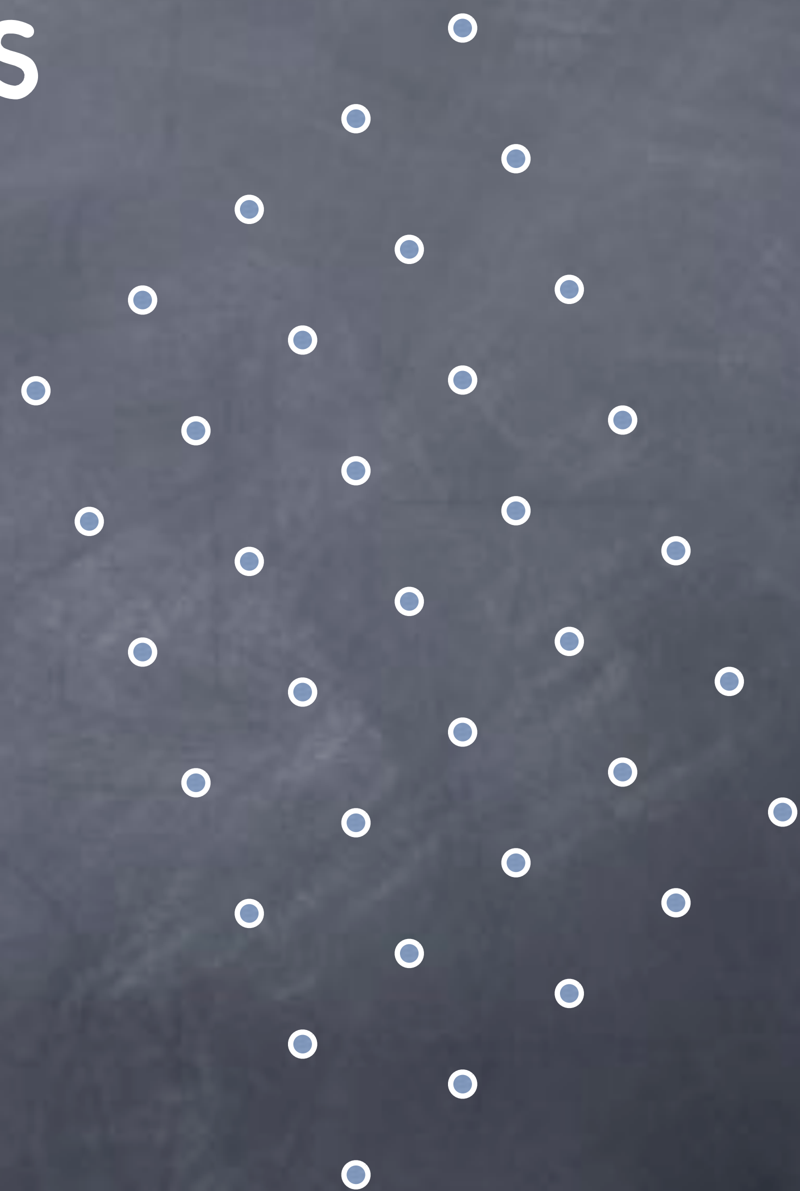
Lattice Cryptography

Lecture 21

Lattices



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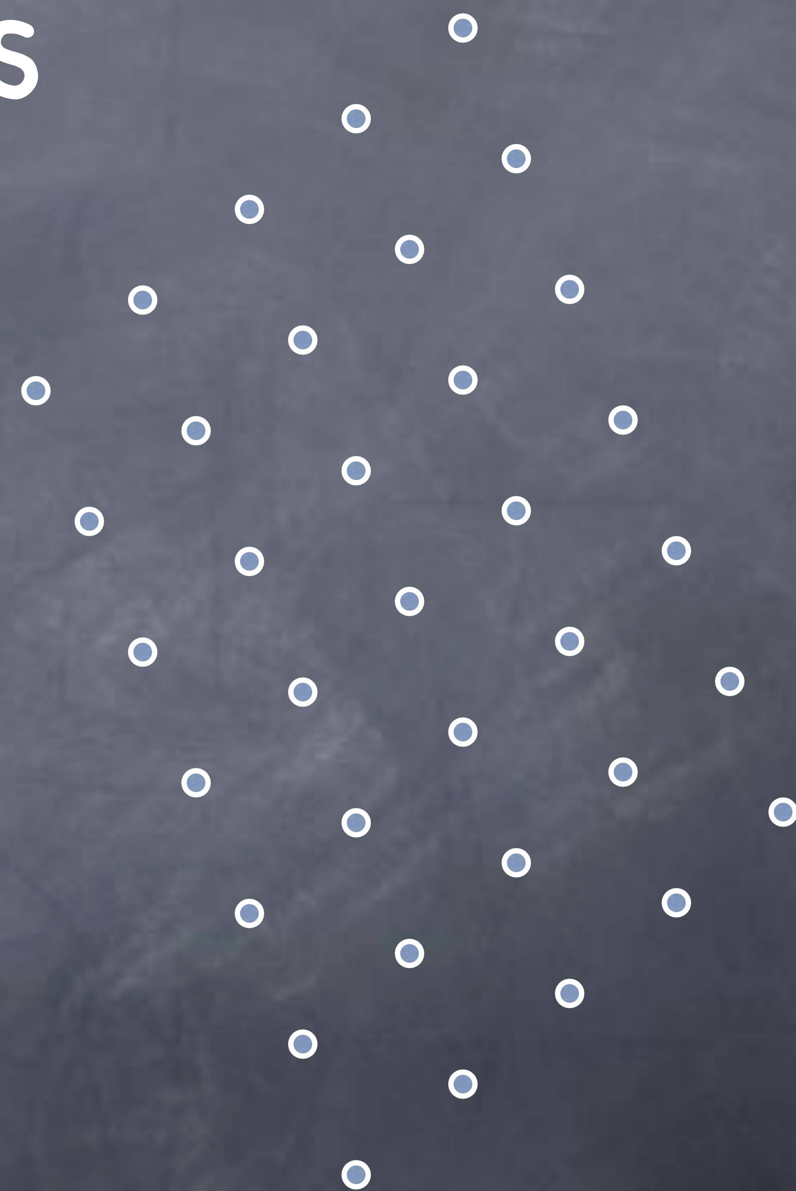

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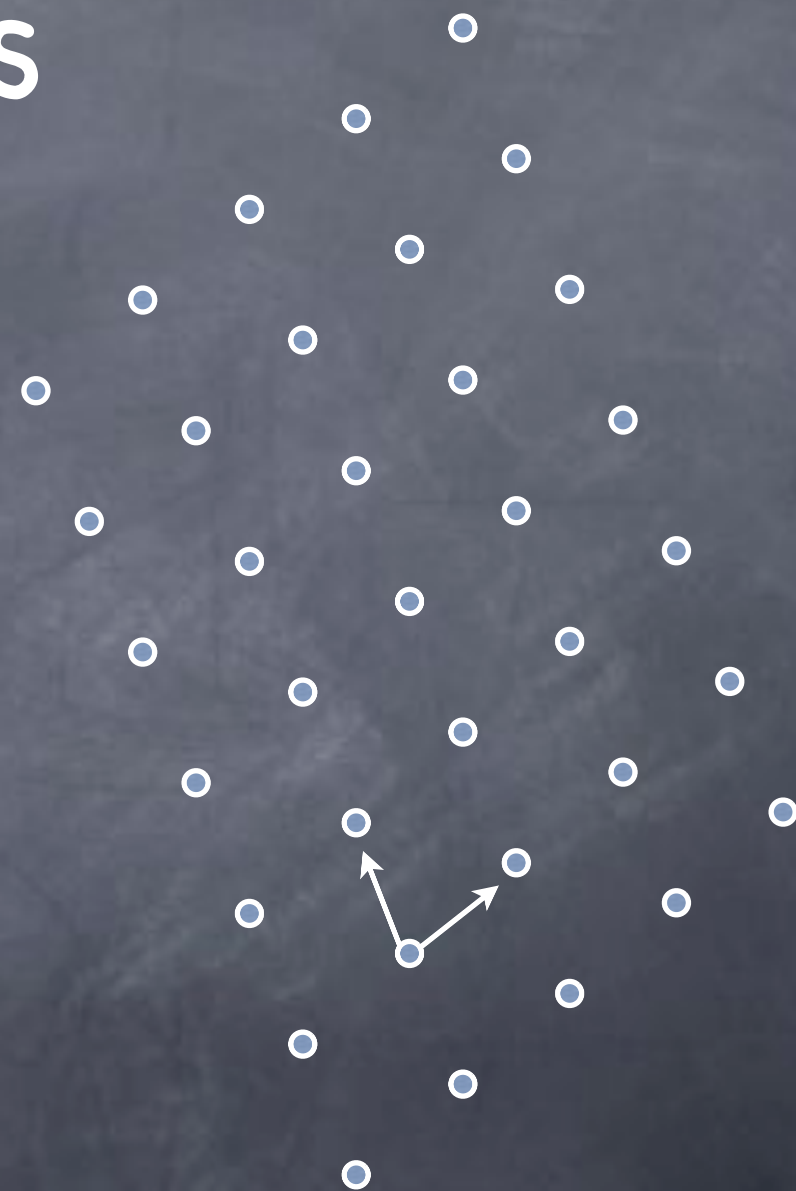

