Public-Key Cryptography

Public-Key Cryptography

Lecture 6

Public-Key Encryption

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Public-Key Encryption

Diffie-Hellman Key-Exchange, El Gamal Encryption

PKE scheme

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- SKE:
  - Syntax
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    - Enc: $\mathcal{M} \times \mathcal{K} \rightarrow \mathcal{C}$
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  - Security (IND-CPA, PKE version)

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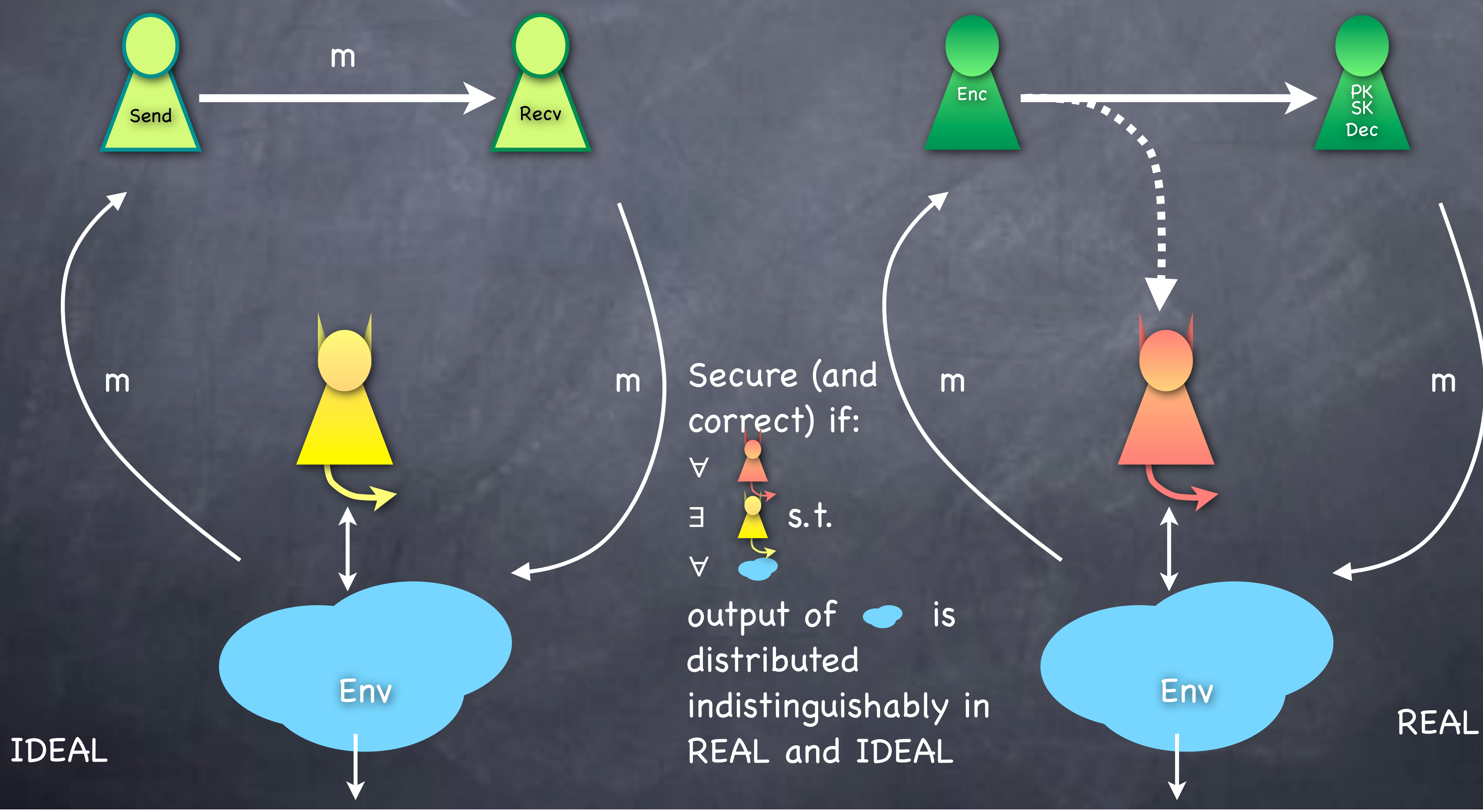
SIM-CPA (PKE Version)



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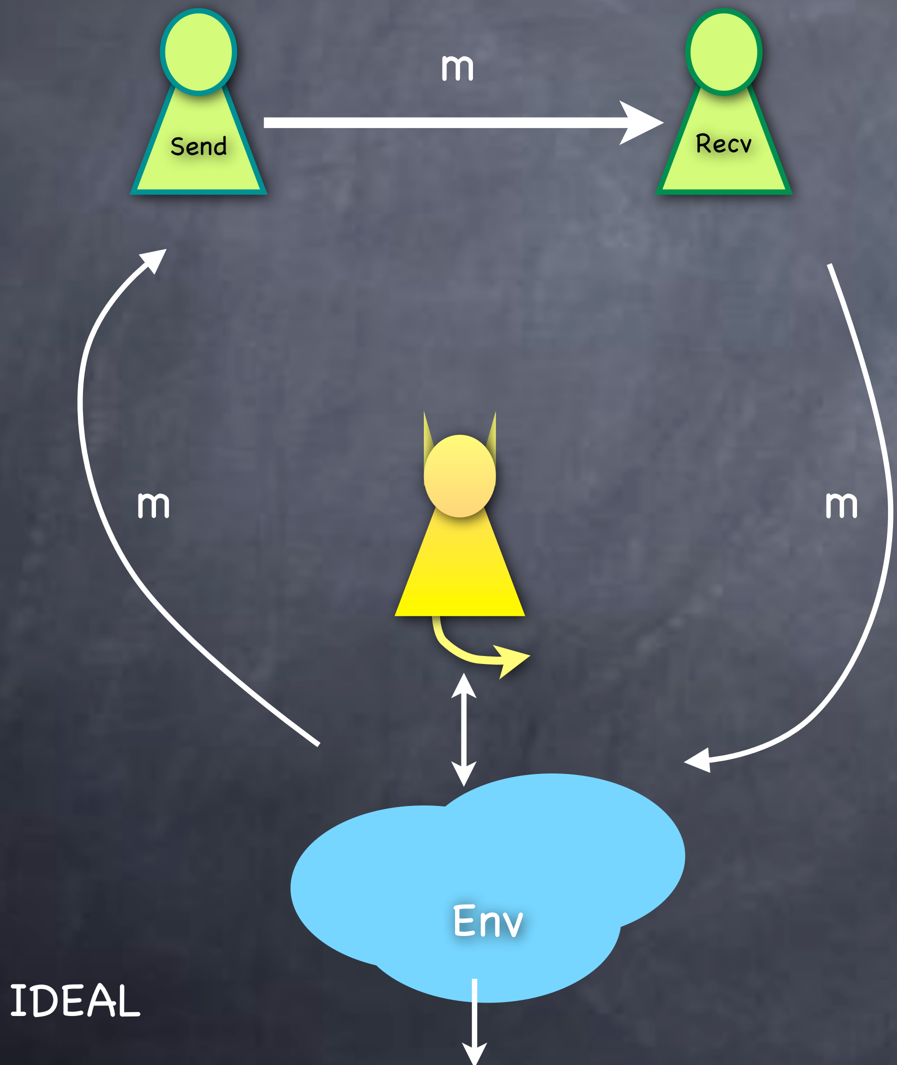


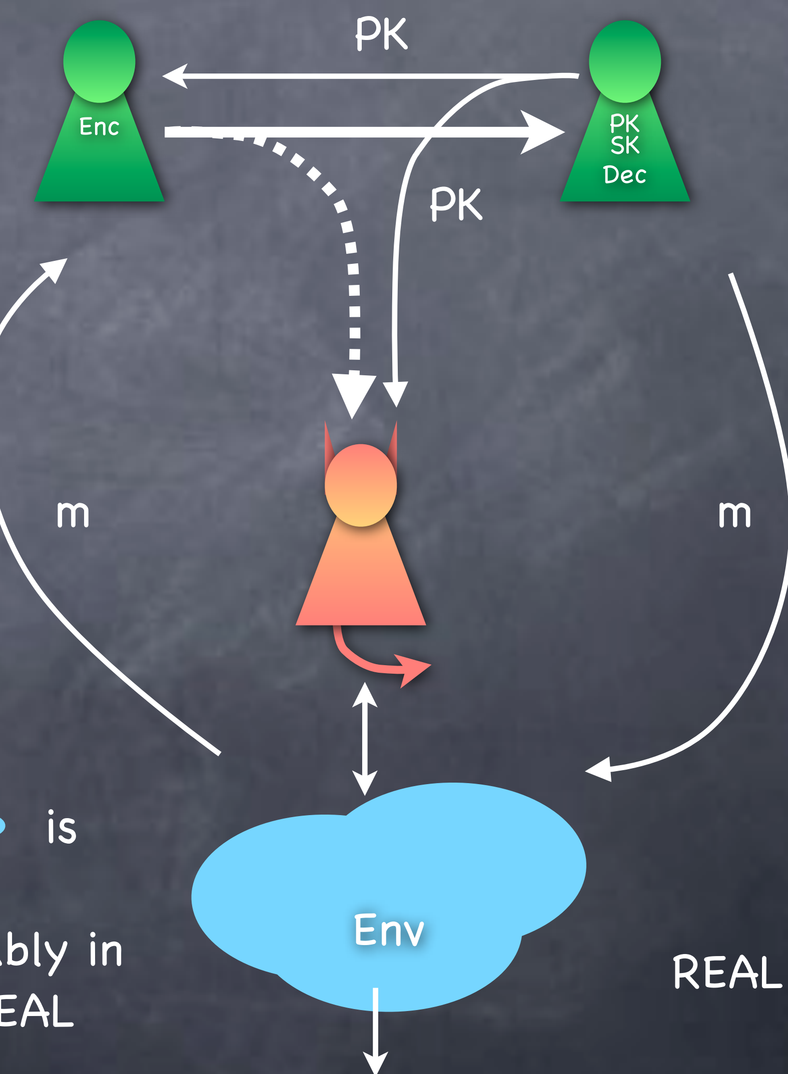
Secure (and correct) if:

∀ 

∃  s.t.

∀ 

output of  is distributed indistinguishably in REAL and IDEAL



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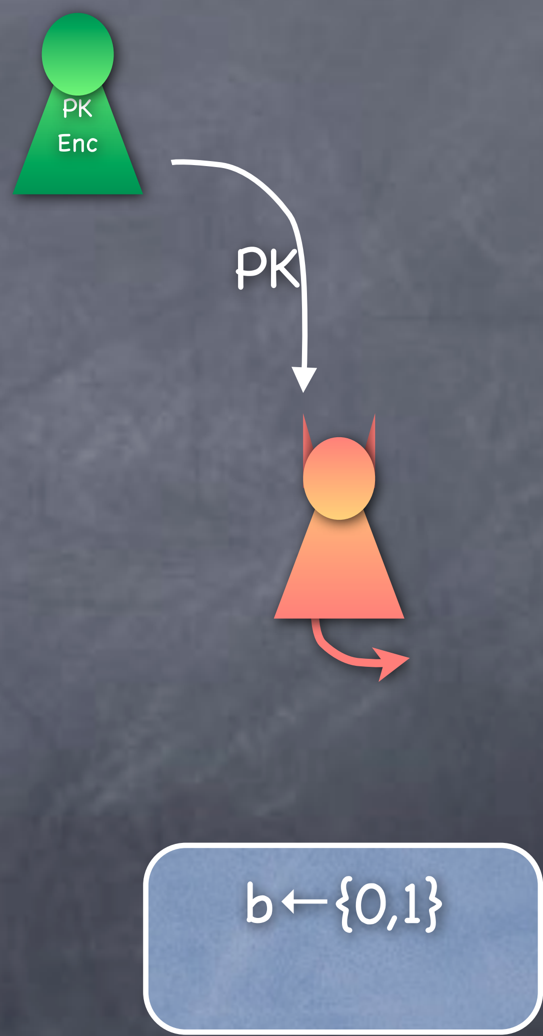

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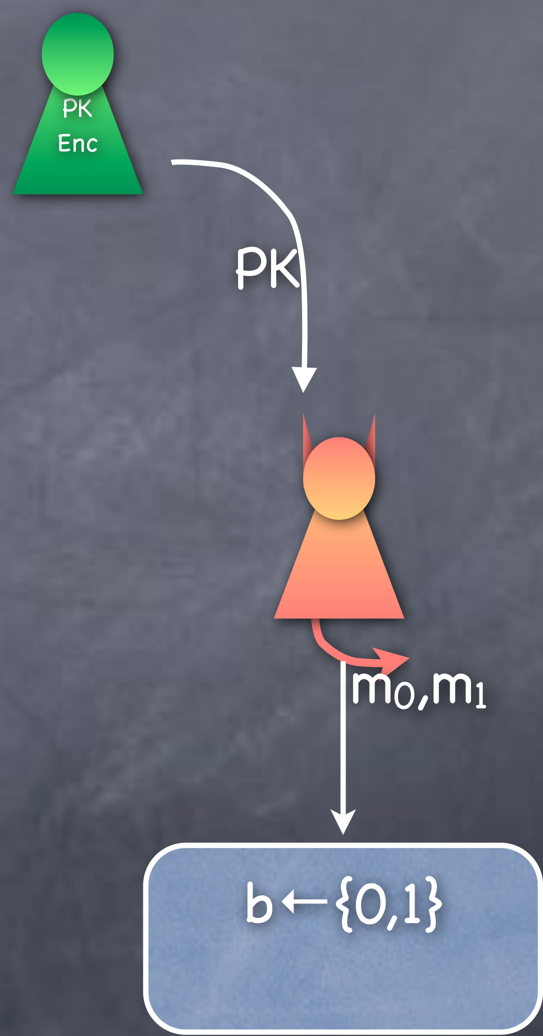

