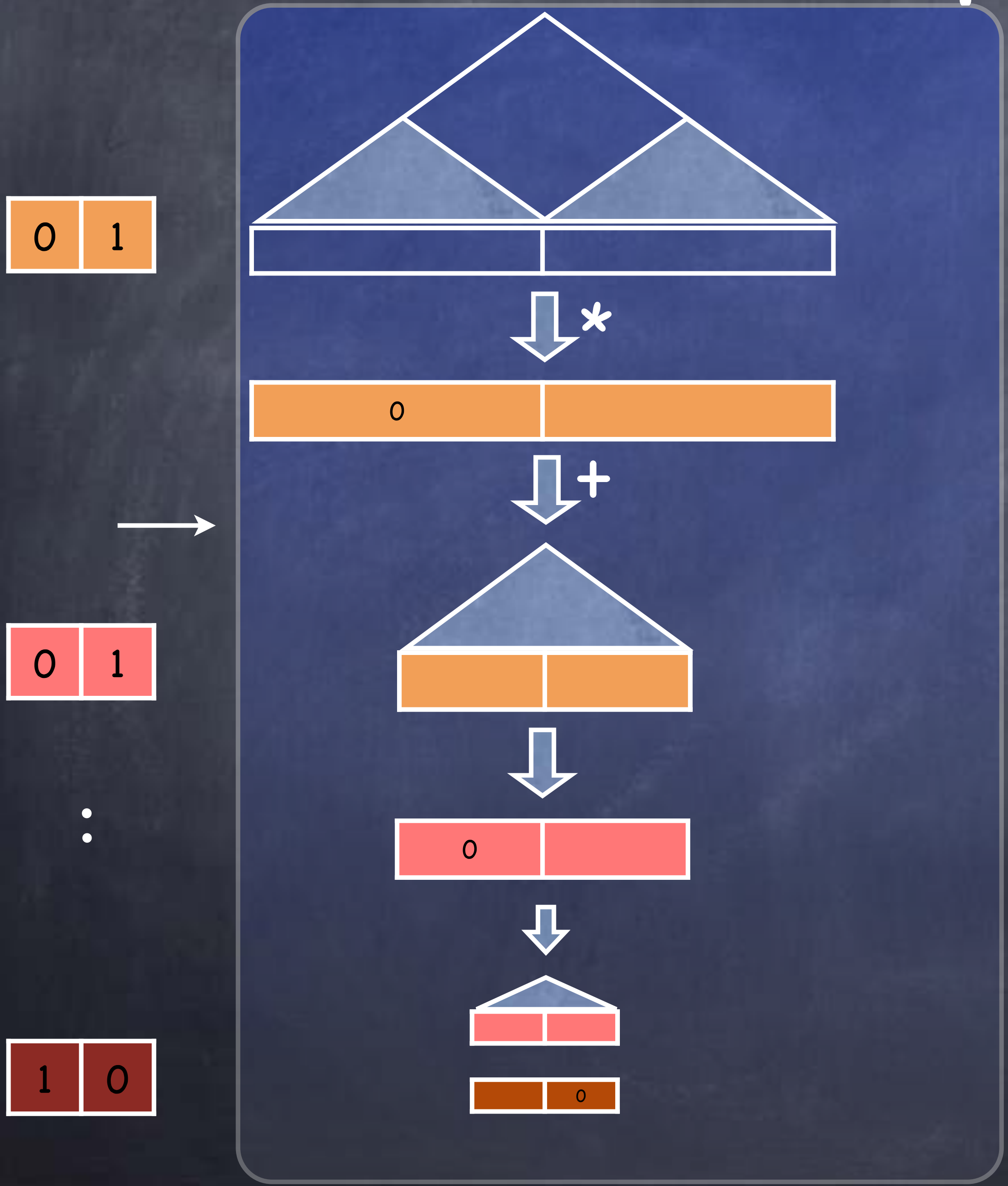

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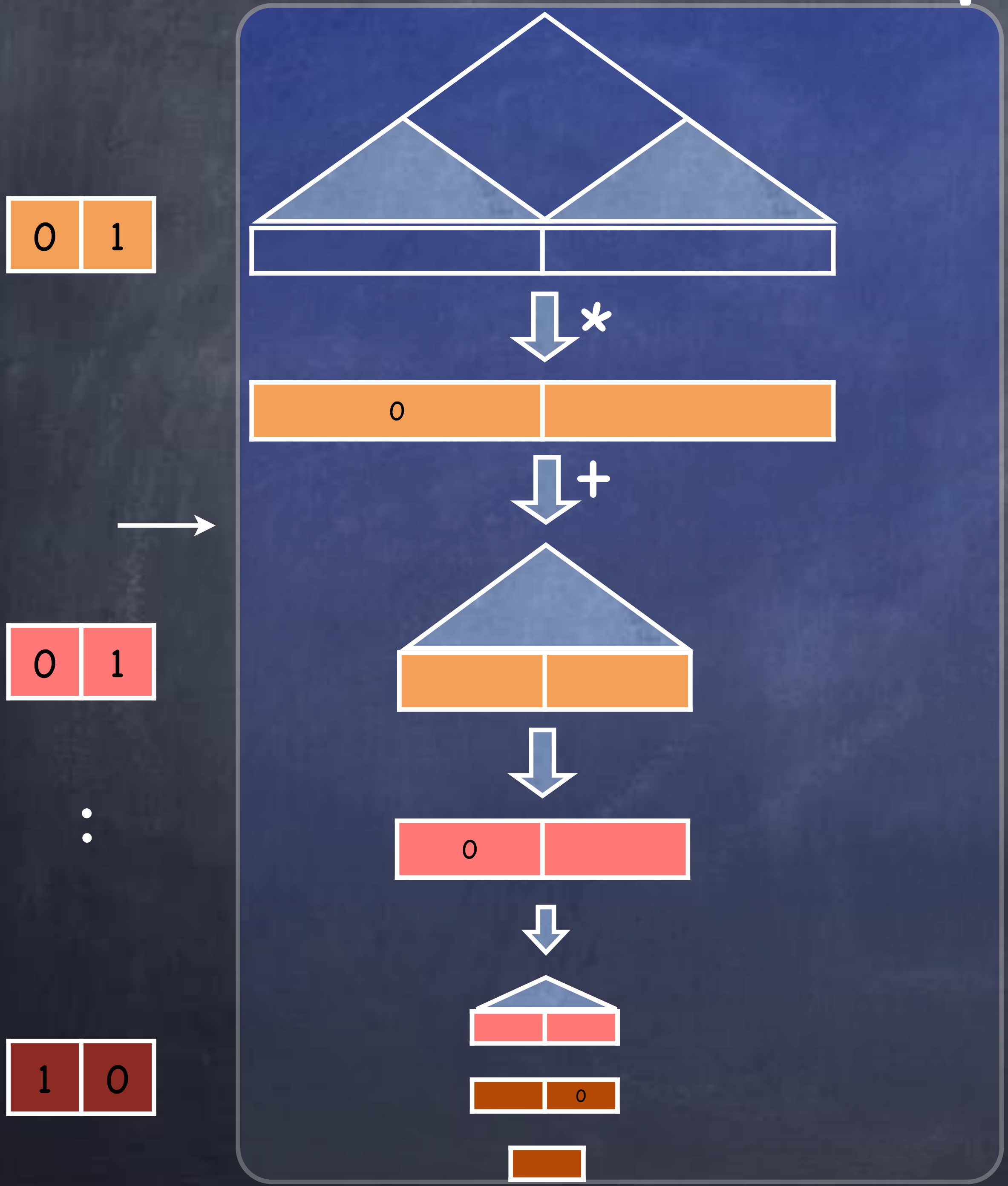

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