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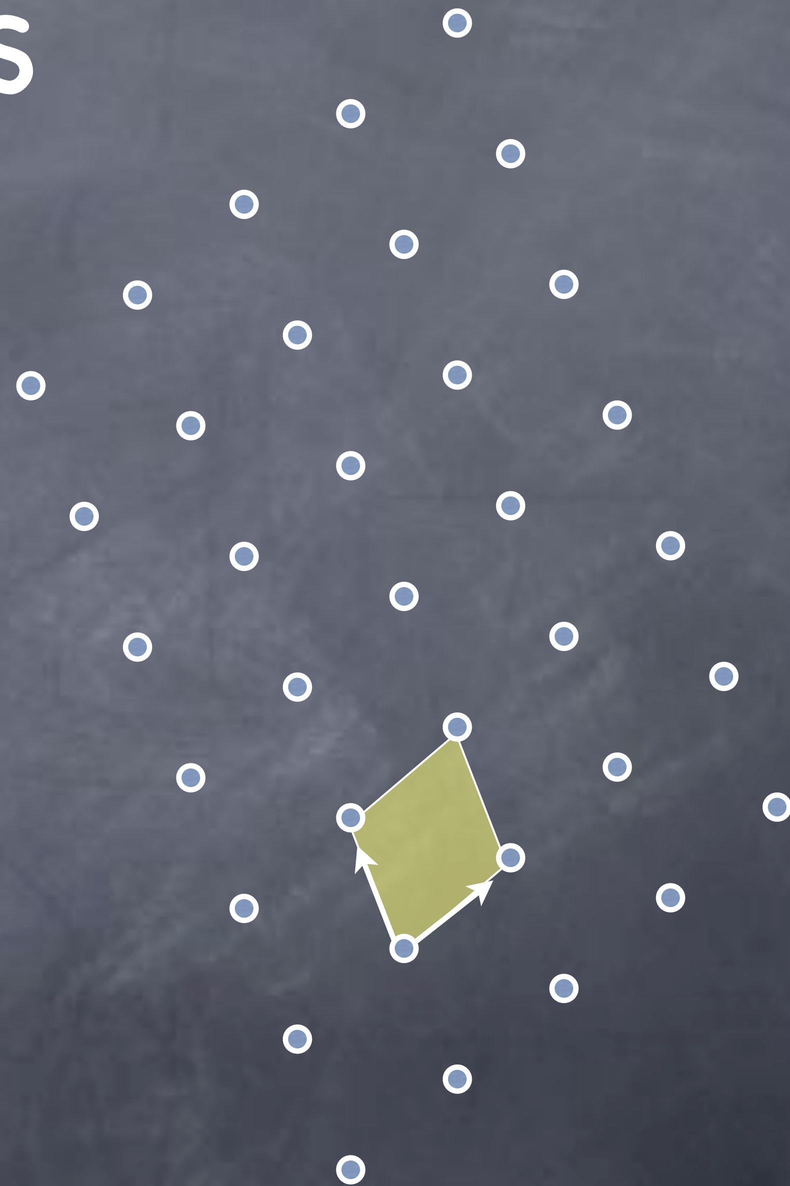

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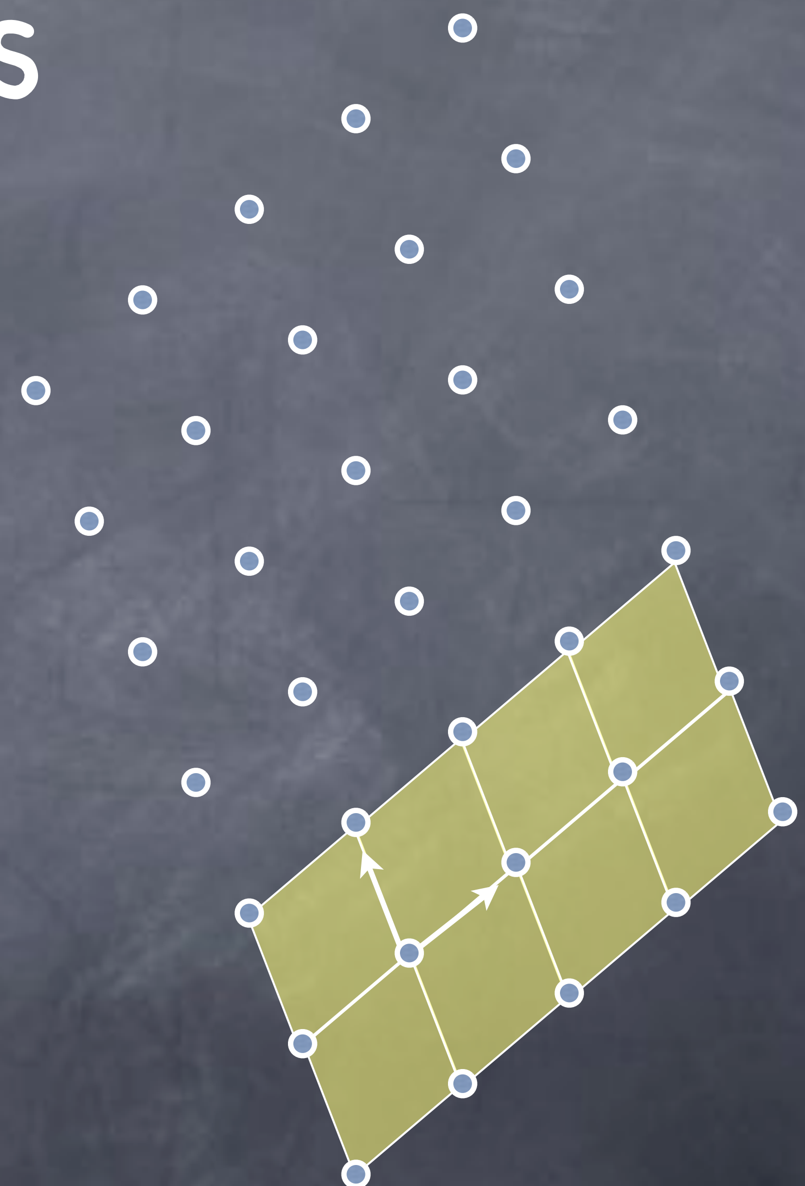

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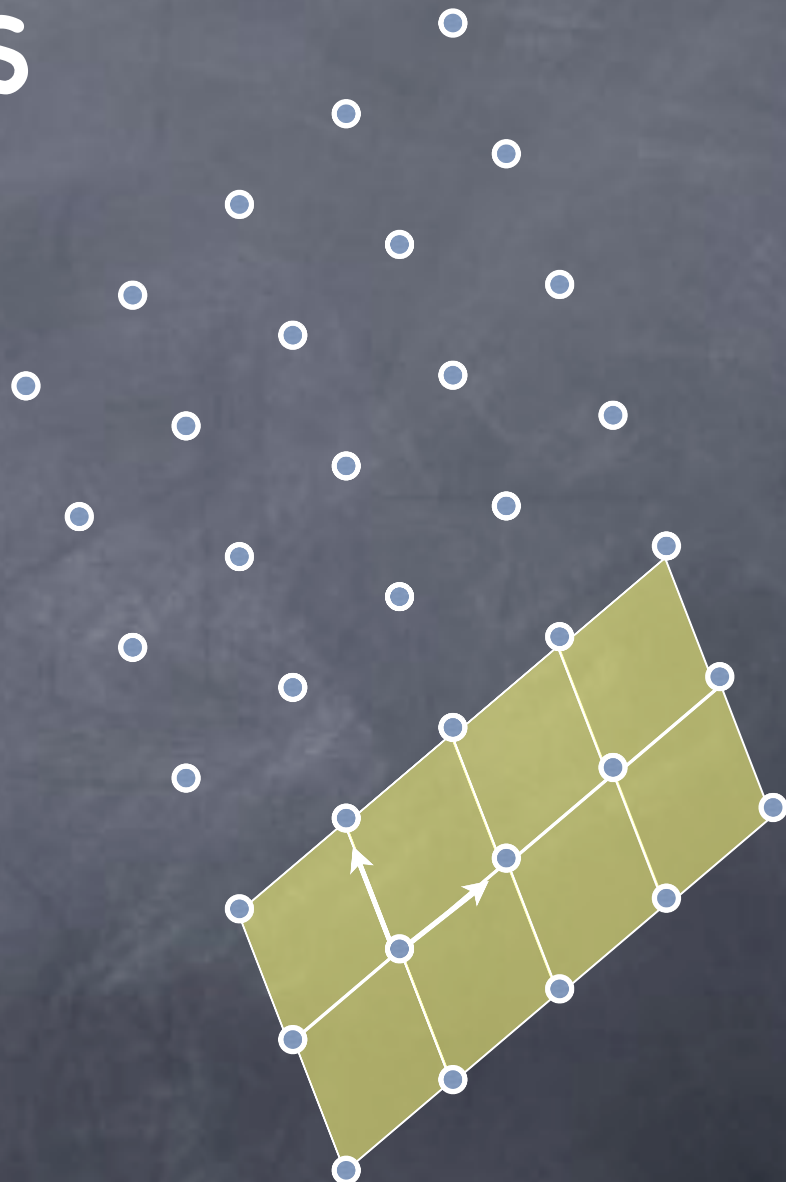