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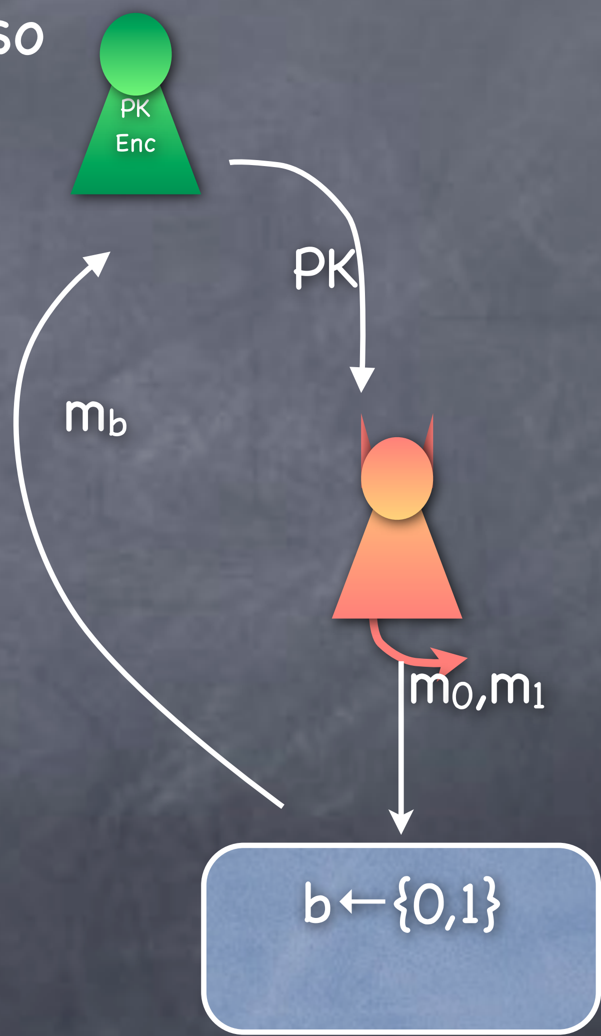

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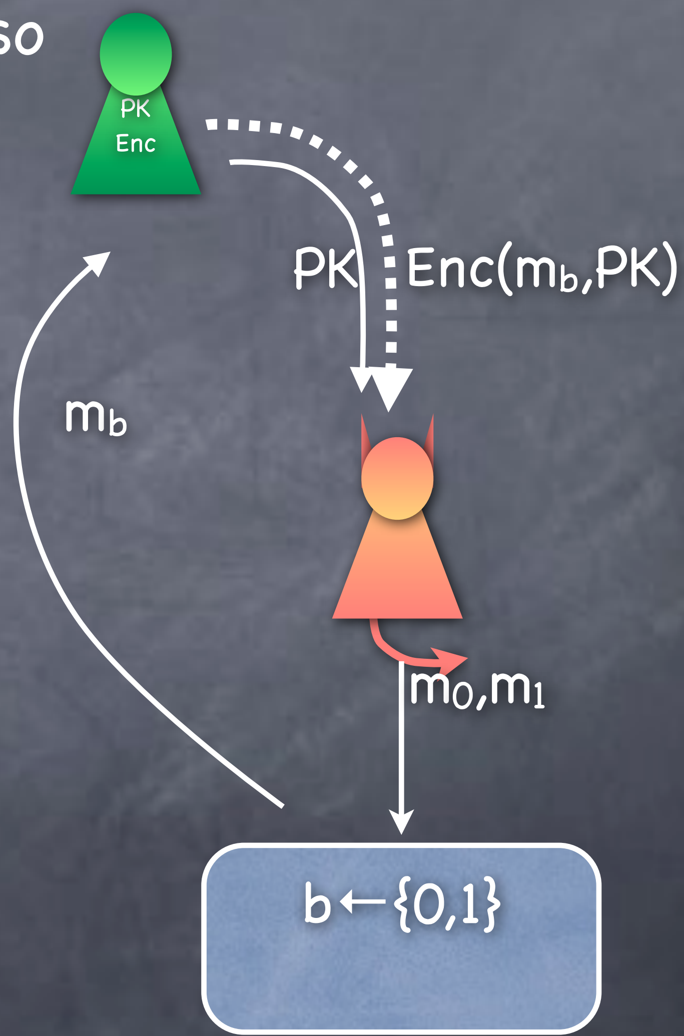

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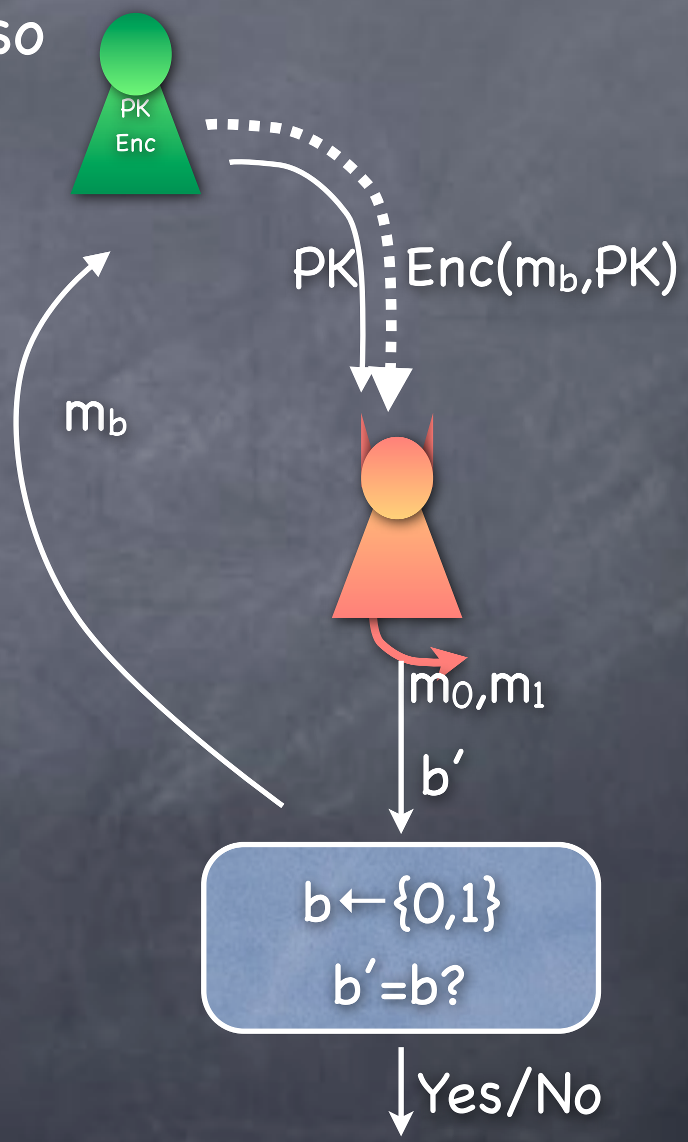

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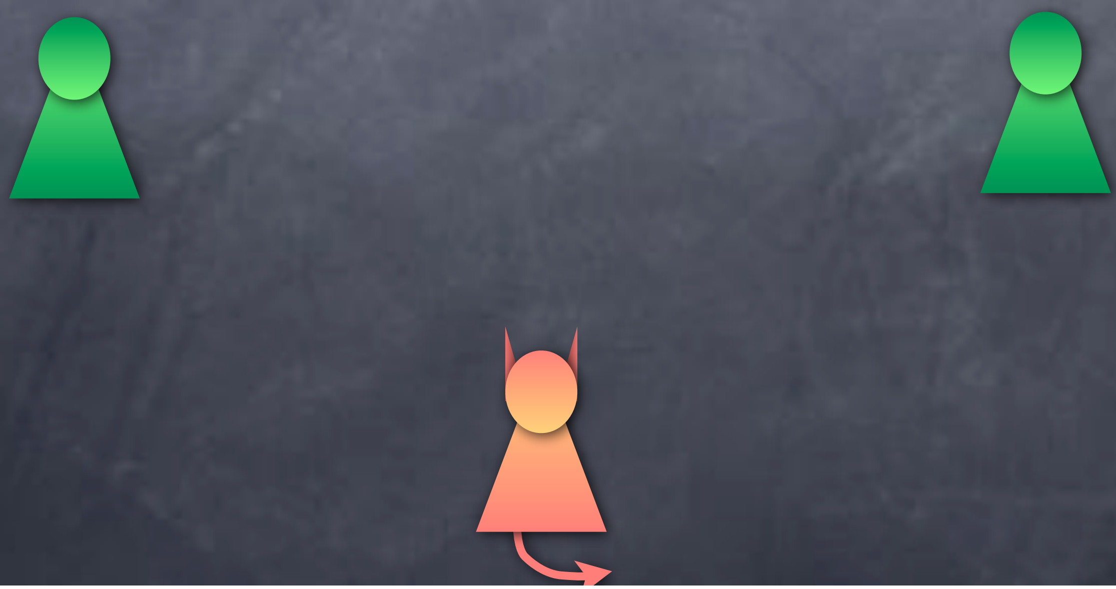

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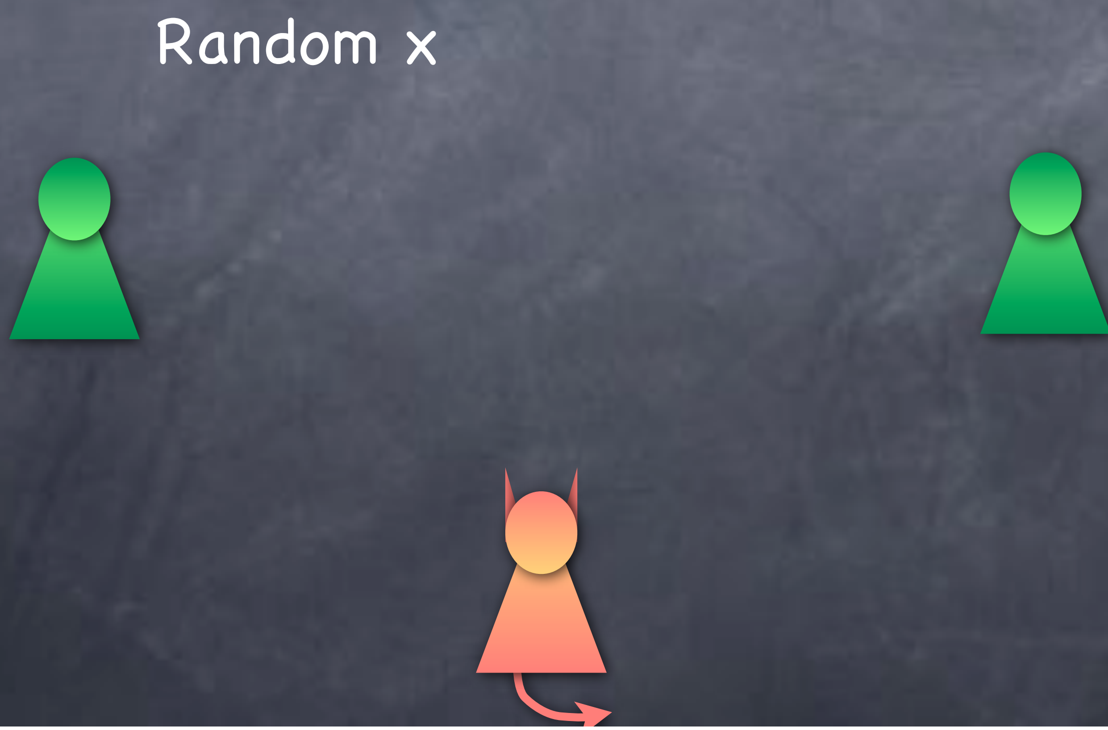

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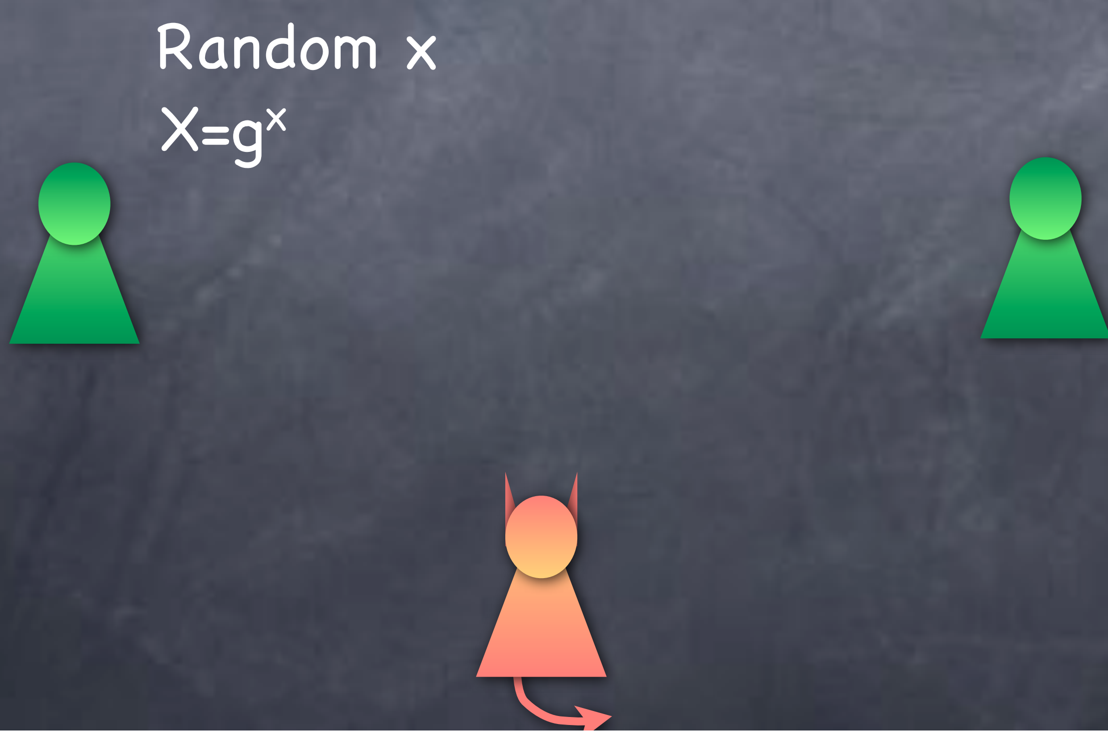