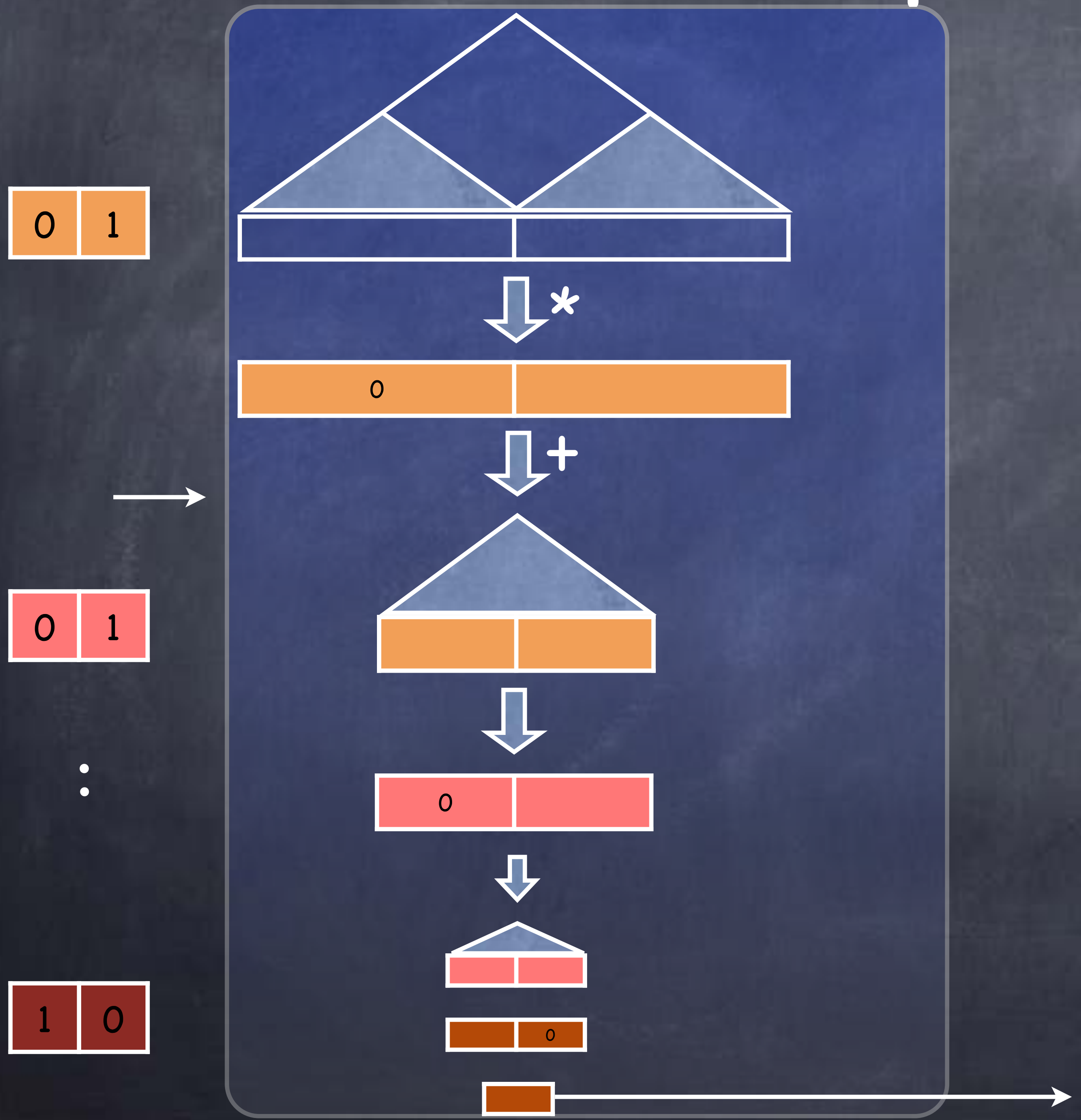

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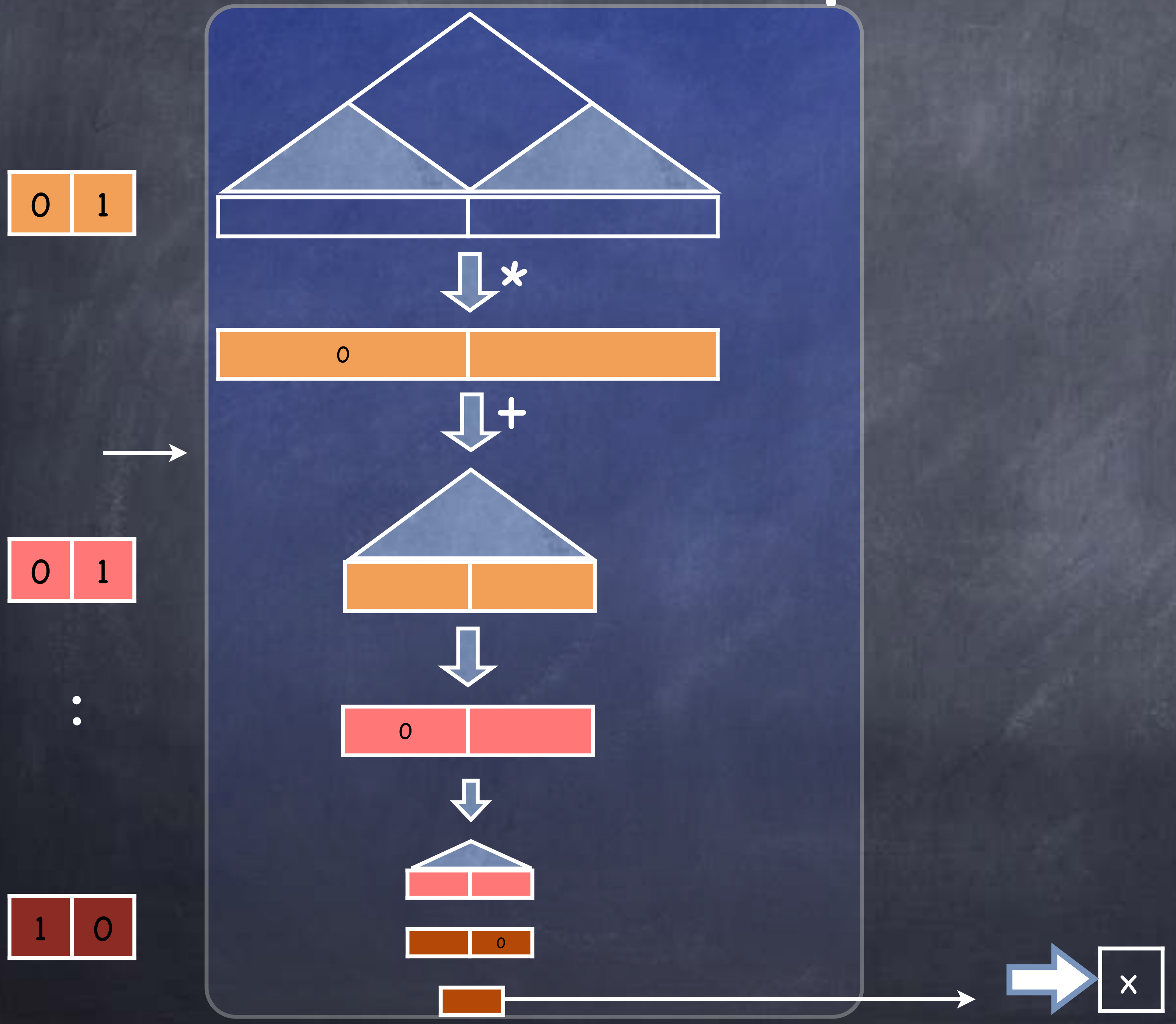