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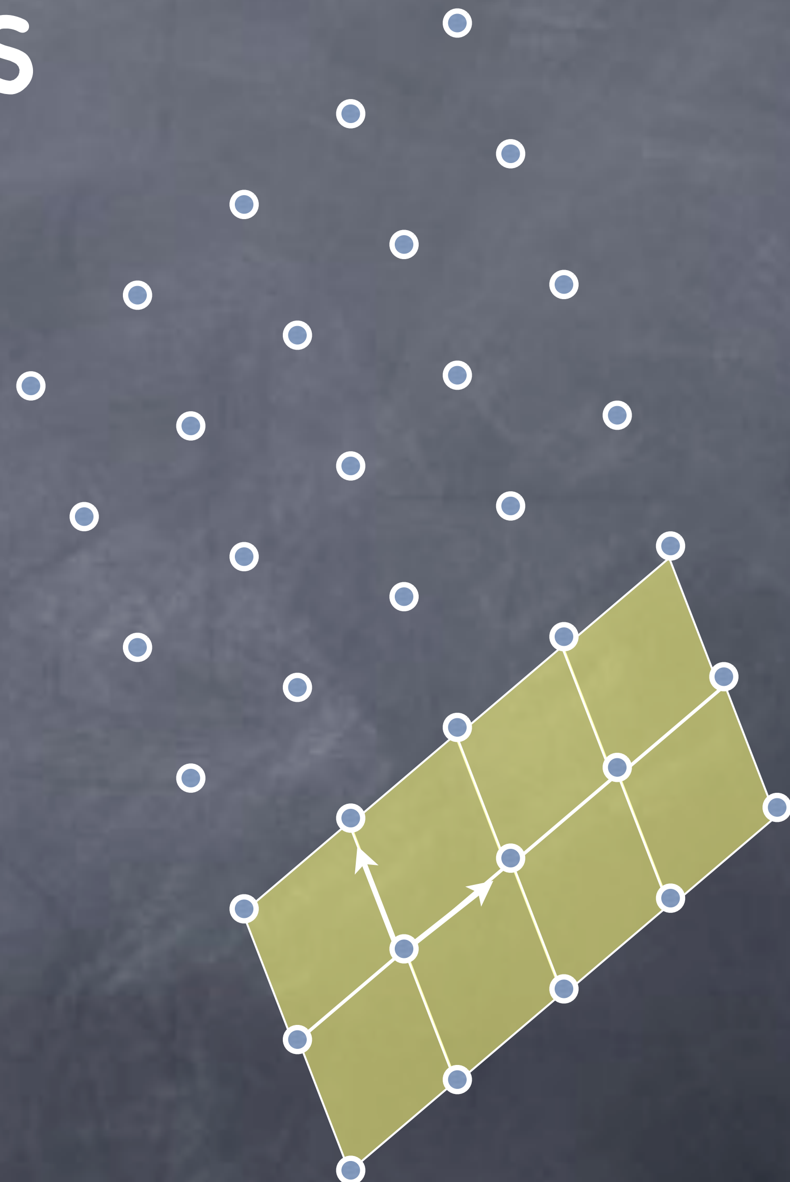

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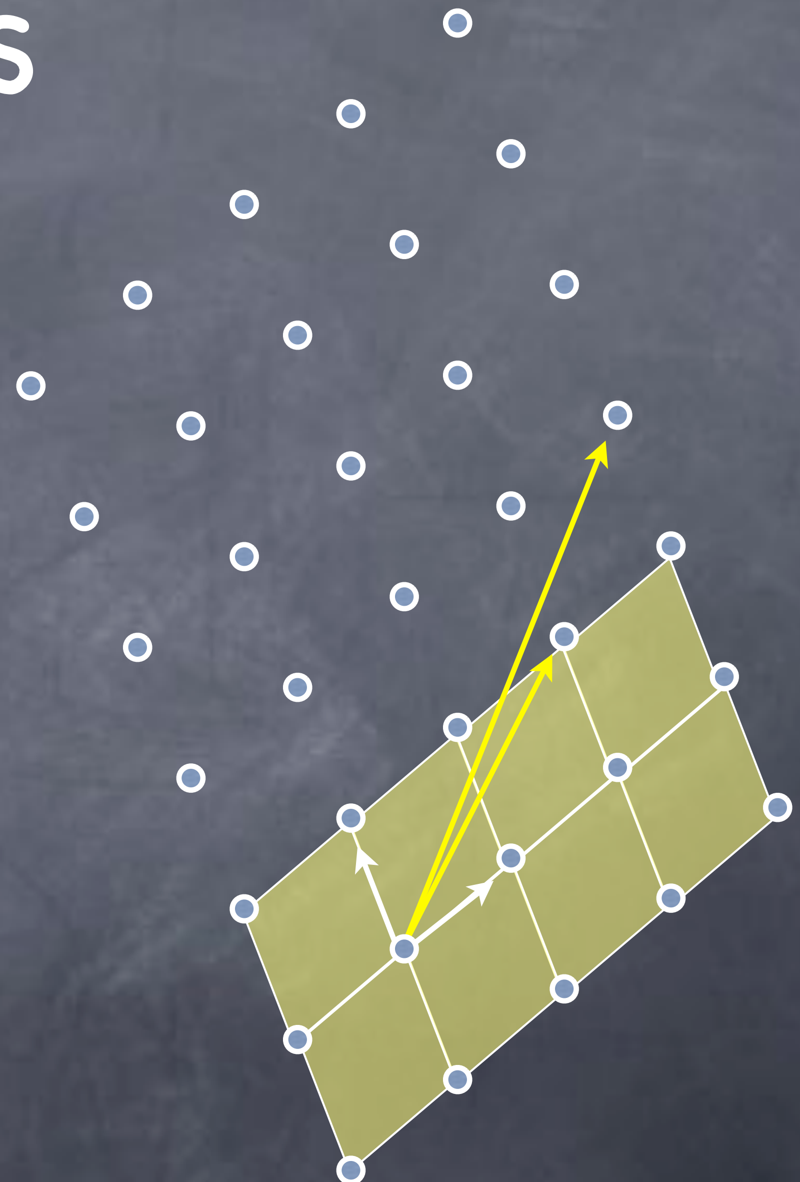

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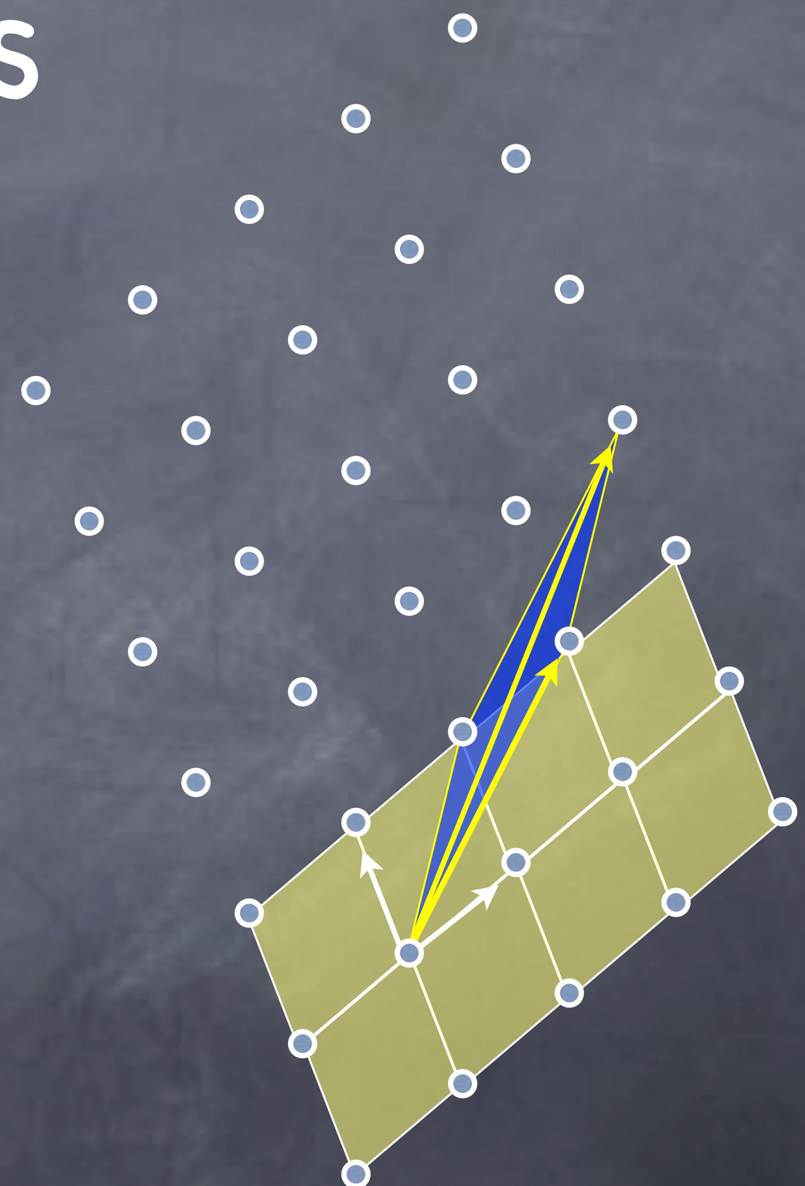