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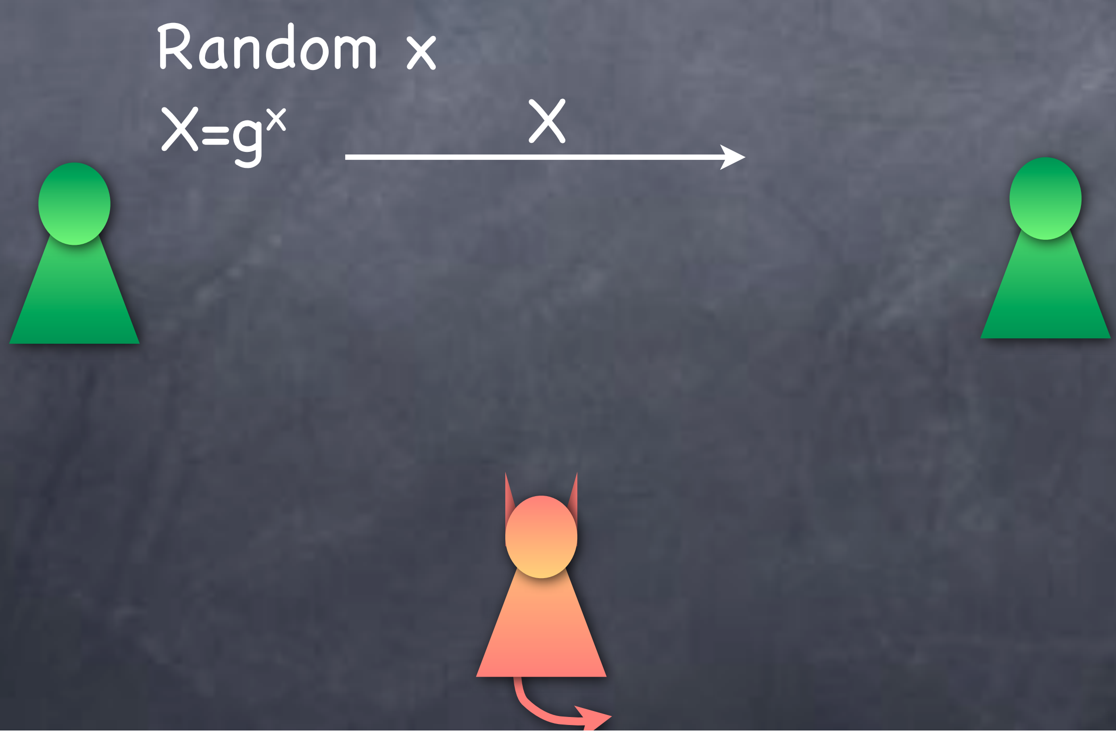

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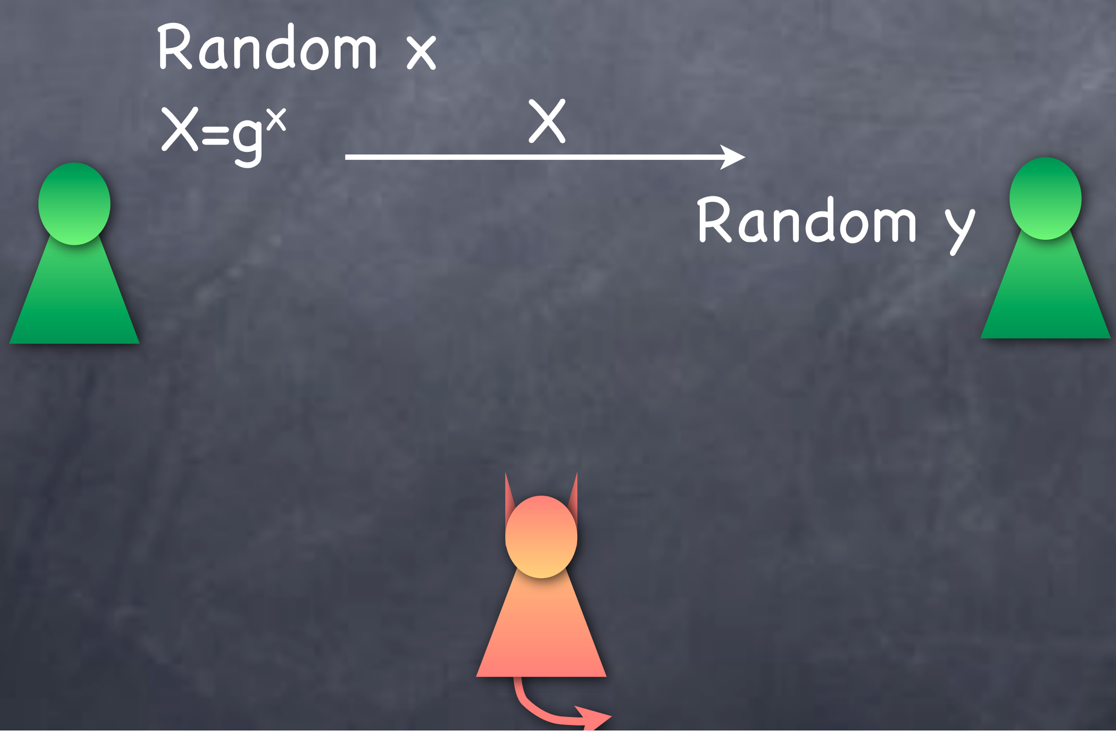

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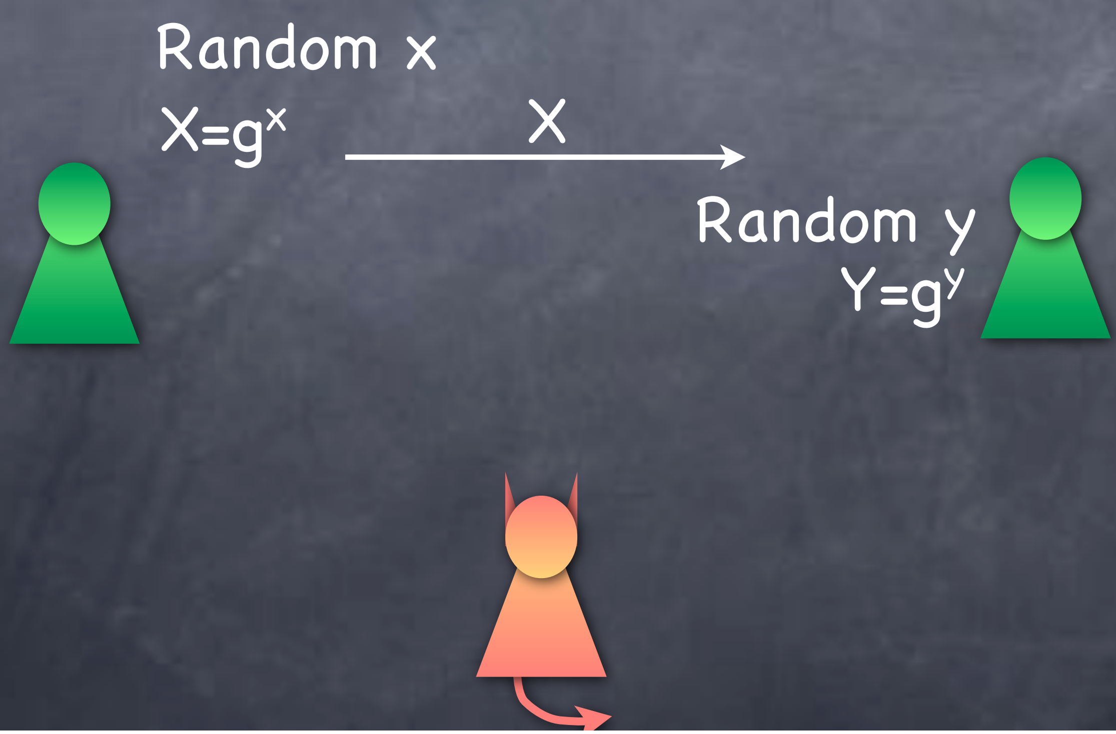

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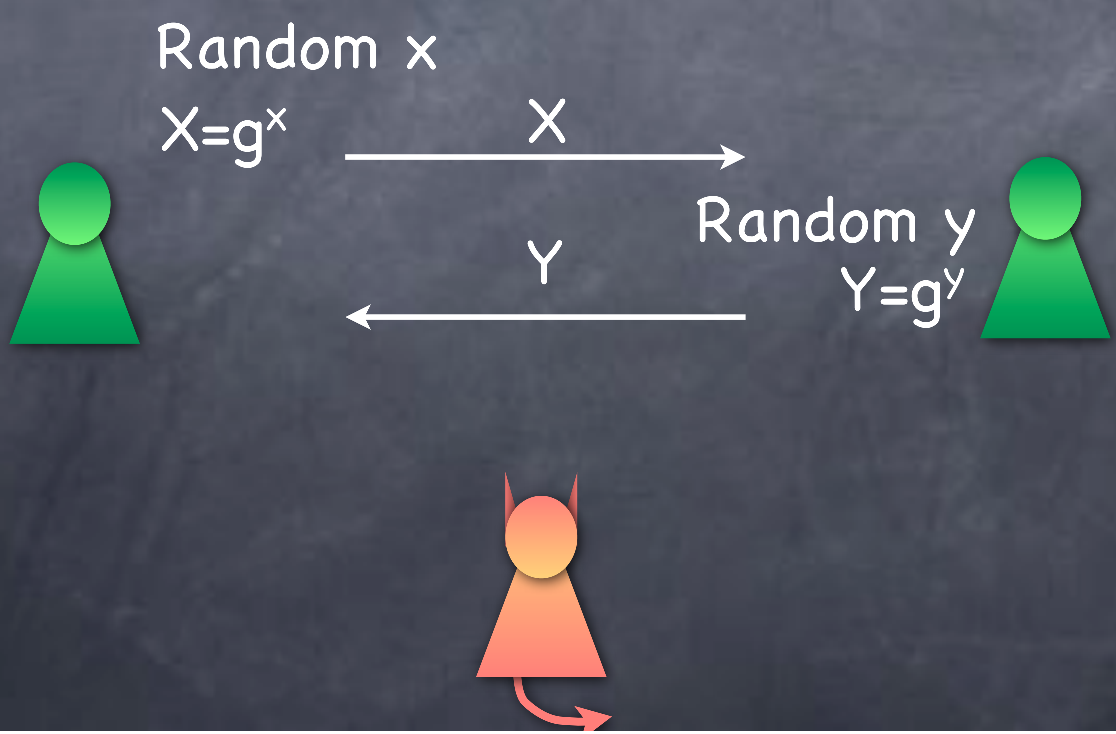

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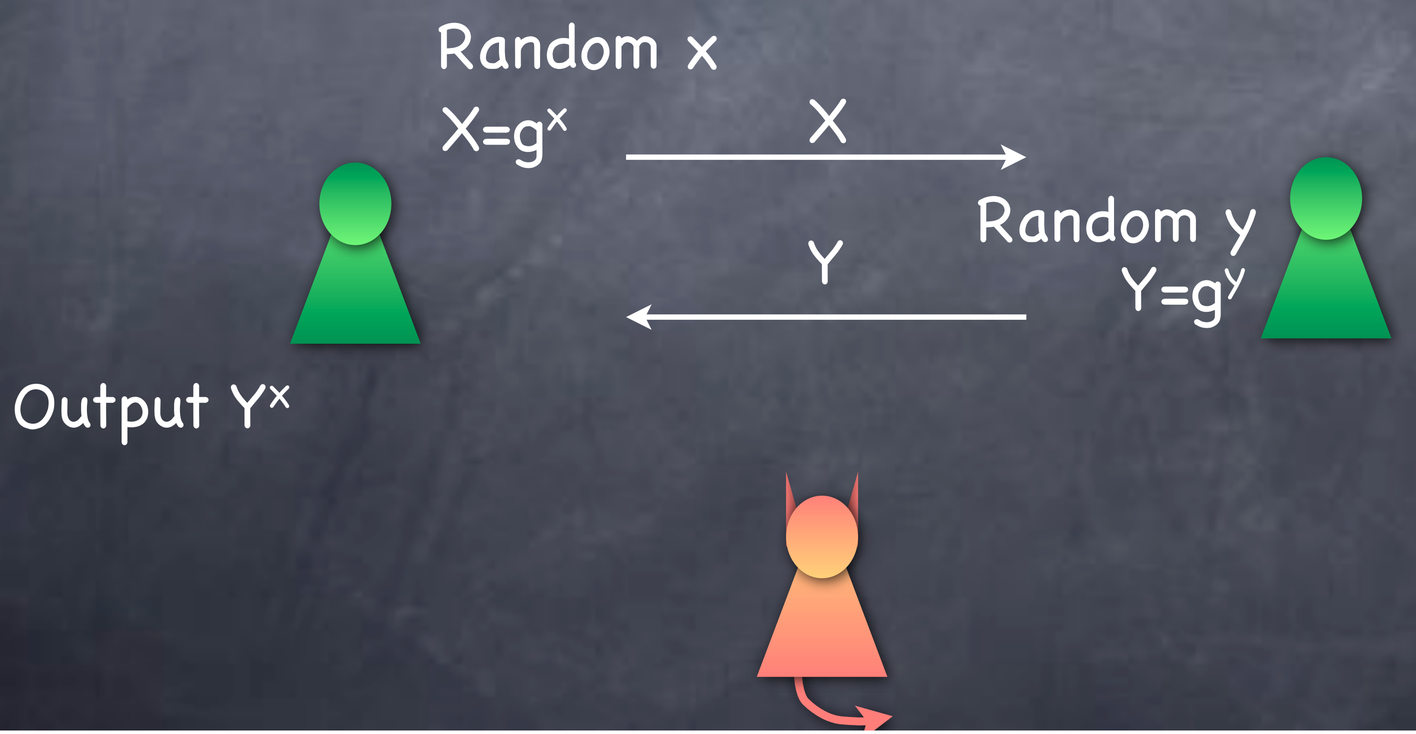