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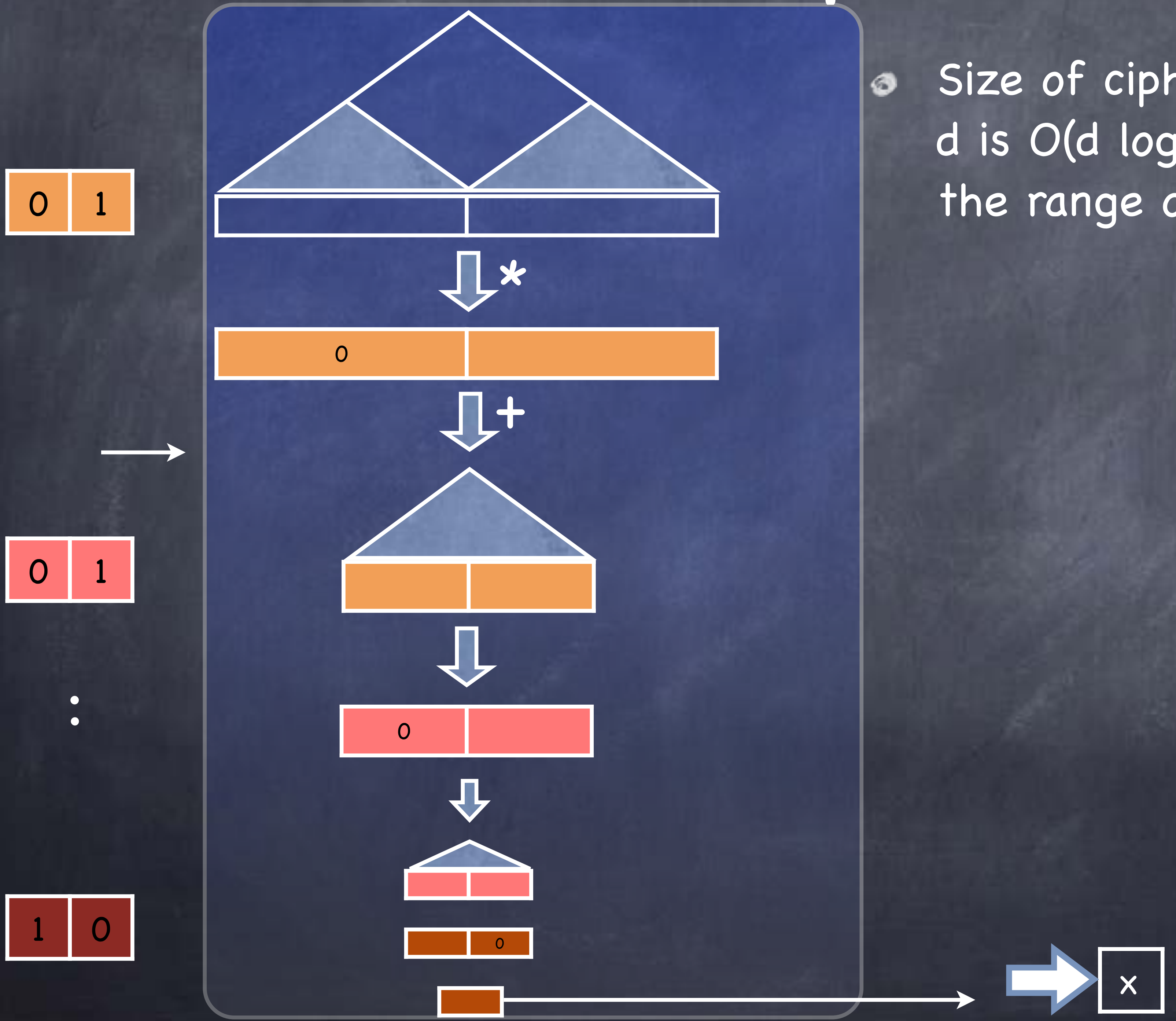

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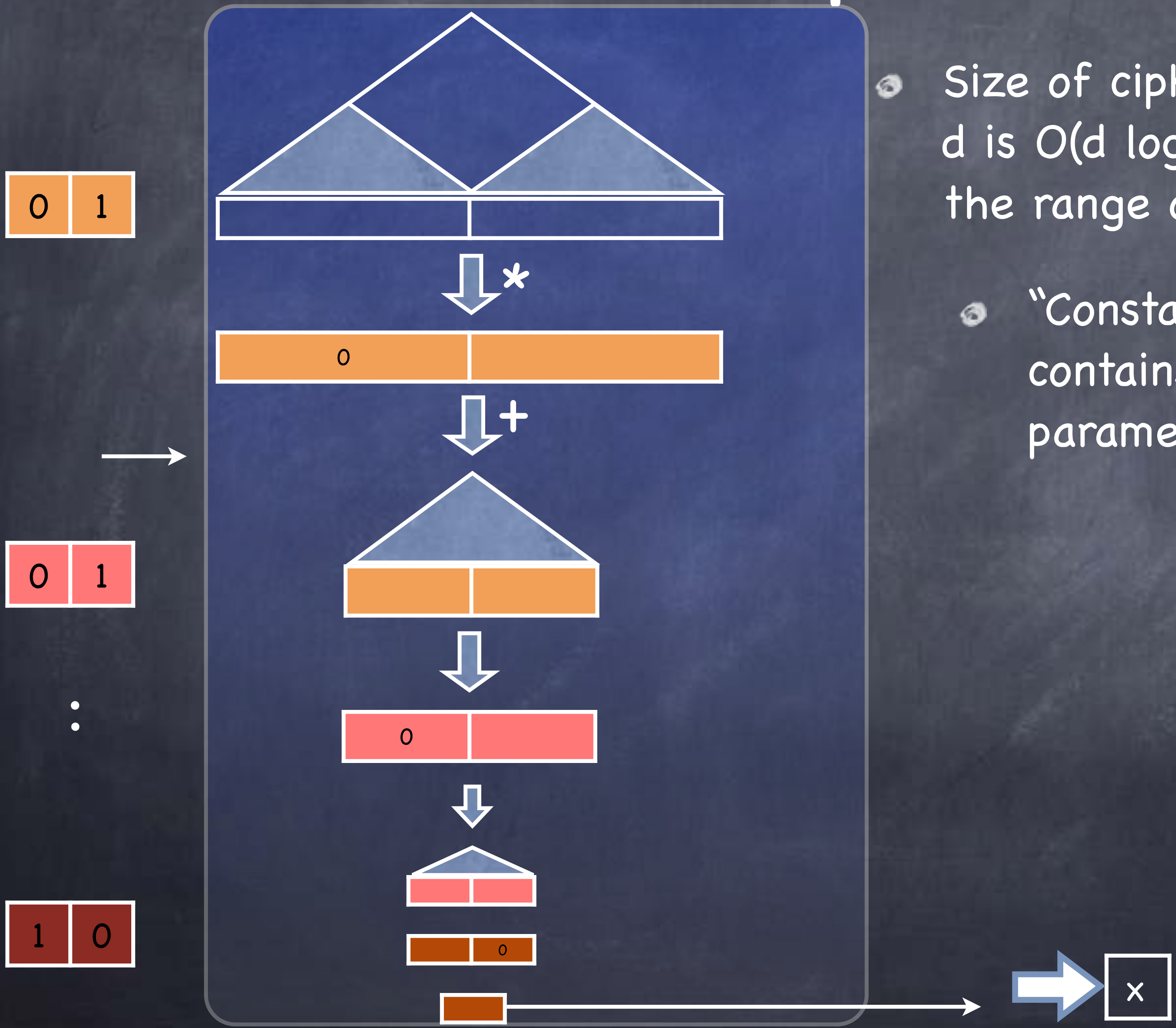