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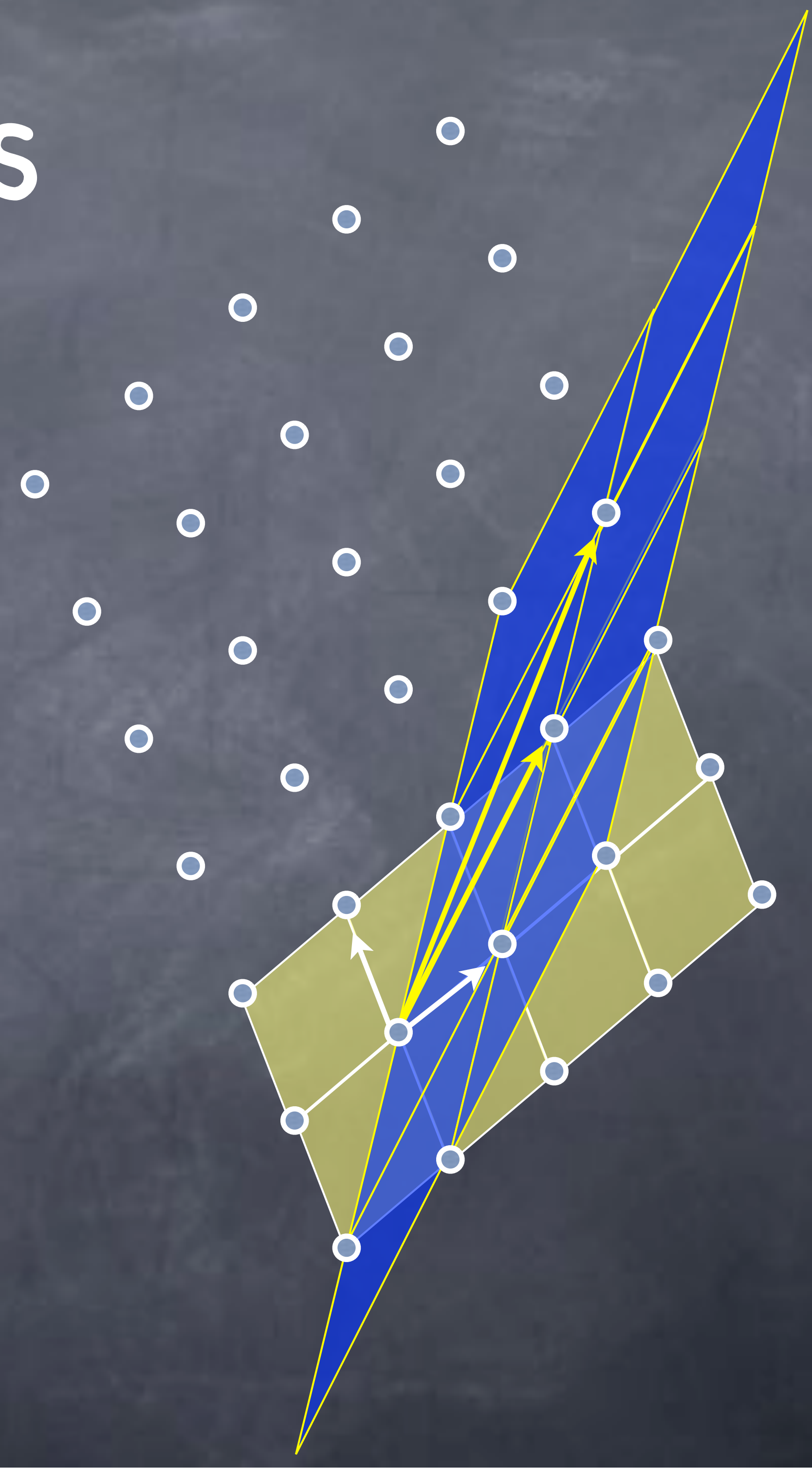

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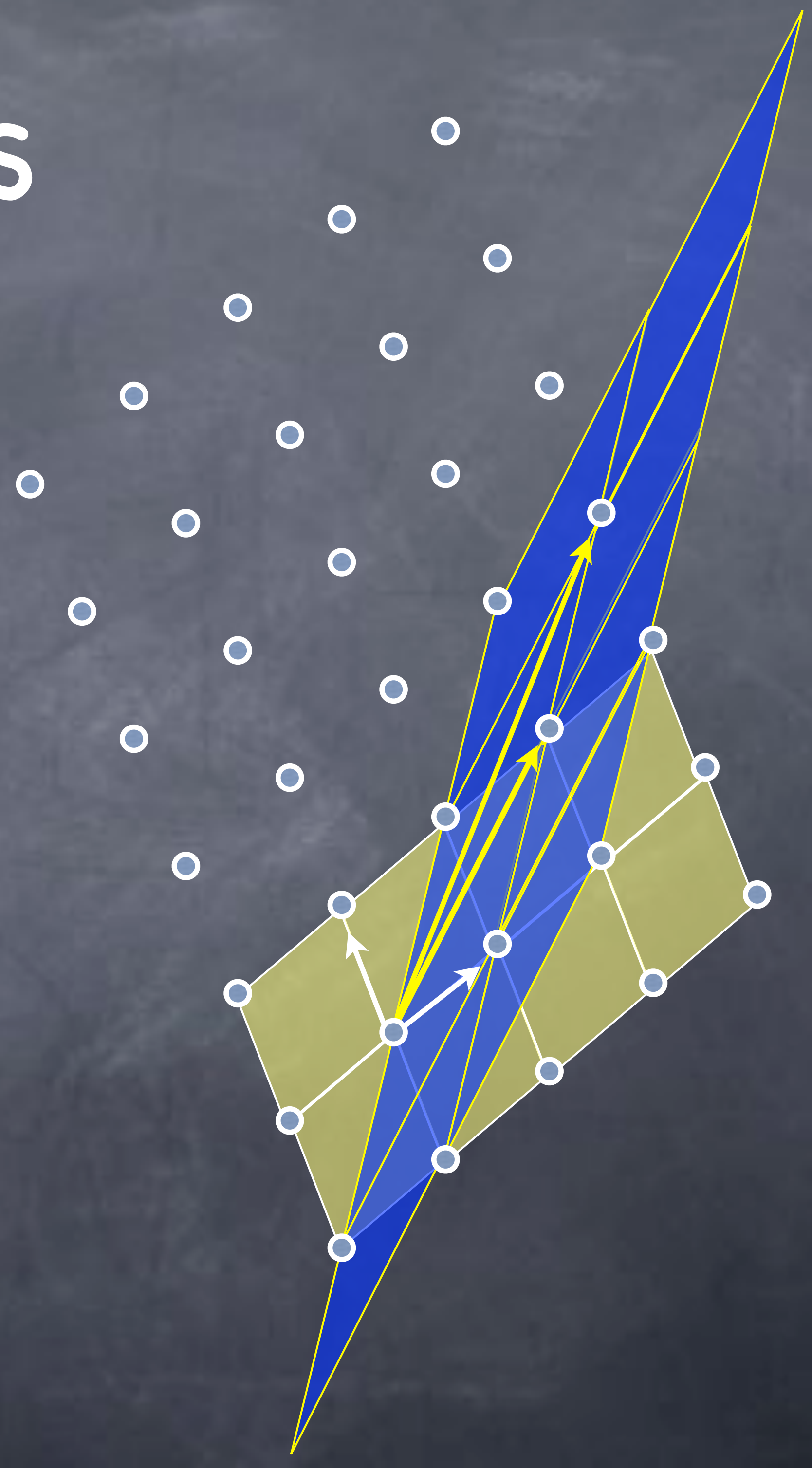