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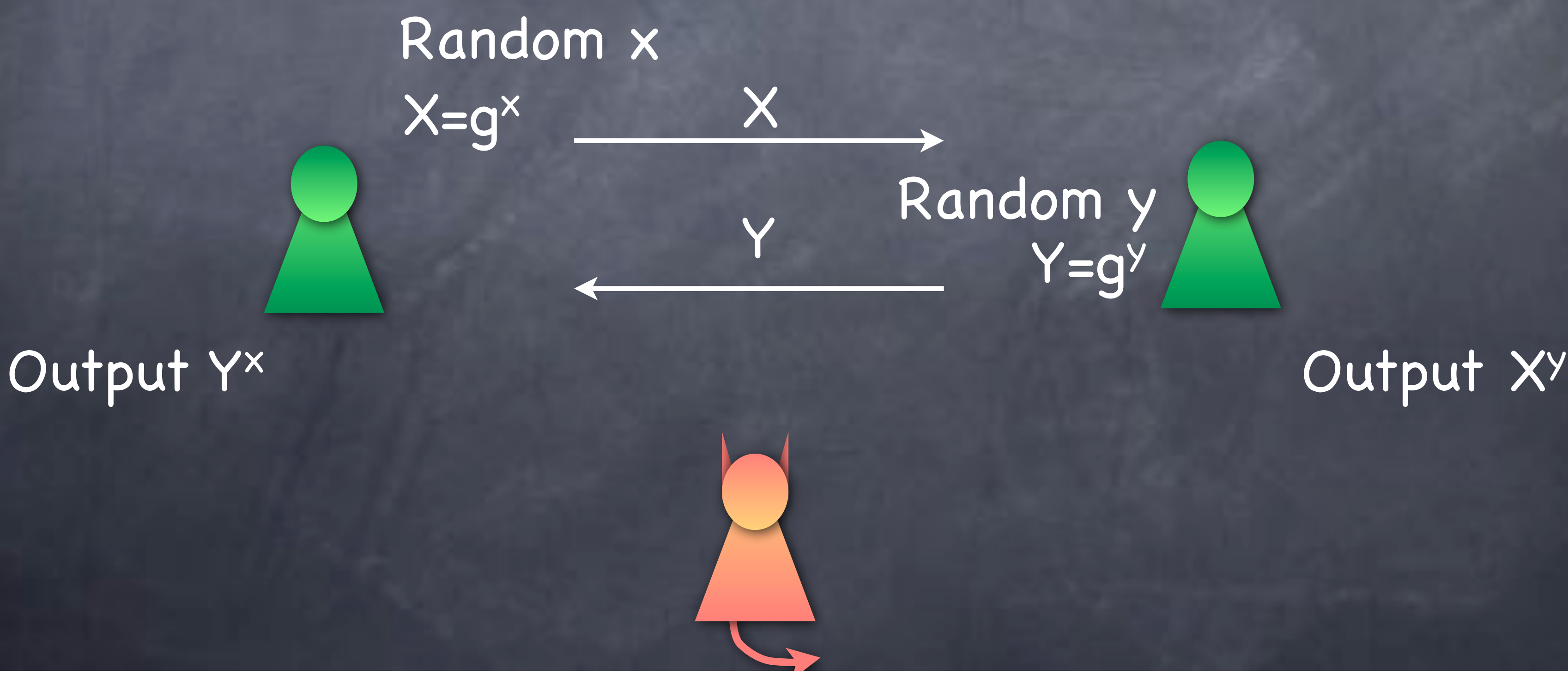

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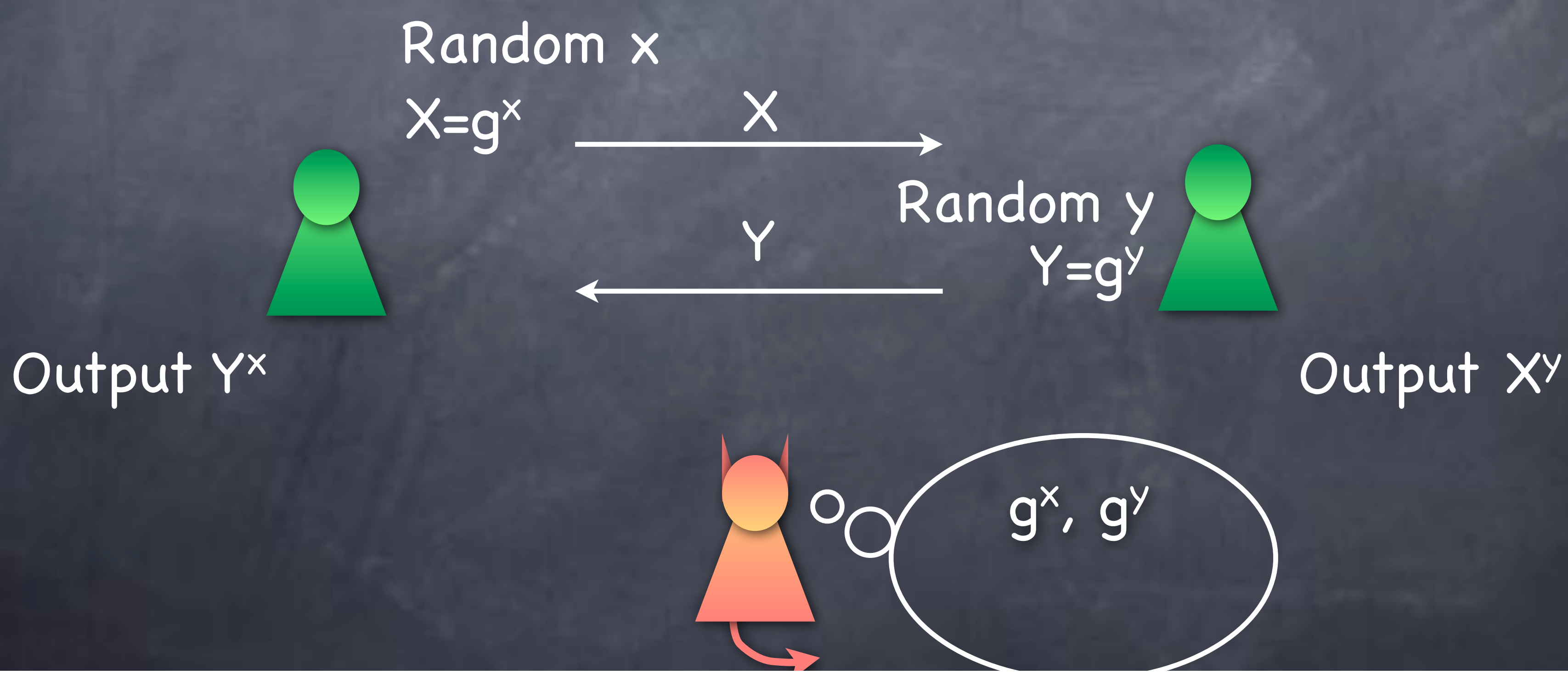

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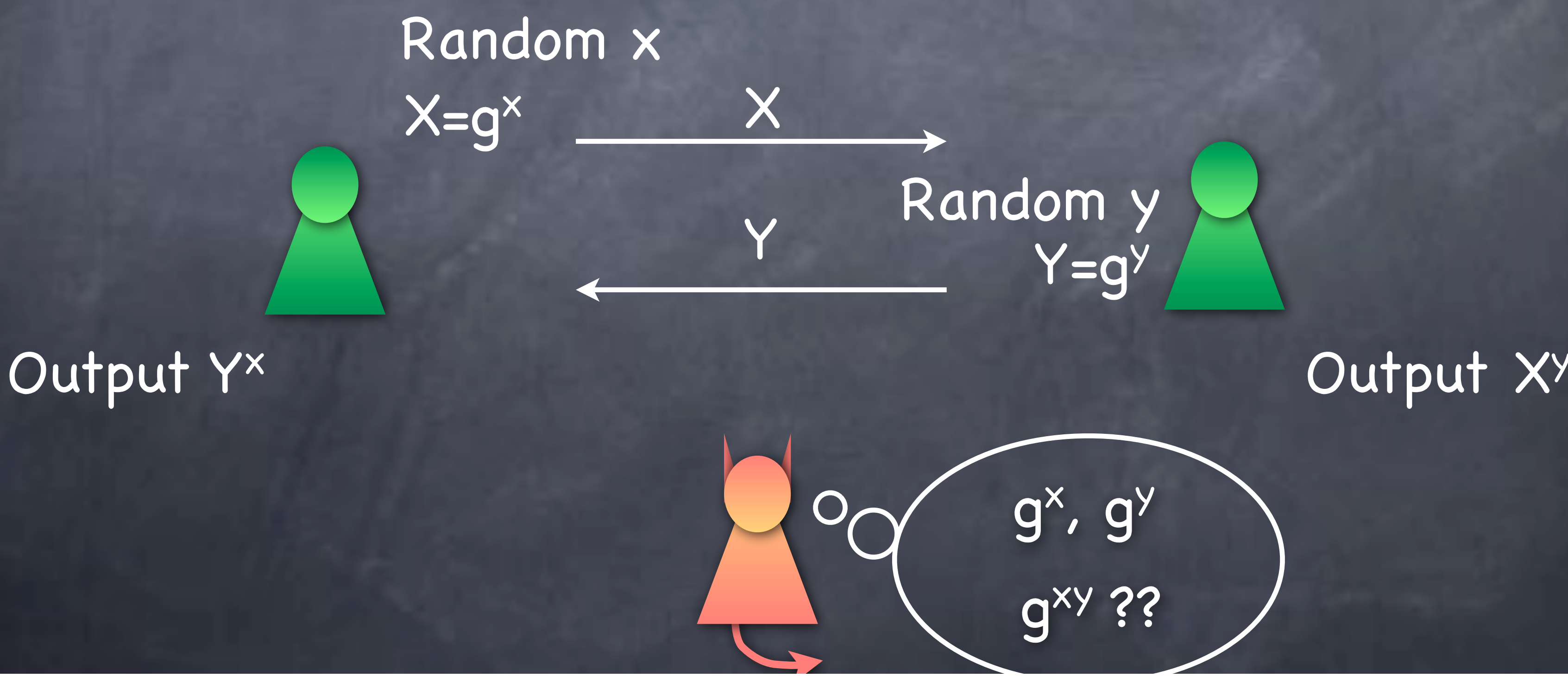

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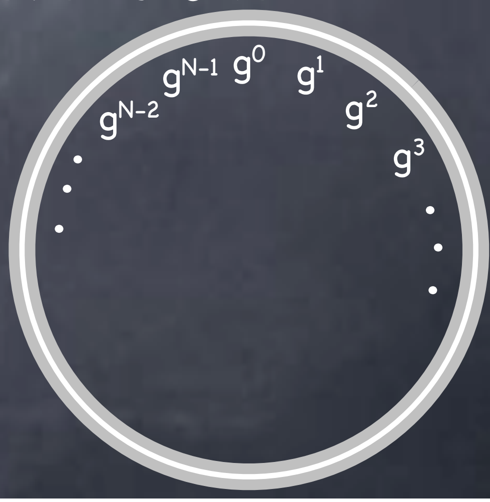

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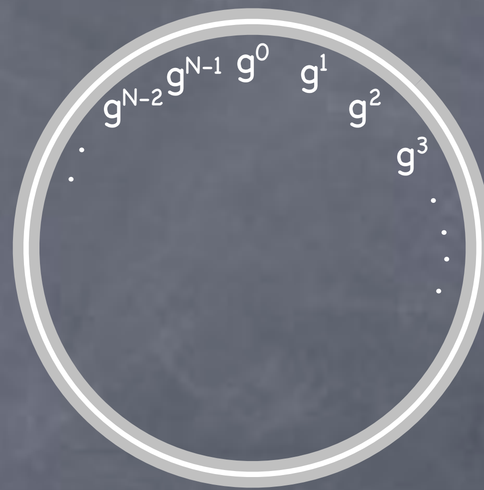

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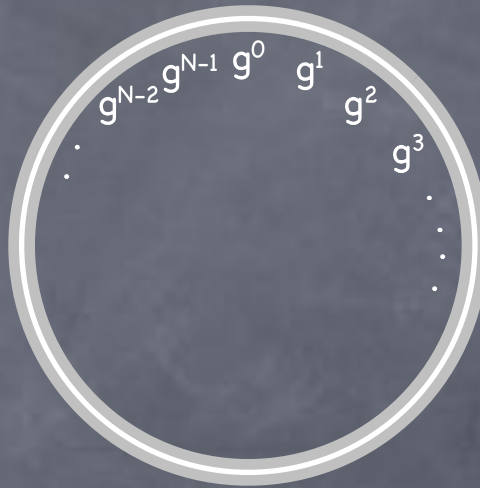