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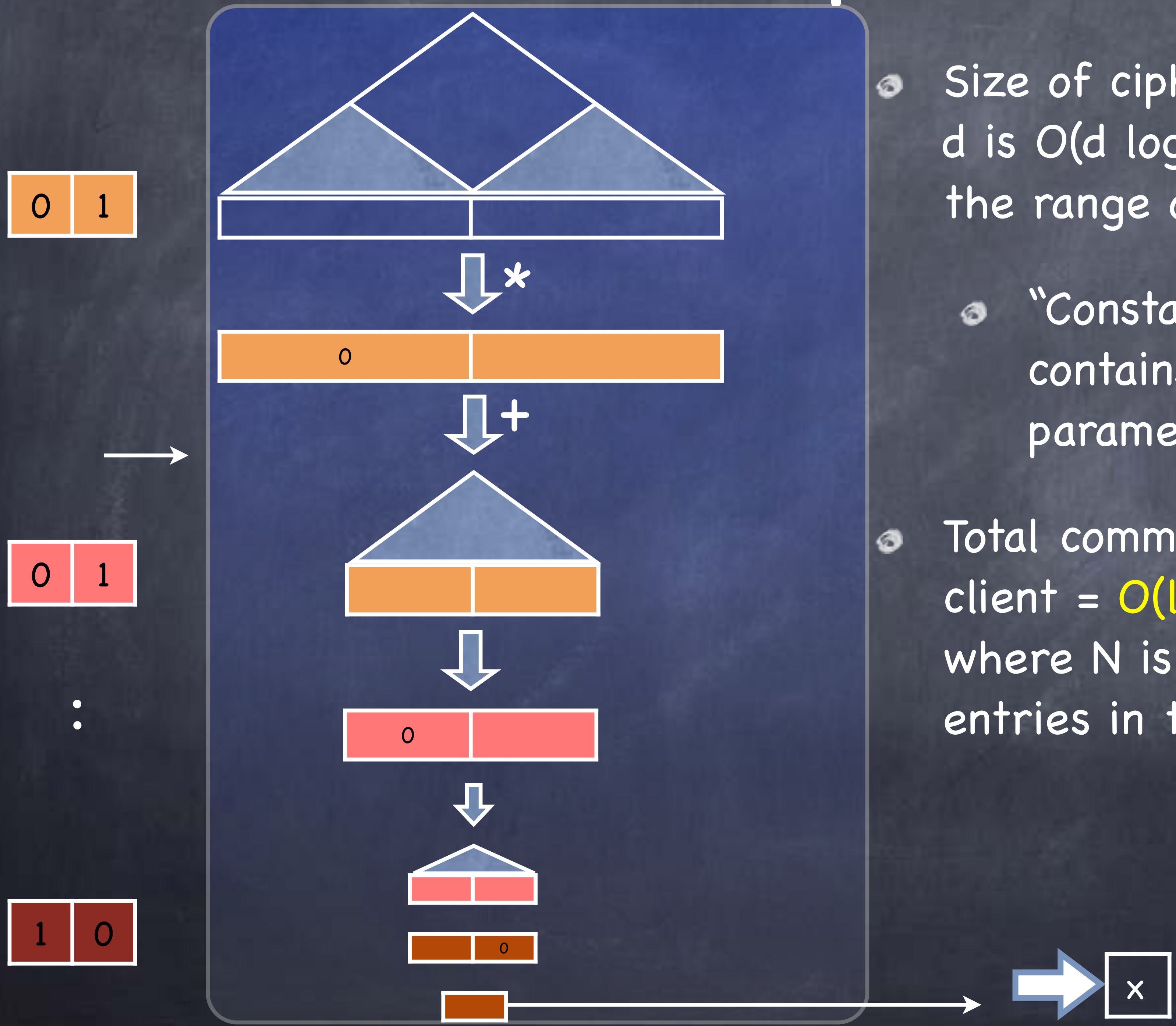

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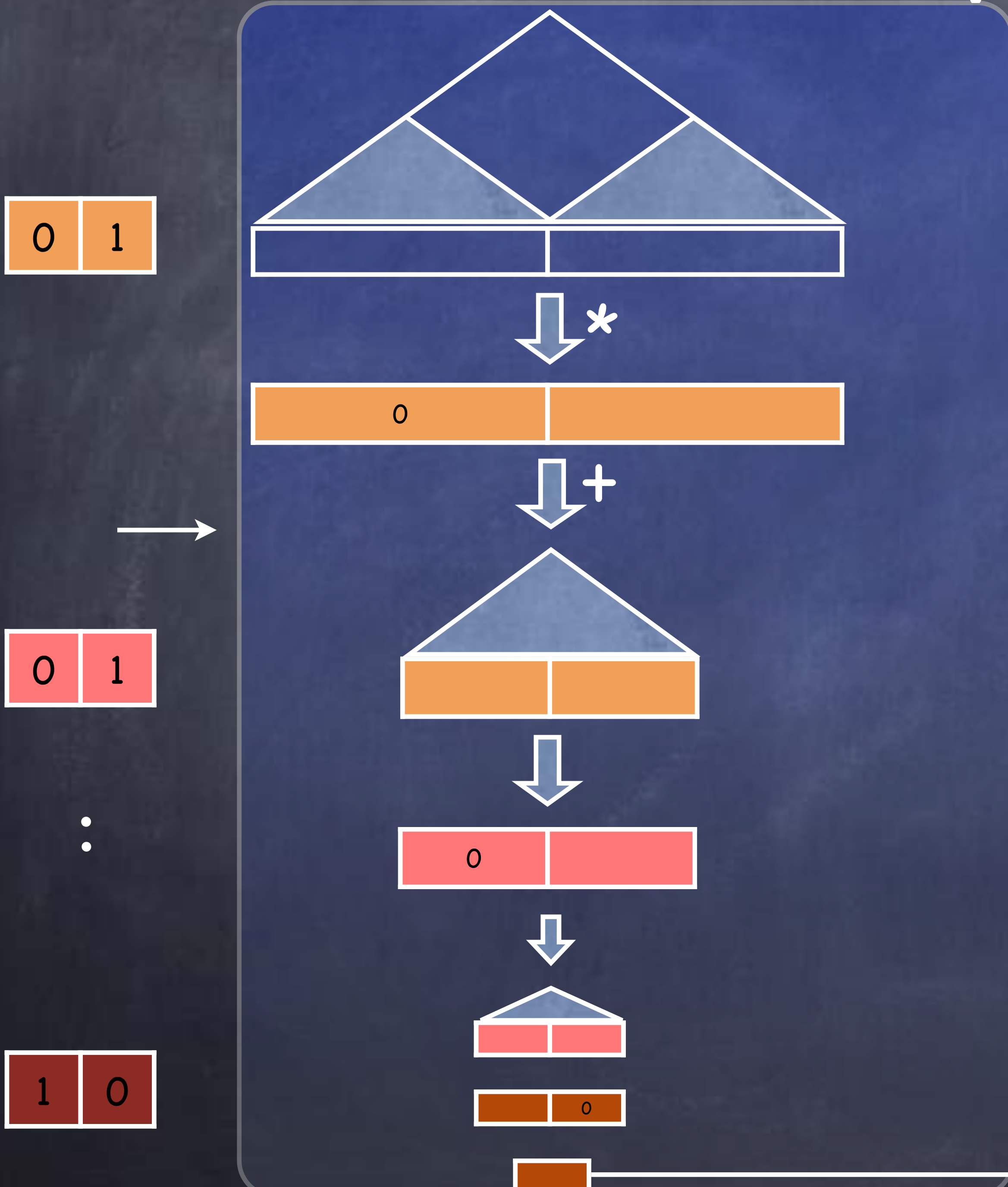

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Homomorphic Encryption for MPC

For active-security, include ZK proofs of correctness/ knowledge of plaintext, when publishing

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- But not good for homomorphic encryption: say, an application needs to use addition modulo 10; can we use Paillier?

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    - Each time add a large random multiple of 10 (but not large enough to cause overflow): 9+3+10r and 2+10r are statistically close if r drawn from a large range

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- Coming up: more applications - in voting