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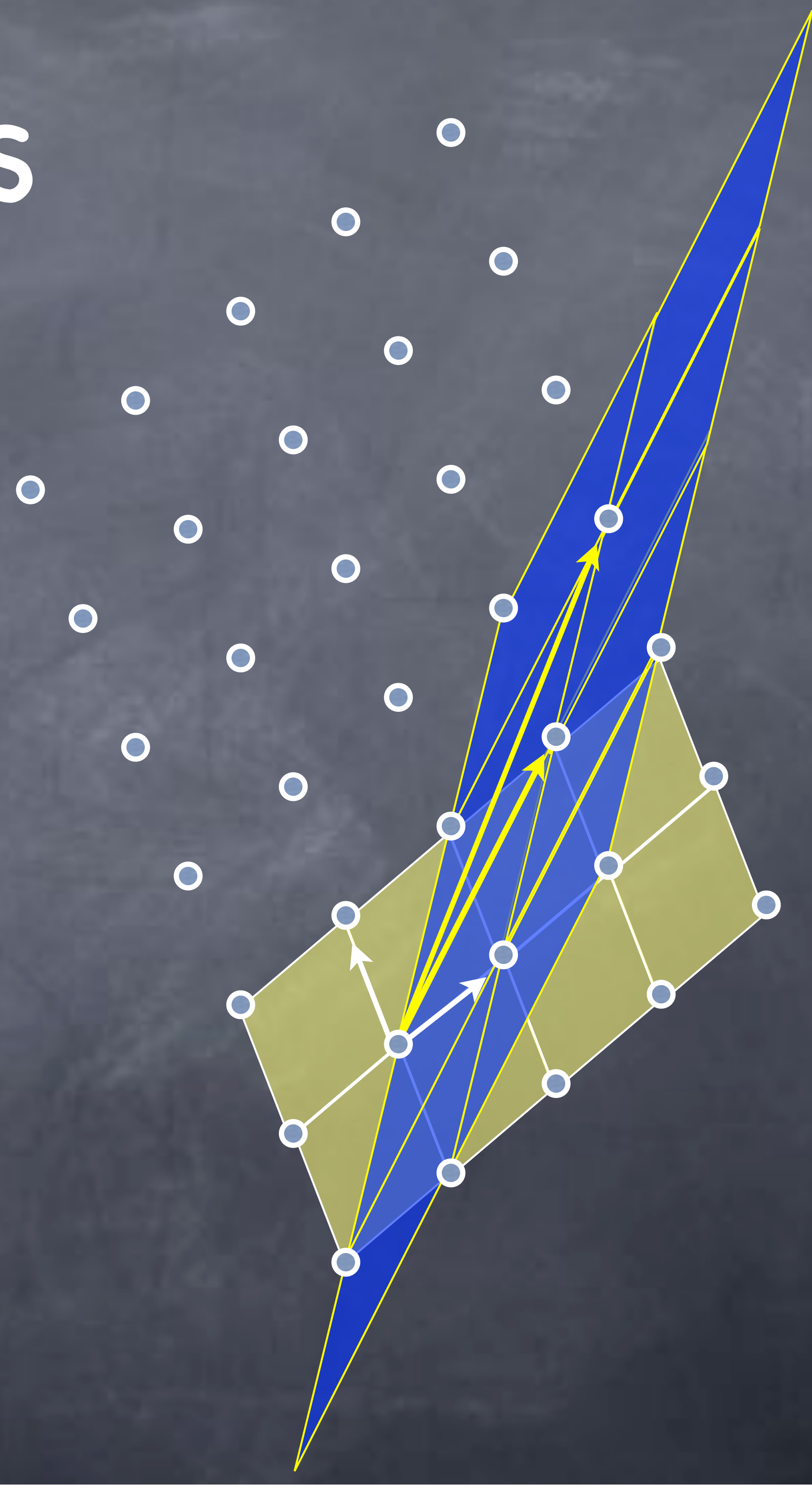
  - Believed to hold even against quantum computation: "Post-Quantum Cryptography"



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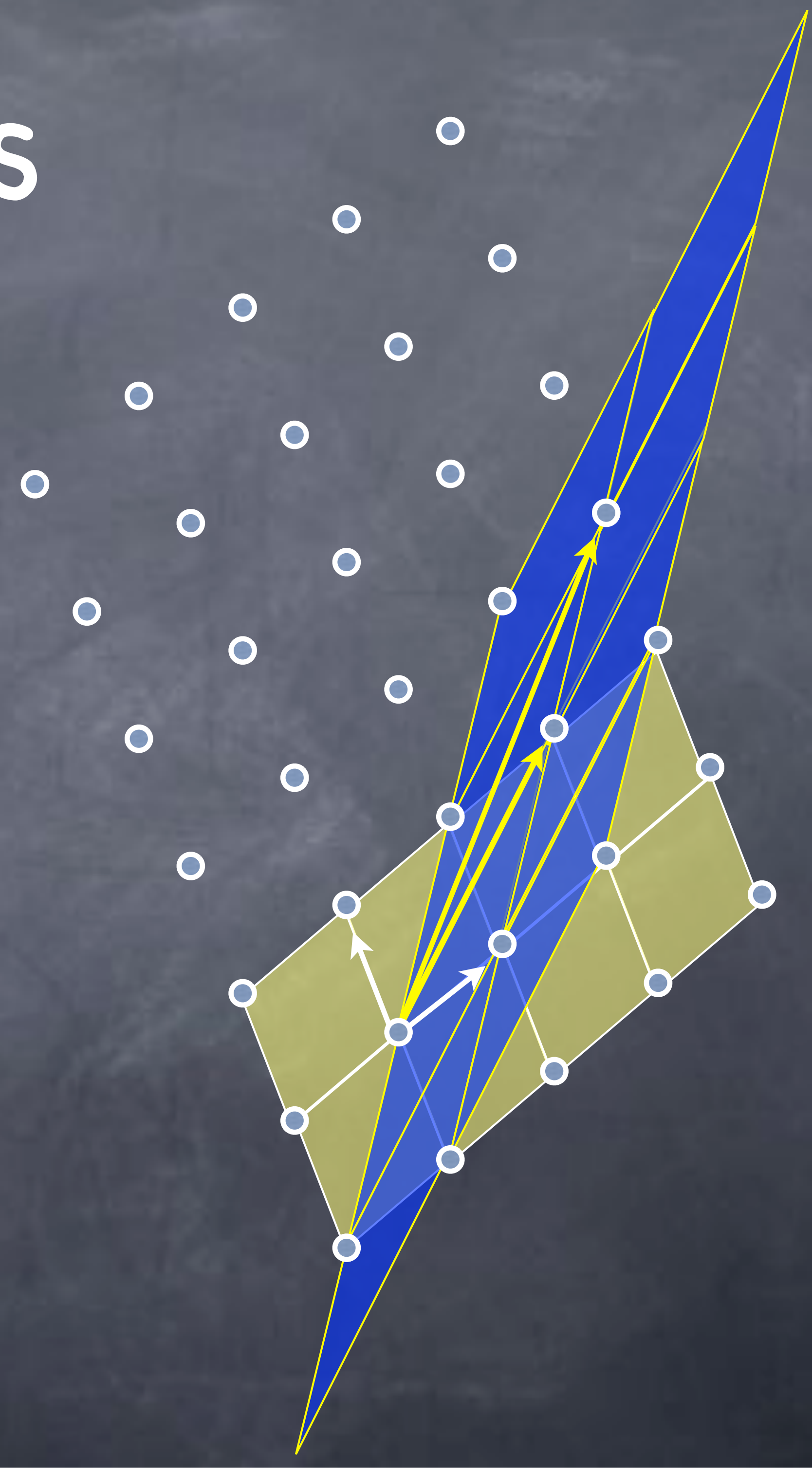

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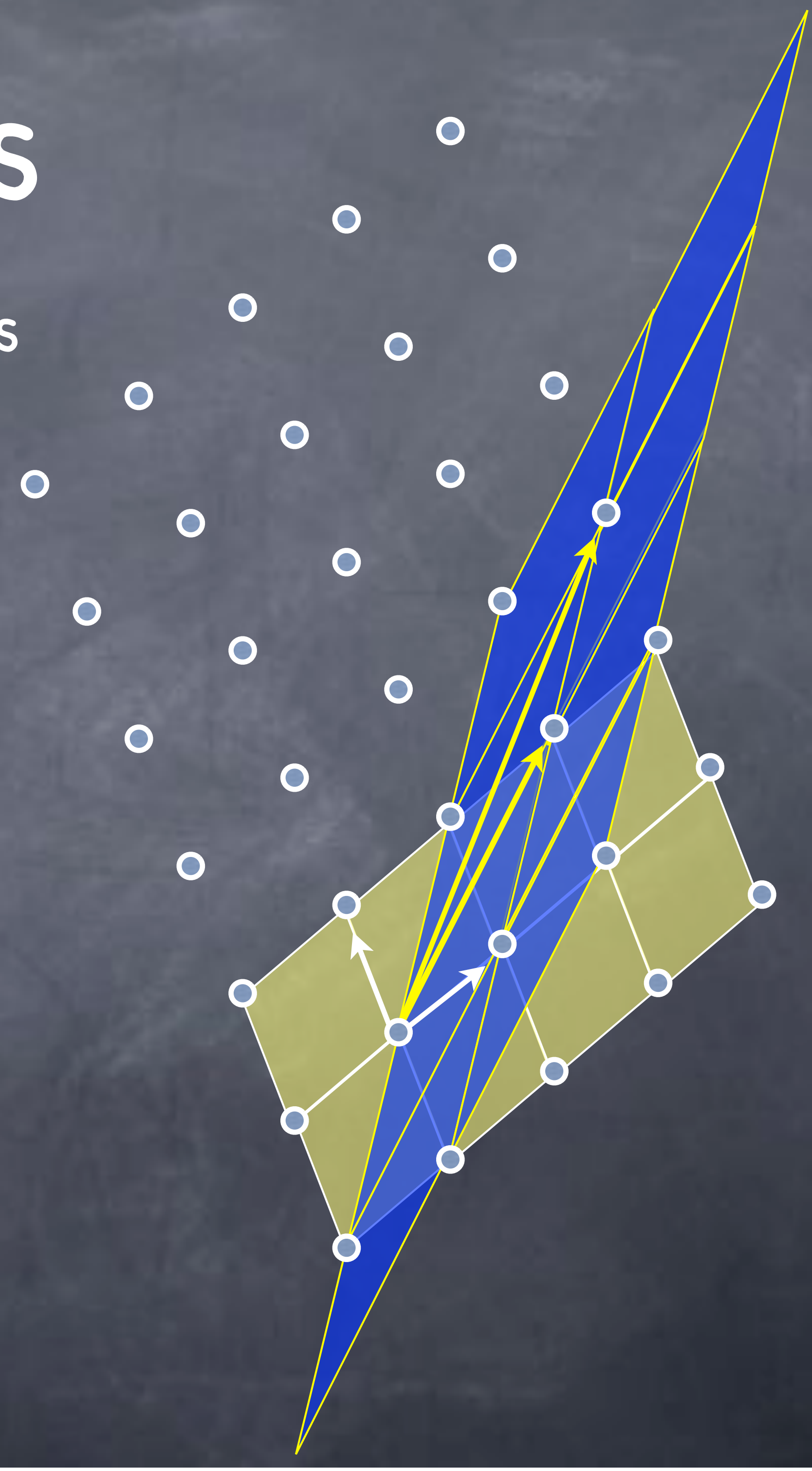