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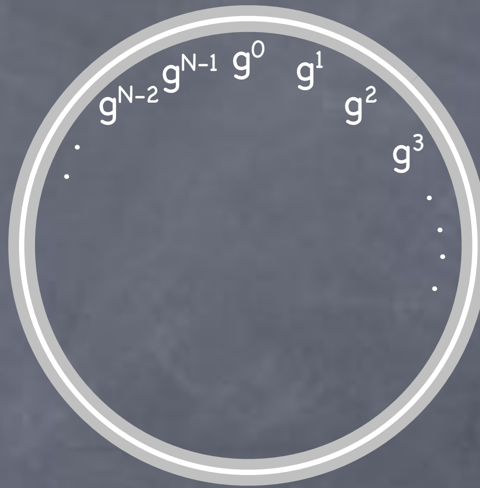

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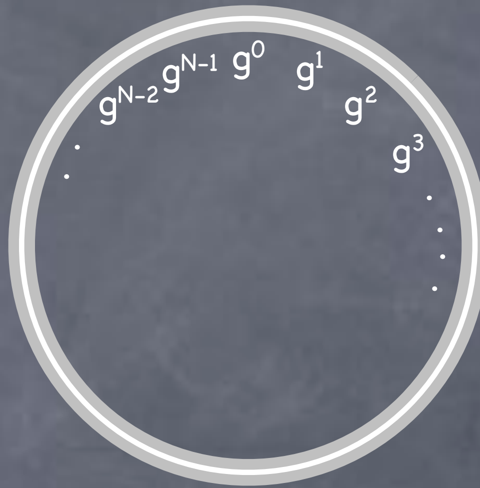

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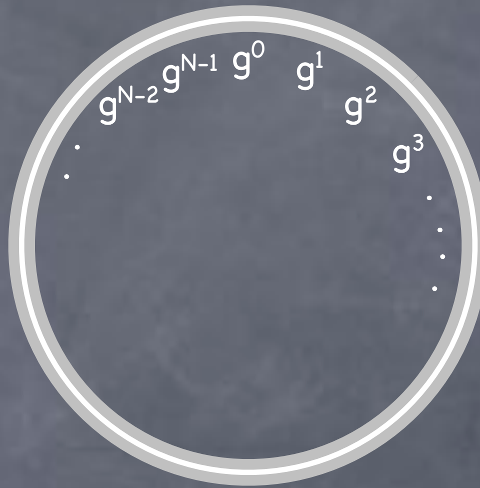

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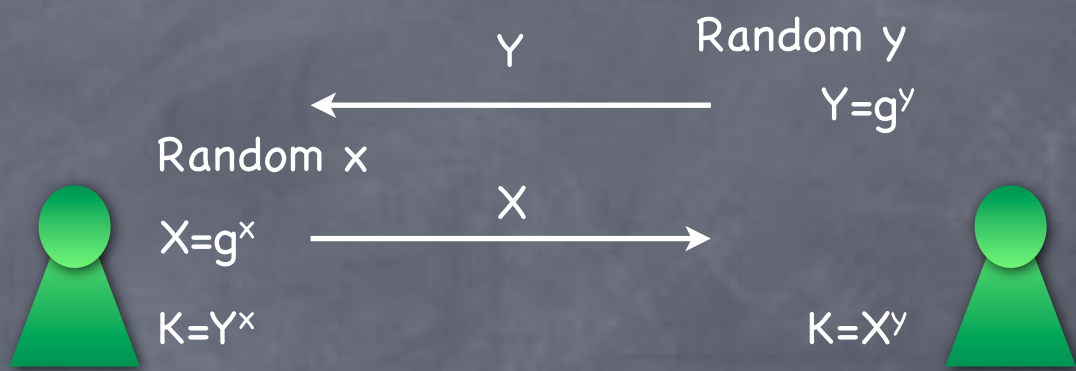

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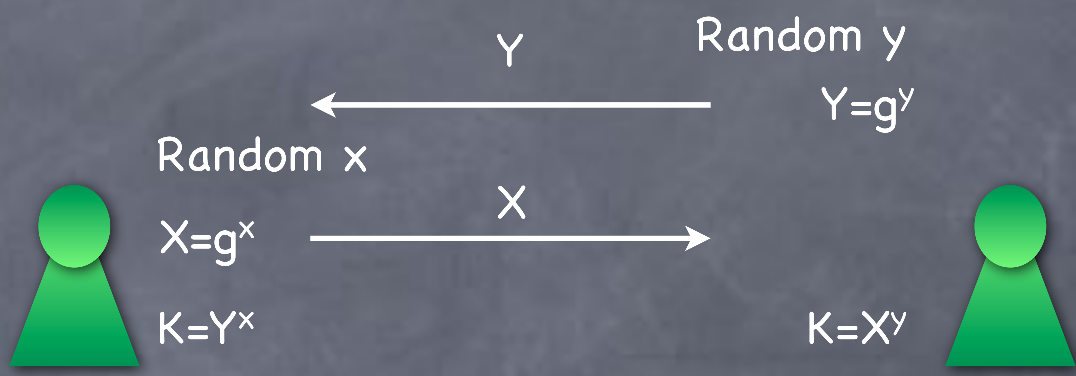

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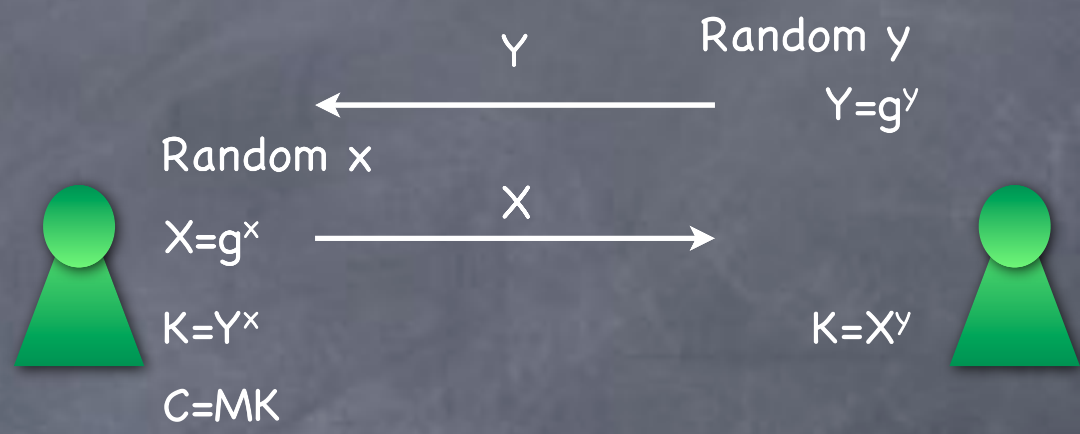

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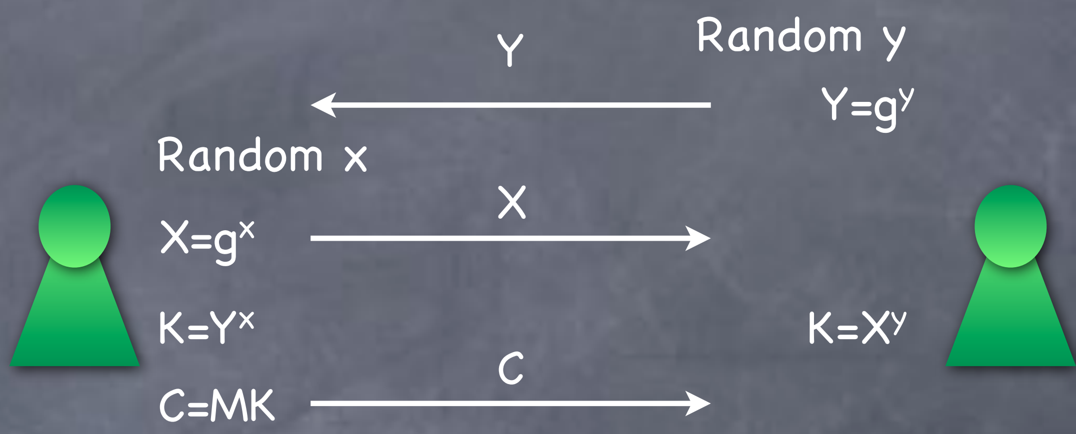

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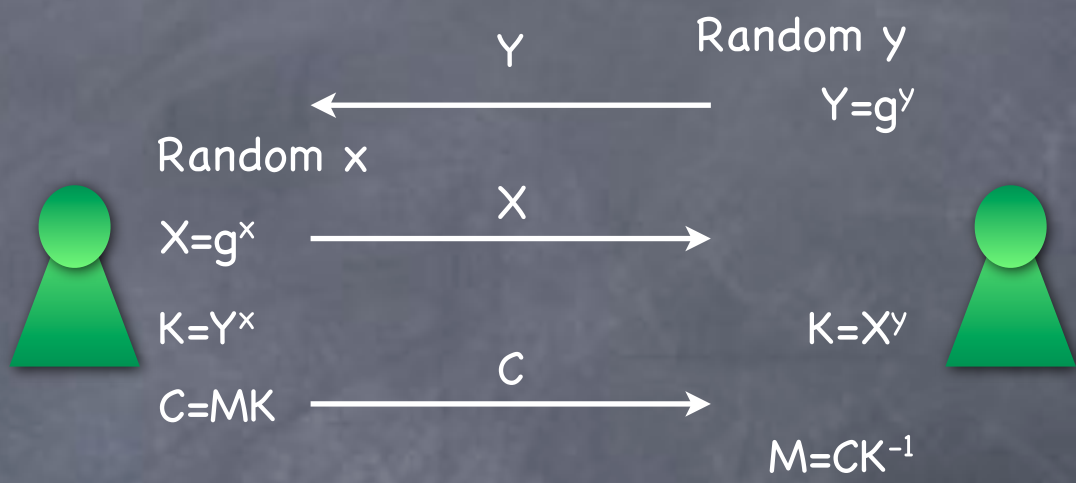

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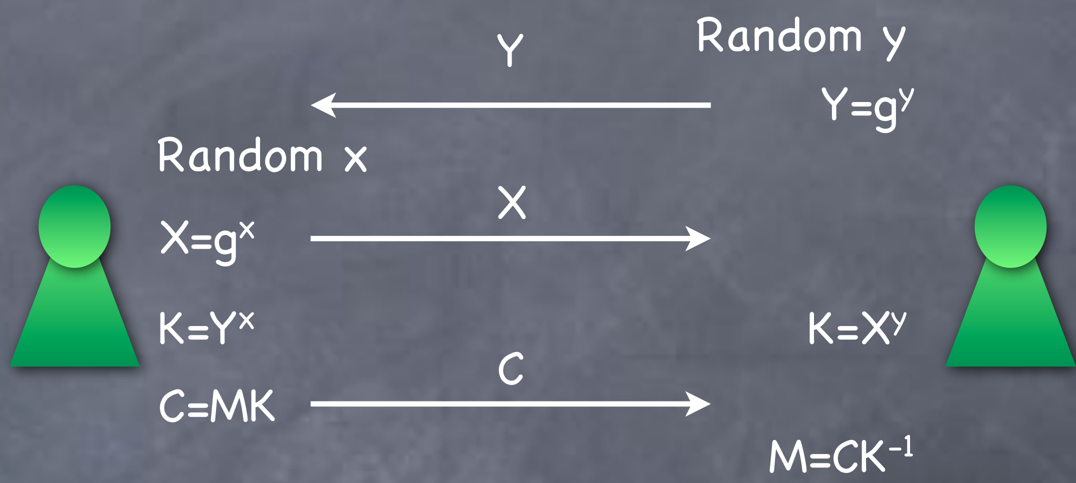

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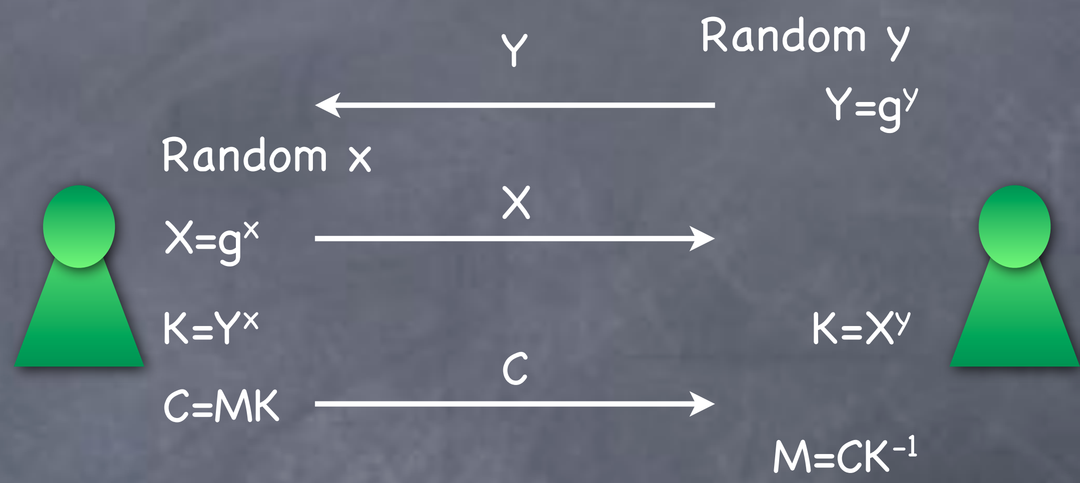

