Hash Functions

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Lecture 10

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Before we talk about digital signatures...

A Tale of Two Boxes

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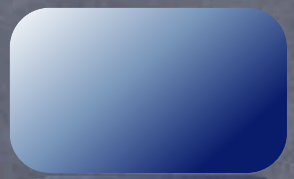

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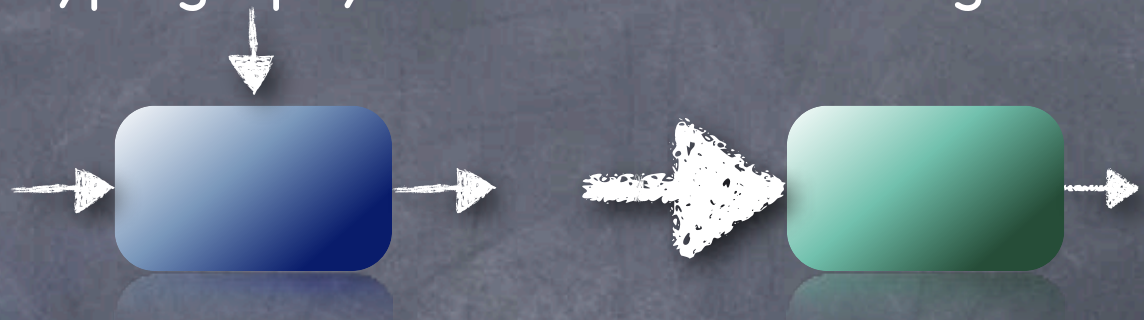

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  - Today: understanding security requirements on hash functions

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  - Also sometimes: some kind of unpredictability

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  - Main goal: to "avoid collisions". Will see several variants of the problem

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- Alternately, could be considered as have already been randomly chosen from a family (and security parameter fixed too)
  - Usually involves hand-picked values (e.g. "I.V." or "round constants") built into the standard

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- CRHF the strongest; UOWHF still powerful (will be enough for digital signatures)

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- CRHF implies second pre-image collision resistance and, if sufficiently compressing, then pre-image collision resistance [Exercise]

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  - Birthday attack effectively halves the hash length (say security parameter) over "naïve attack"

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- Inefficient example: $\mathcal{H}$ set of all functions from X to Z

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  - $\forall x_1..x_k$ (distinct), $z_1..z_k$, $\Pr_{h \leftarrow \mathcal{H}}[\forall i\ h(x_i)=z_i\ ] = 1/|Z|^k$
- Inefficient example: $\mathcal{H}$ set of all functions from X to Z
  - But we will need all $h \in \mathcal{H}$ to be succinctly described and efficiently evaluable

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  - But does not compress!

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    - $\Pr_h[\ \text{Chop}(h(x)) = w,\ \text{Chop}(h(y)) = z]$
      $= \Pr_h[\ h(x) = w0 \text{ or } w1,\ h(y) = z0 \text{ or } z1] = 4/|Z|^2 = 1/|Z'|^2$

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  - Gives a UOWHF that compresses by 1 bit (same as the UHF)
    - Will see next, how to extend the domain to arbitrarily long strings (without increasing output size)

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  - All candidates use mathematical structures that are considered computationally expensive in practice

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- Current practice: much less paranoid; faith on efficient, ad hoc (and unkeyed) constructions (though increasingly under attack)

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