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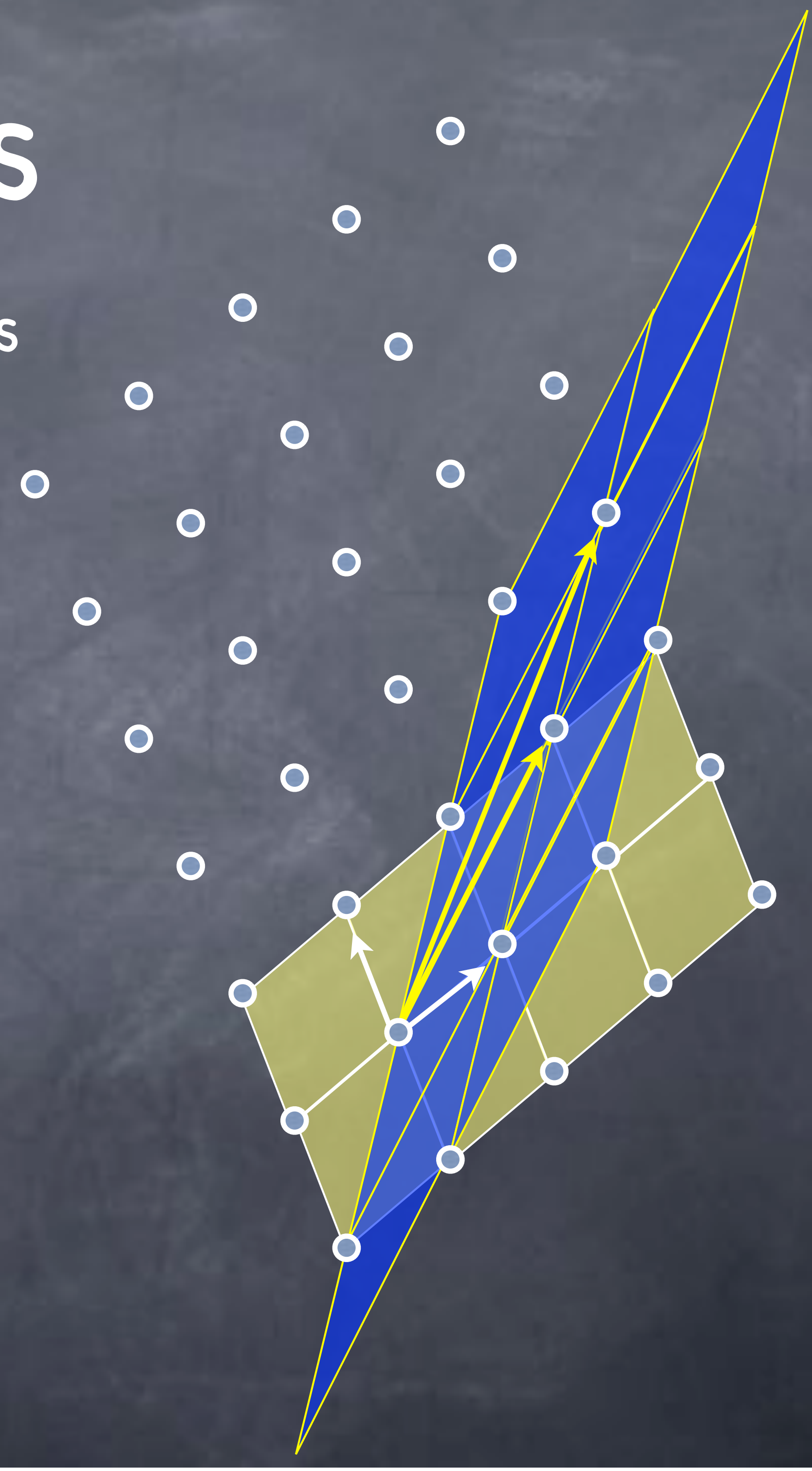

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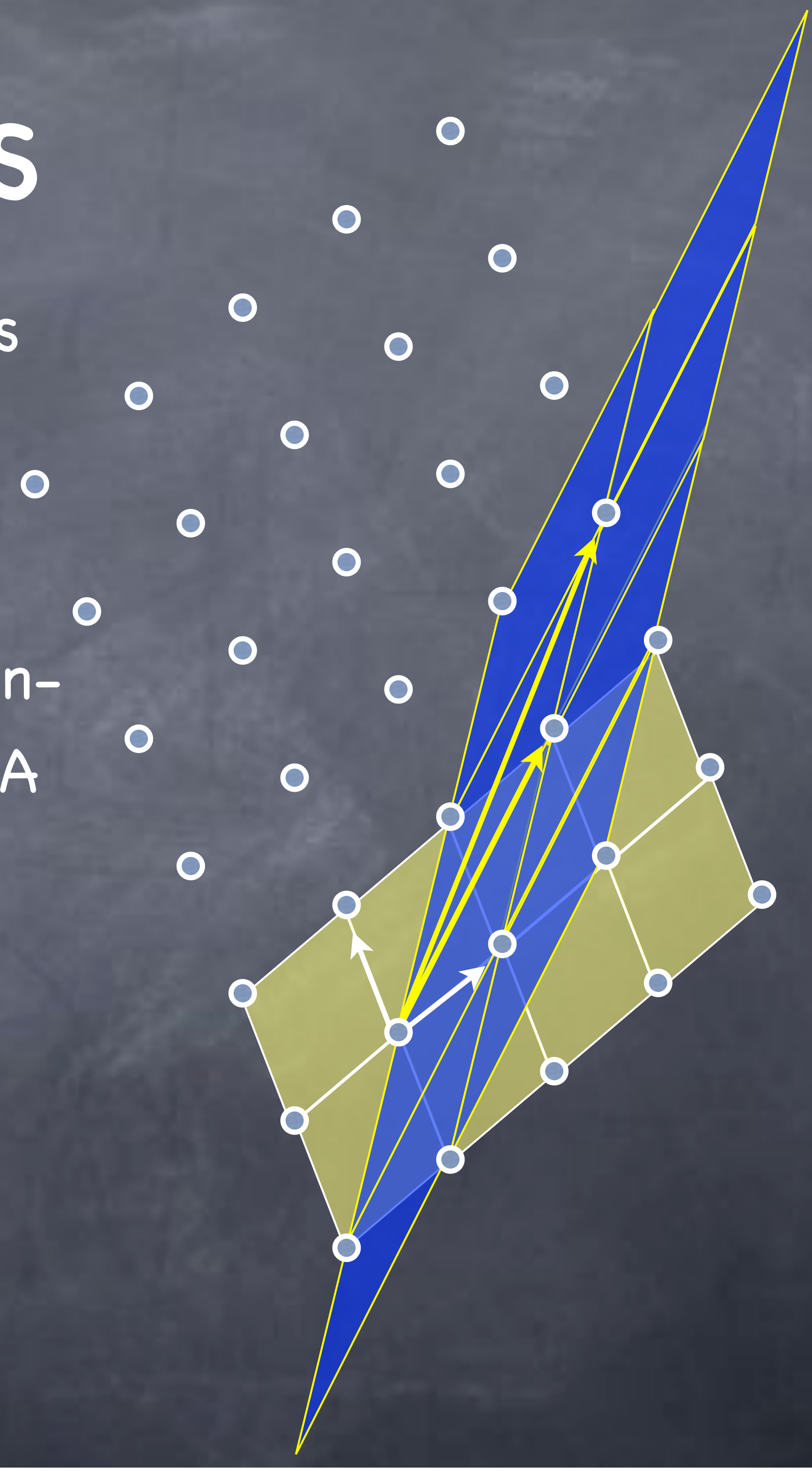