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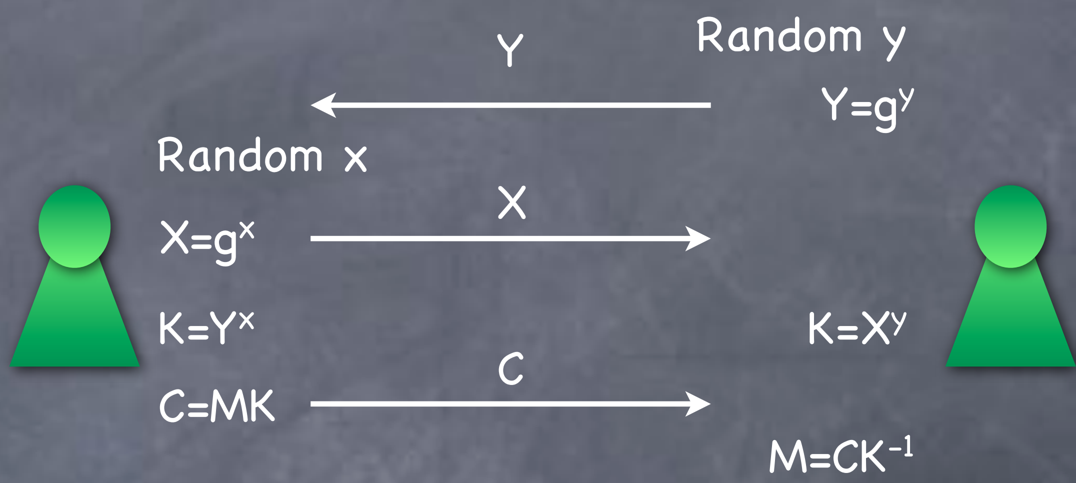

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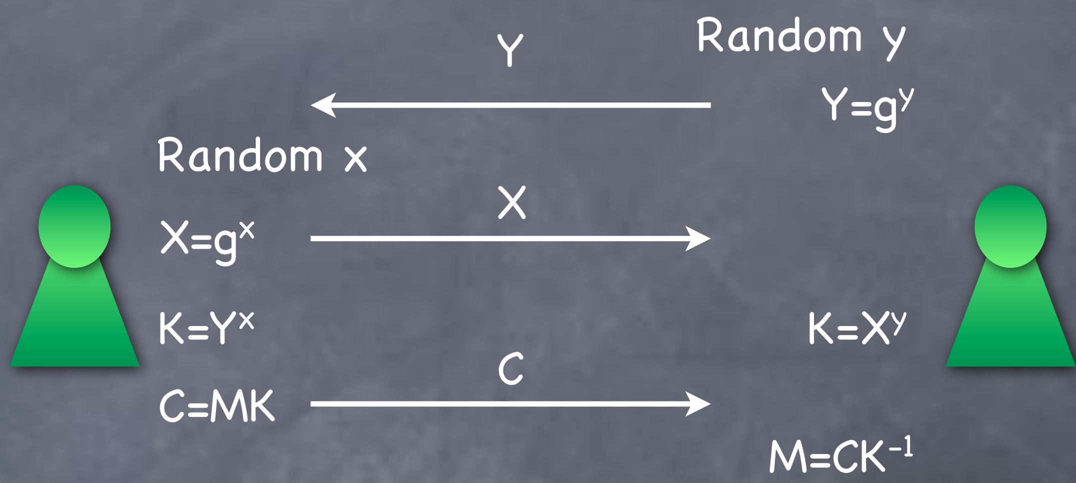

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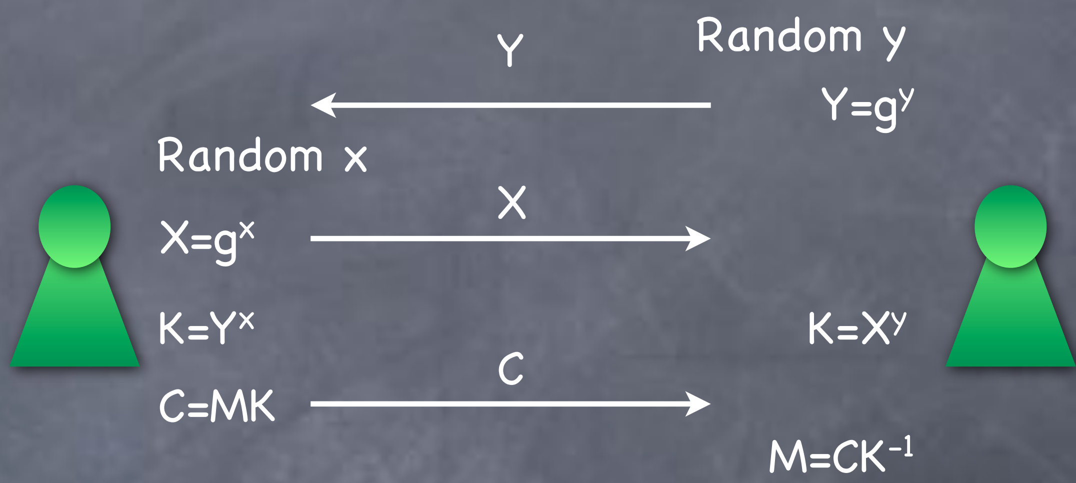

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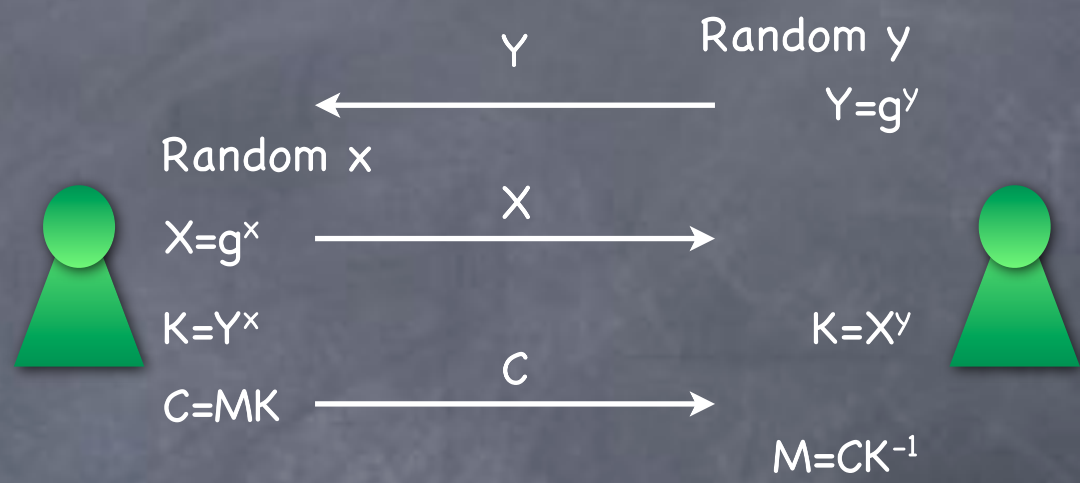

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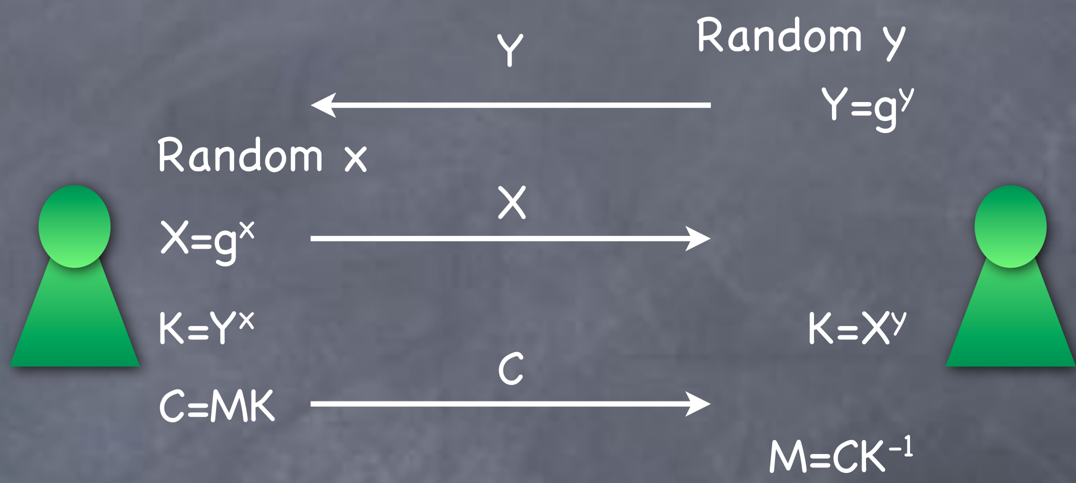

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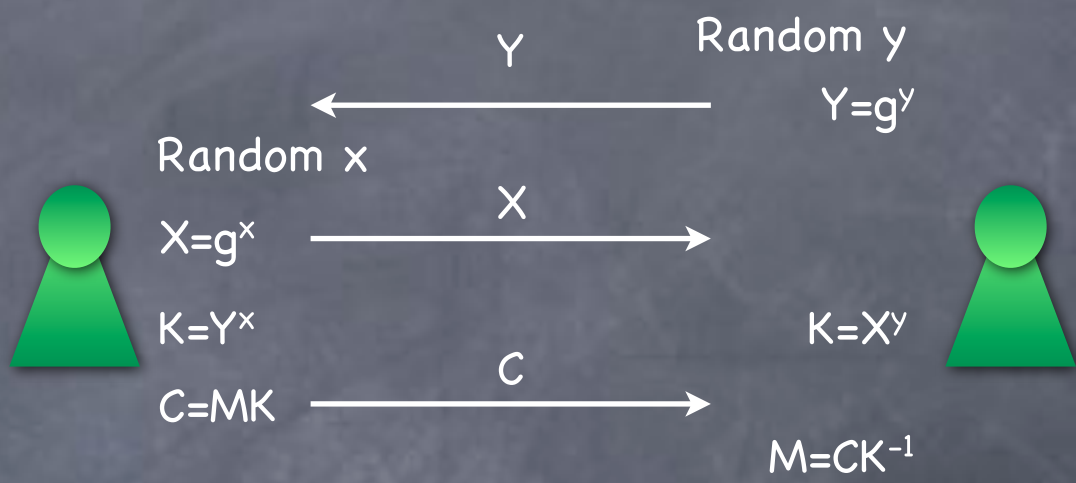

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- Message encoded into group element, and decoded

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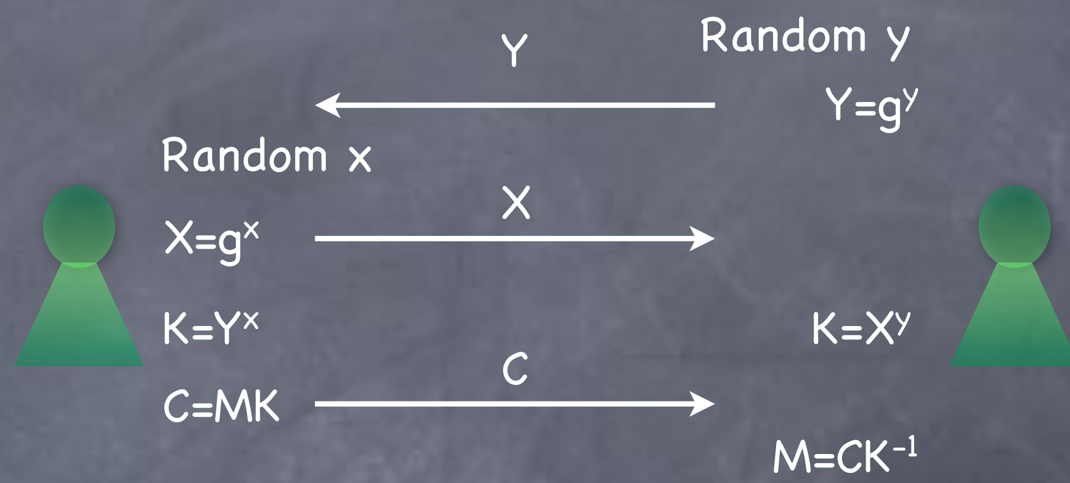
  - When z=xy, exactly IND-CPA experiment: A* outputs 1 with probability = 1/2 + advantage of A.