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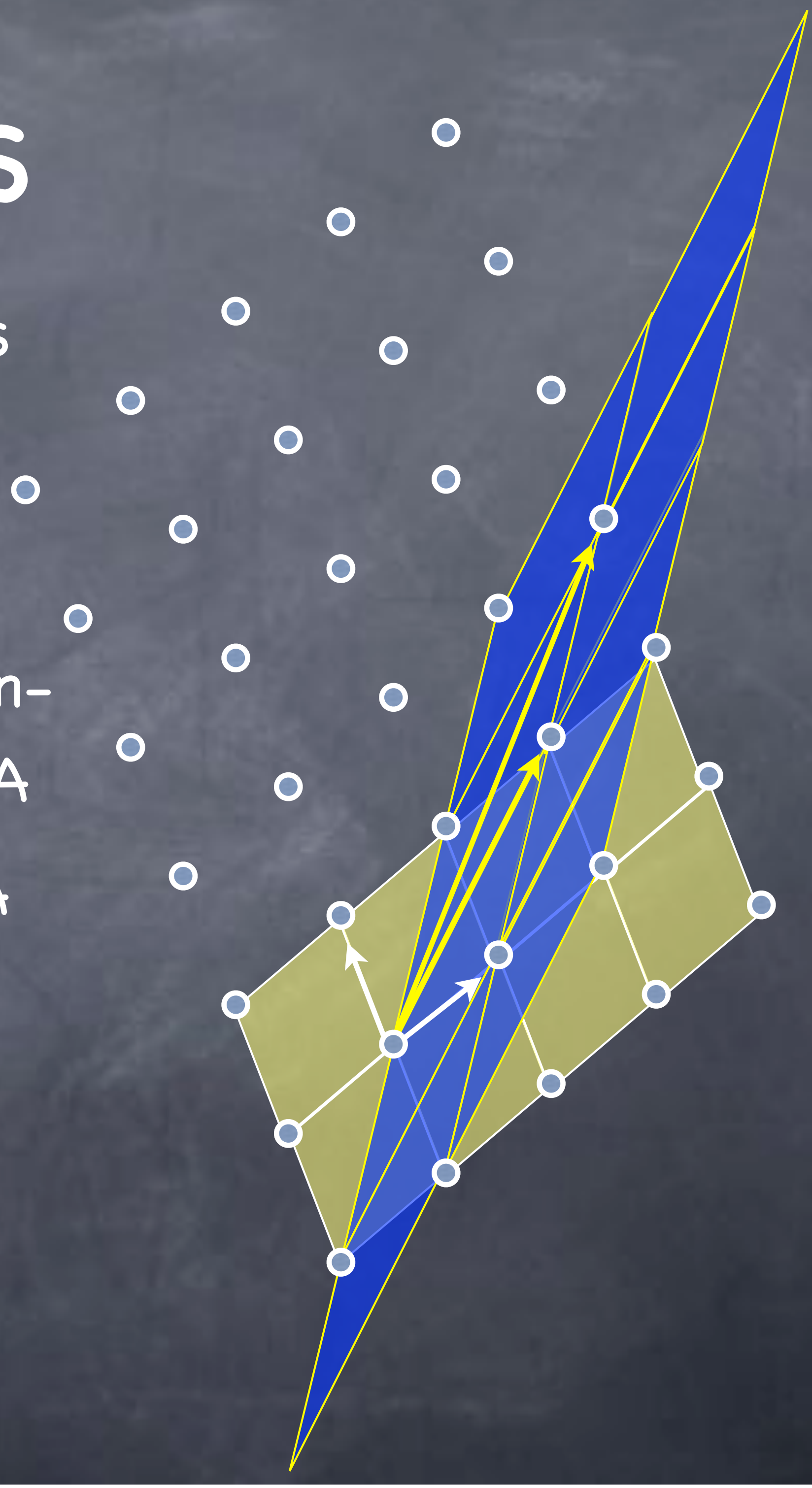

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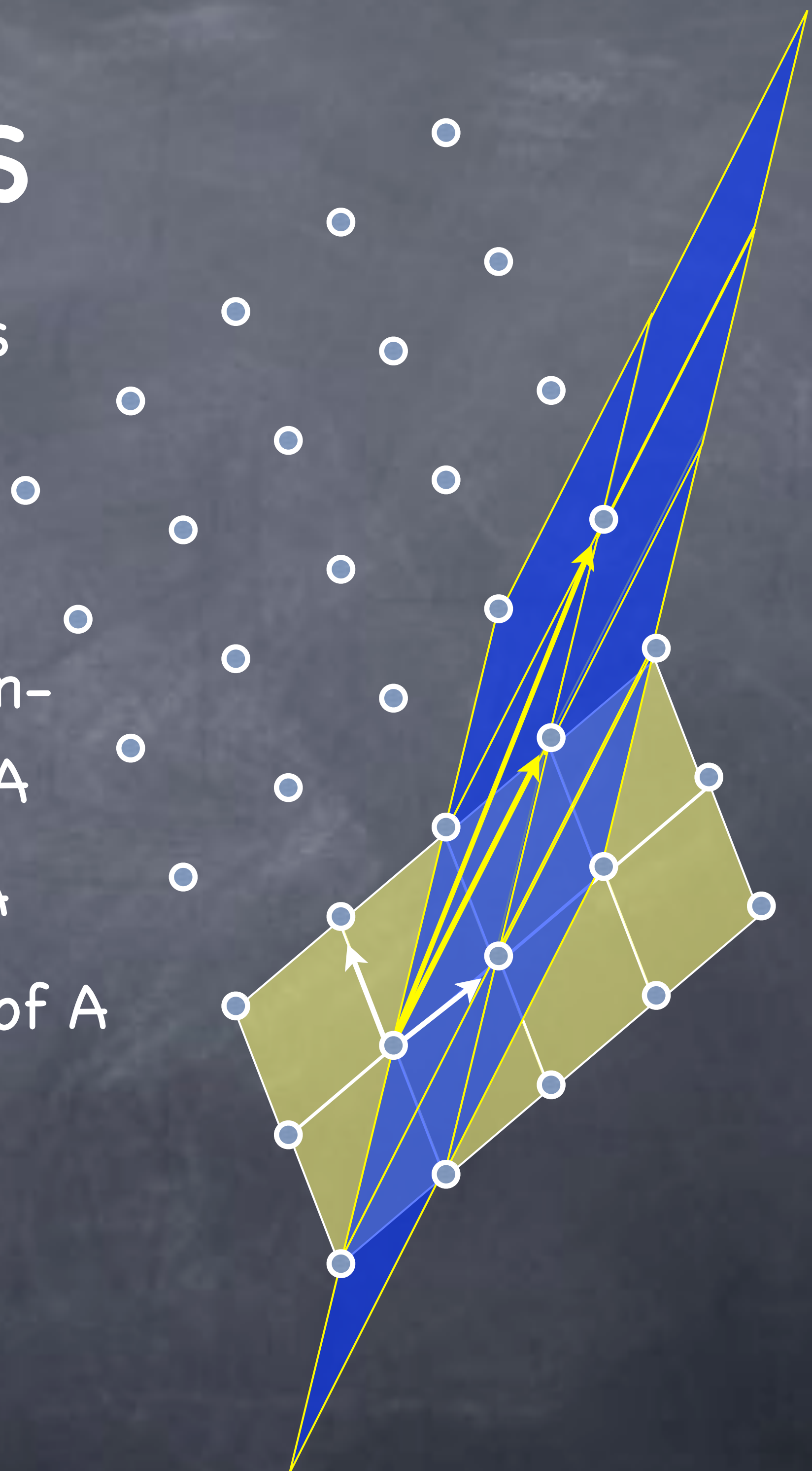

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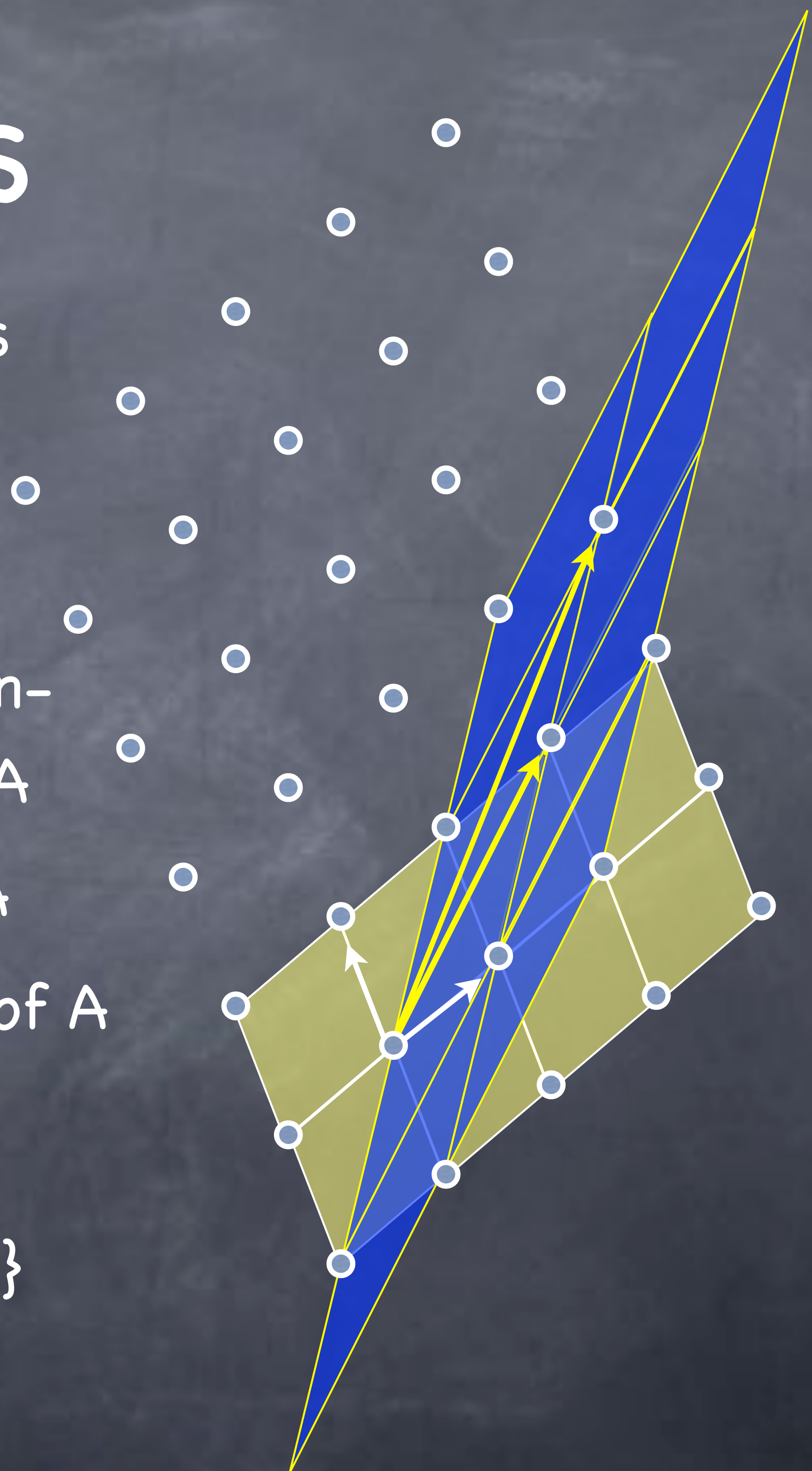

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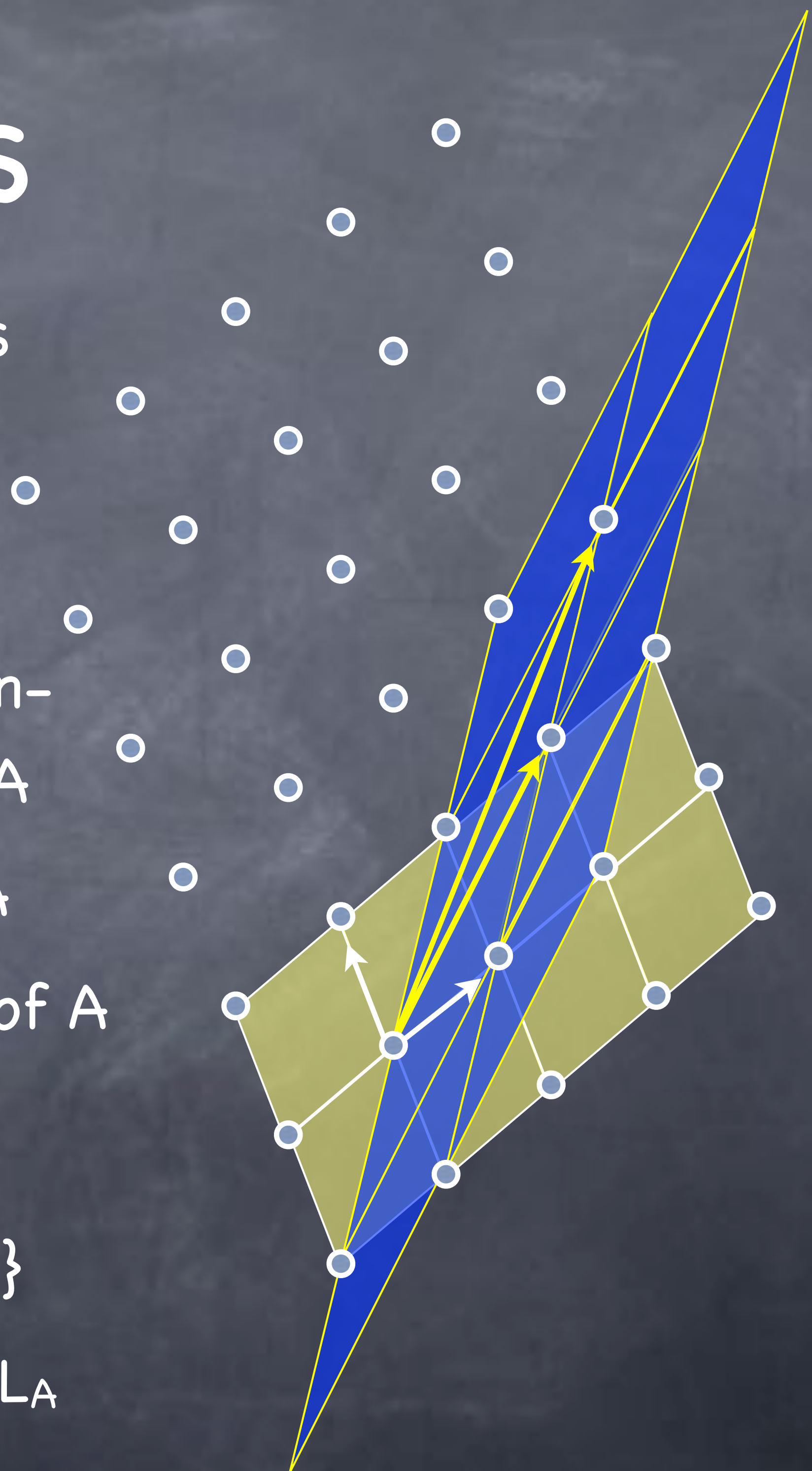
  - e.g. $(L_A)^* = 1/q\ L_A^\perp$ and $(L_A^\perp)^* = 1/q\ L_A$



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- Shortest Independent Vector Problem (SIVP): Find n independent vectors minimizing the longest of them

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NP-hard		in co-NP	in P

γ: 1 $\quad 2^{(\log n)^{(1-\varepsilon)}}$ $\quad \sqrt{n}$ $\quad n$ (crypto regime) $\quad 2^n$

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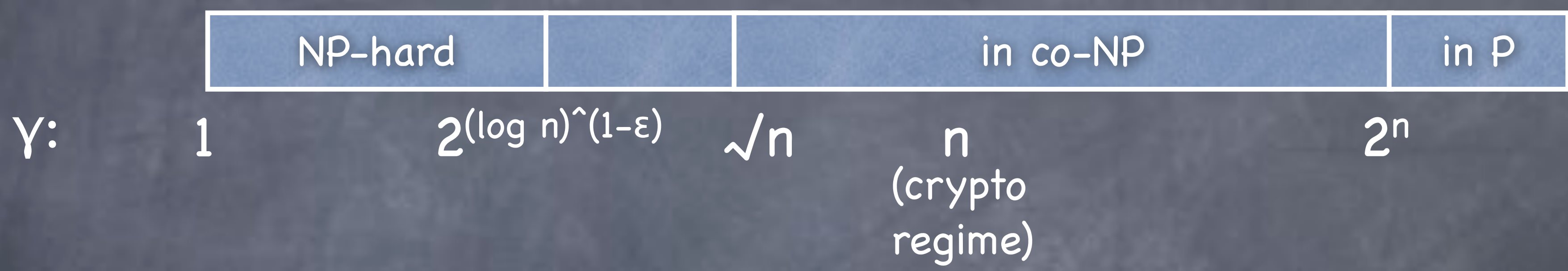

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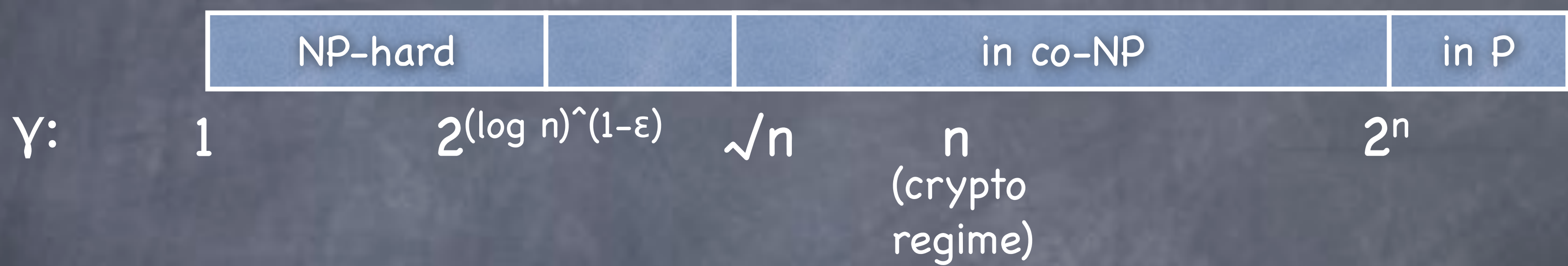

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  - Crypto requires average-case hardness
  - For many lattice problems average-case hardness assumptions are implied by worst-case hardness of related problems (but at regimes not known to be NP-hard)

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  - Turns out to be a very useful assumption

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- If sufficiently compressing (say by half), a CRHF is also a OWF

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  - Considered hard when A is chosen uniformly at random
  - This is as hard as solving certain lattice problems in <u>the worst case</u> (i.e., with good success probability for <u>every instance</u> of the problem)

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- Security depends on worst-case hardness of same problems as before, but when restricted to ideal lattices

Public-Key Encryption

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- Conjectured to be CPA secure. No security reduction known to simple lattice problems

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- LWE also used for CCA secure PKE

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    - Quadratic key size/signing complexity (unlike NTRUSign)

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  - Recall: one-time signatures can be augmented to full-fledged signatures using a CRHF (in fact, a UOWHF)

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  - Useful in building "identification schemes" and potentially in other lattice-based constructions

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