Abstracting El Gamal



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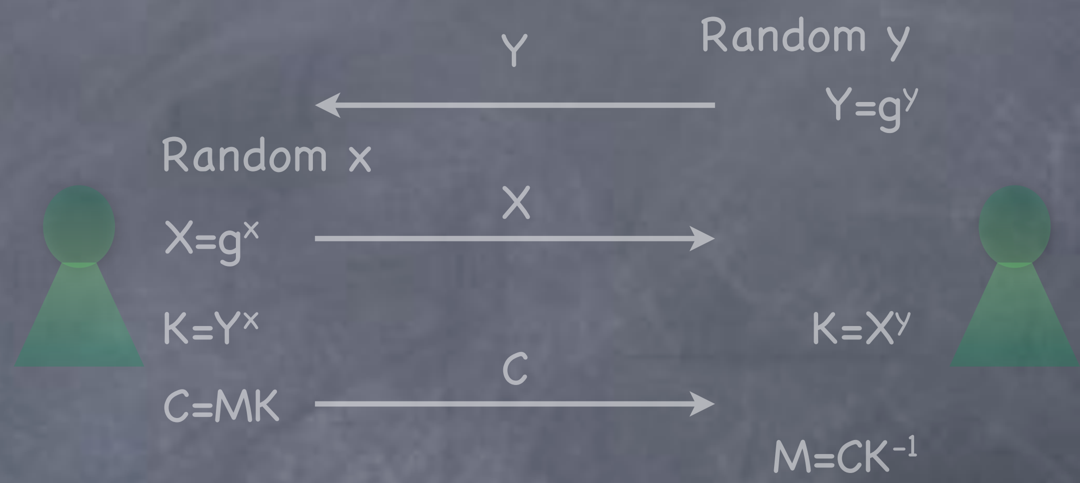

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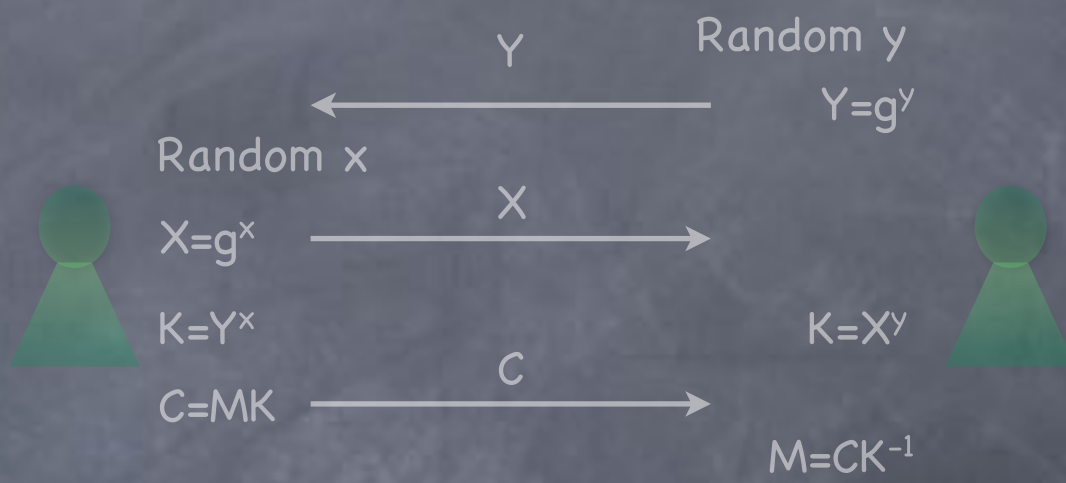

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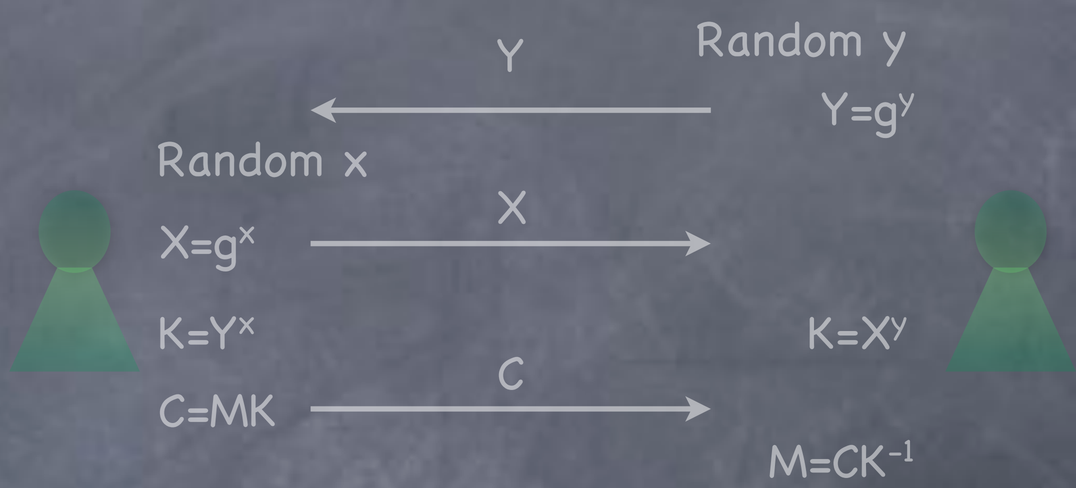

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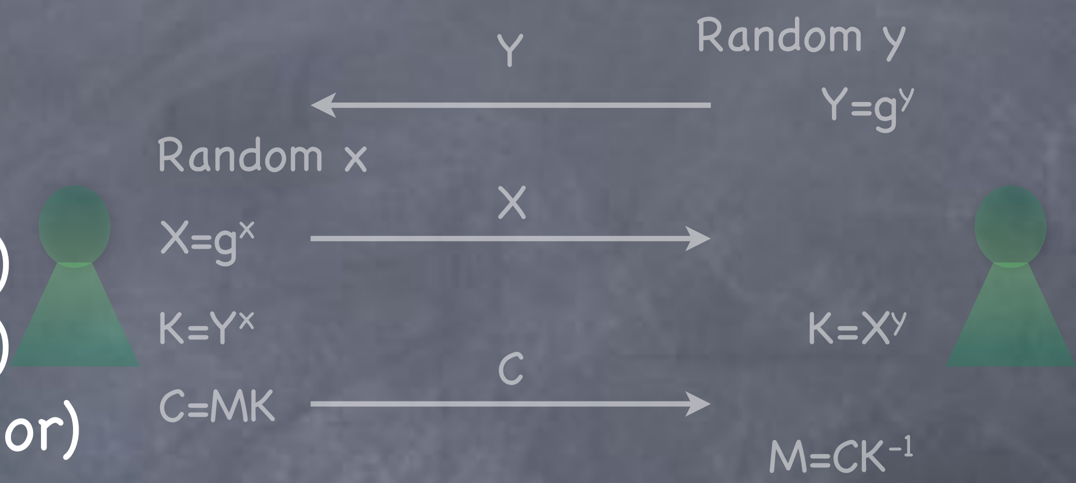

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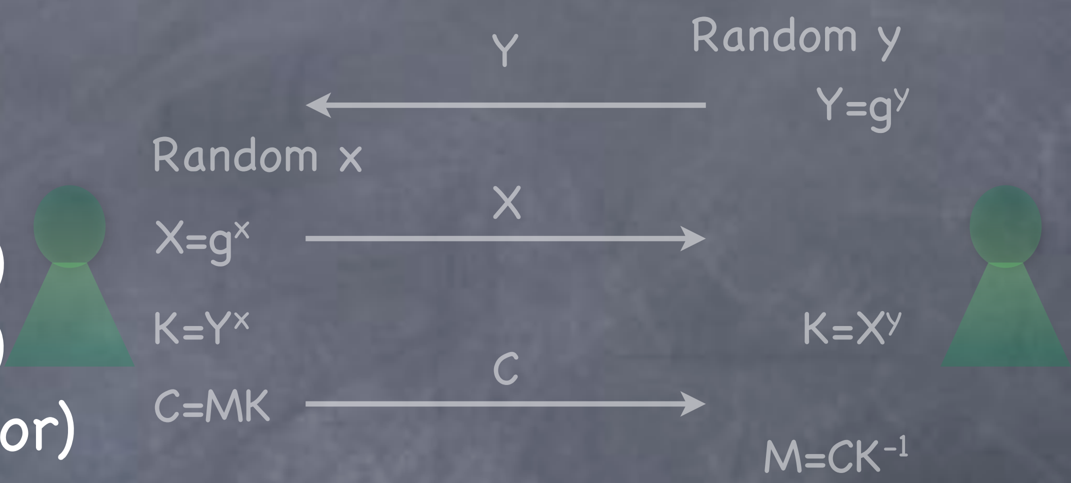

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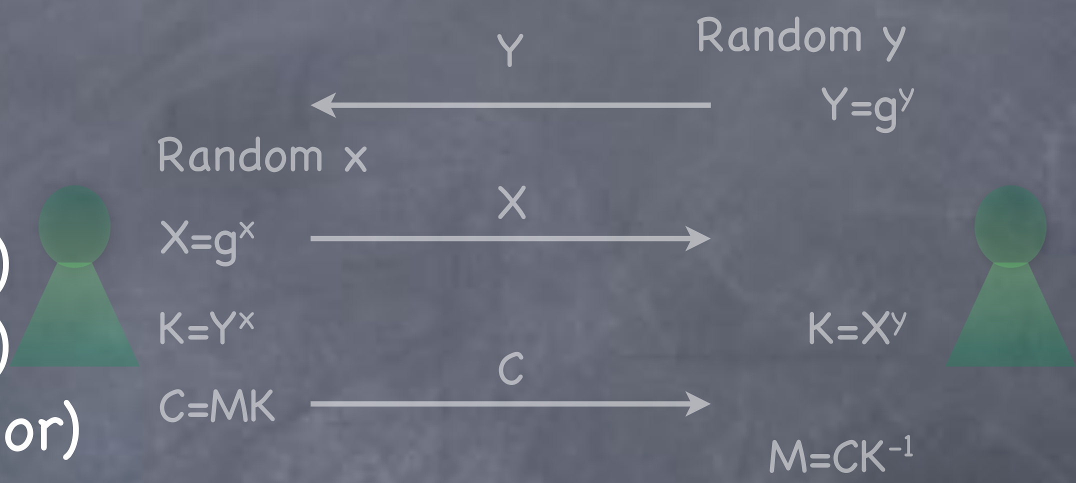

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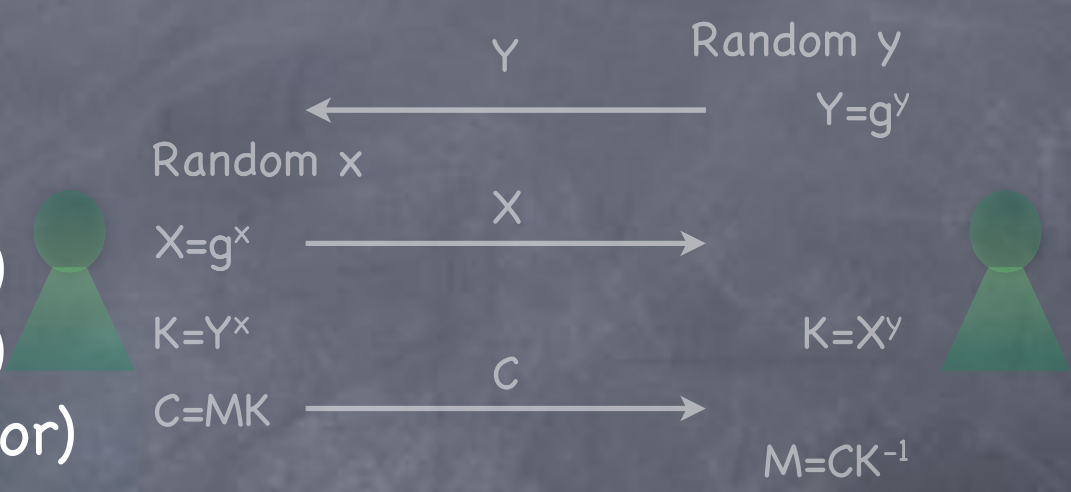

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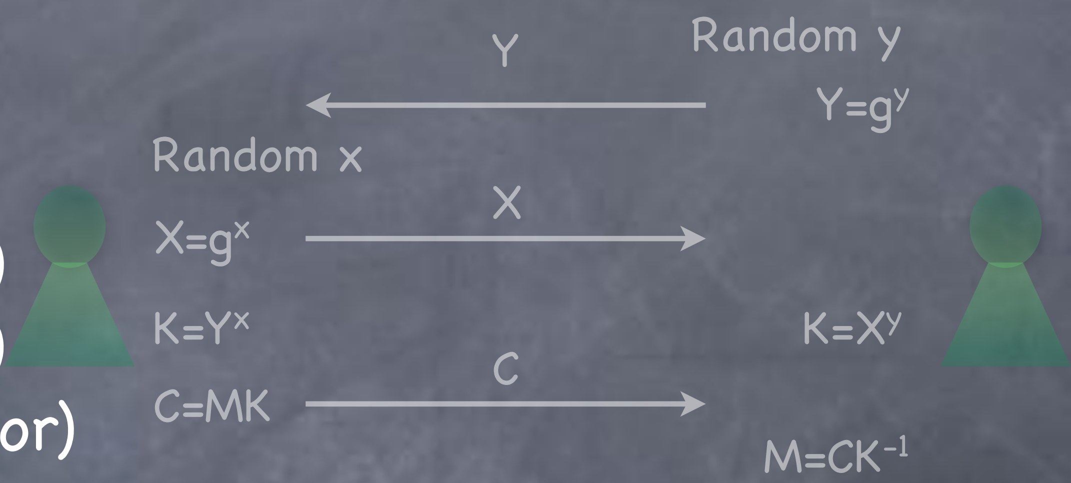

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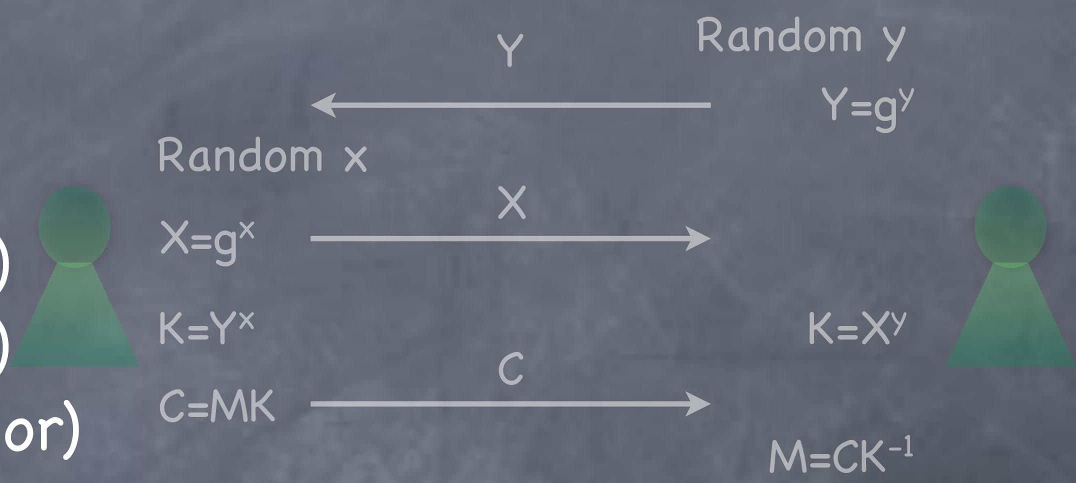

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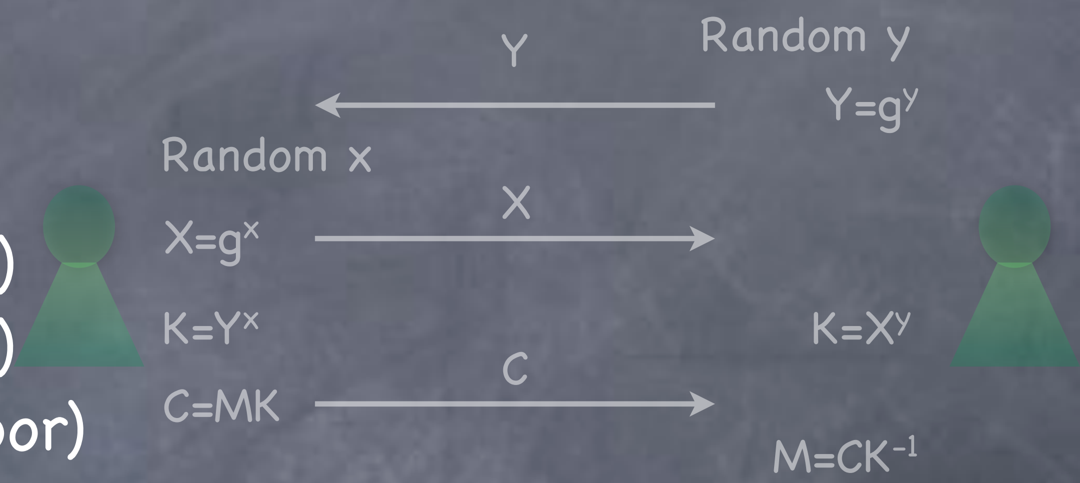

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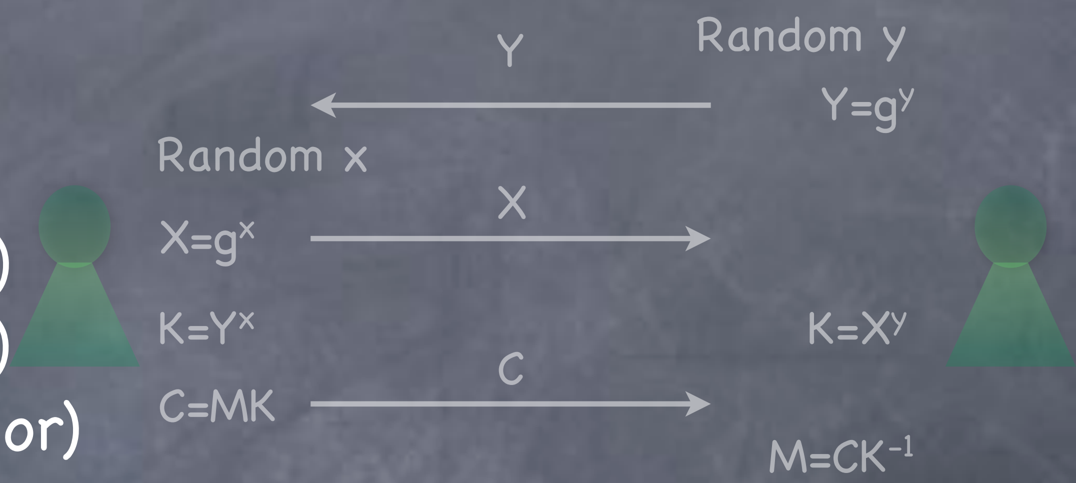

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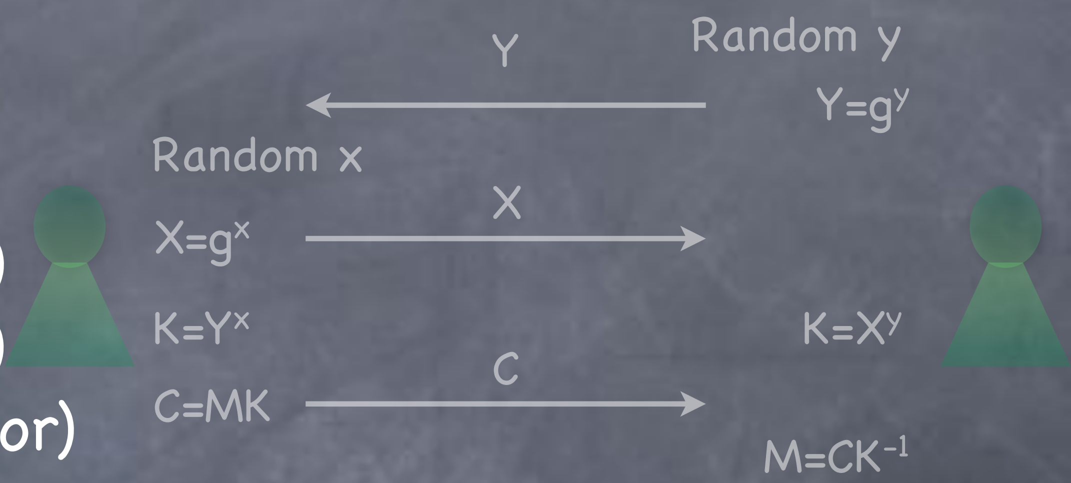

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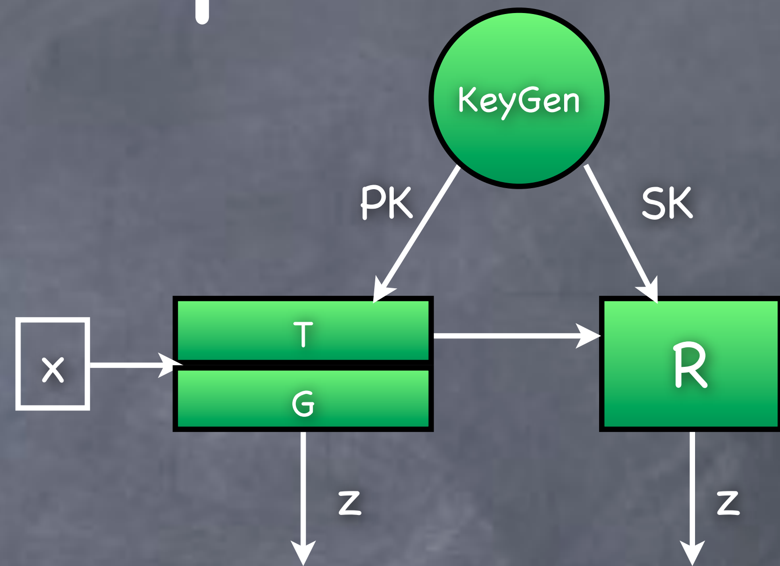
Trapdoor PRG from Generic Assumption?



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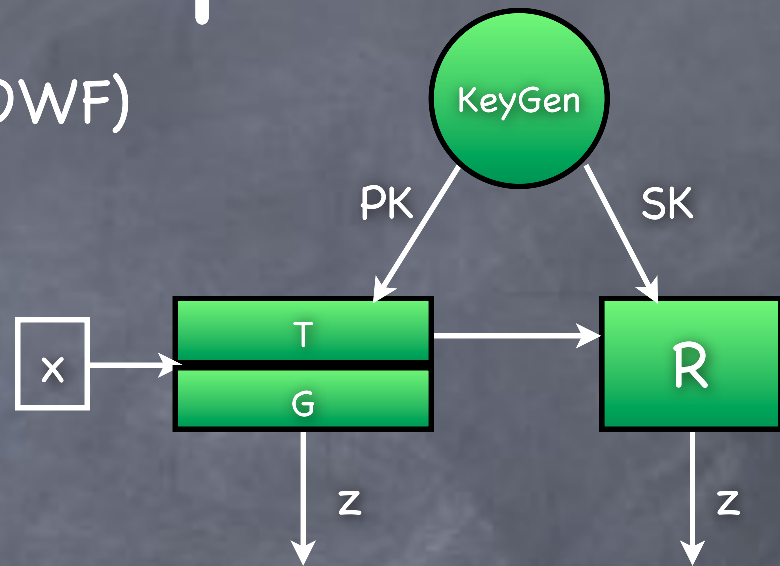

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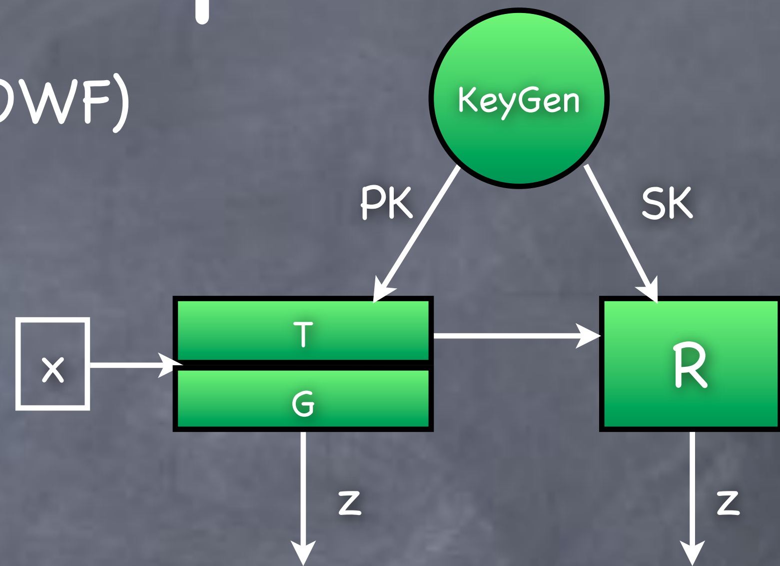

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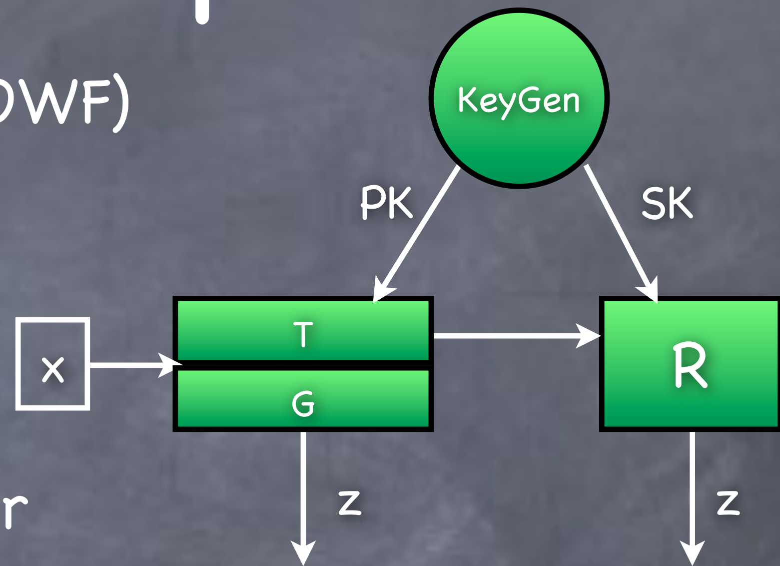

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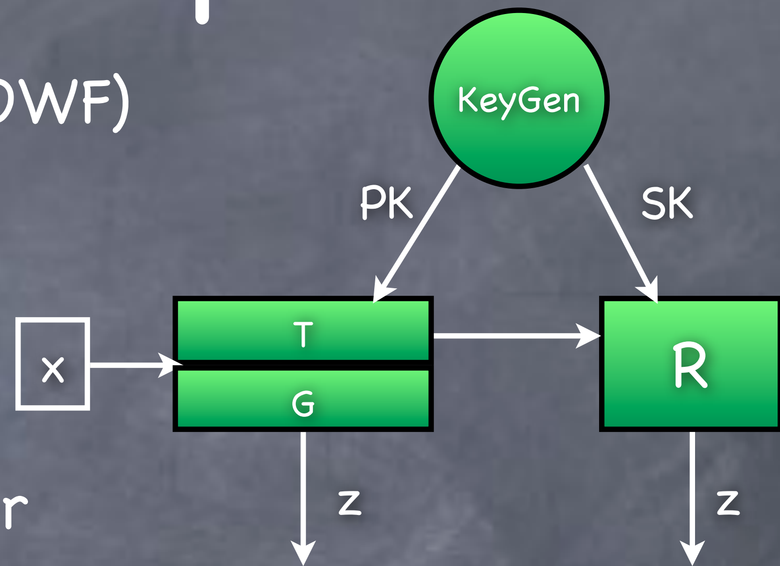

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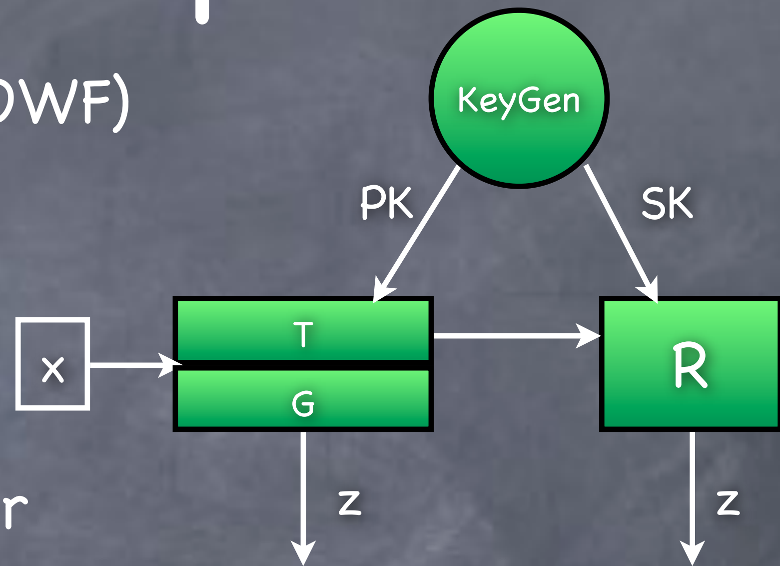
  - Will start with "Trapdoor OWP"



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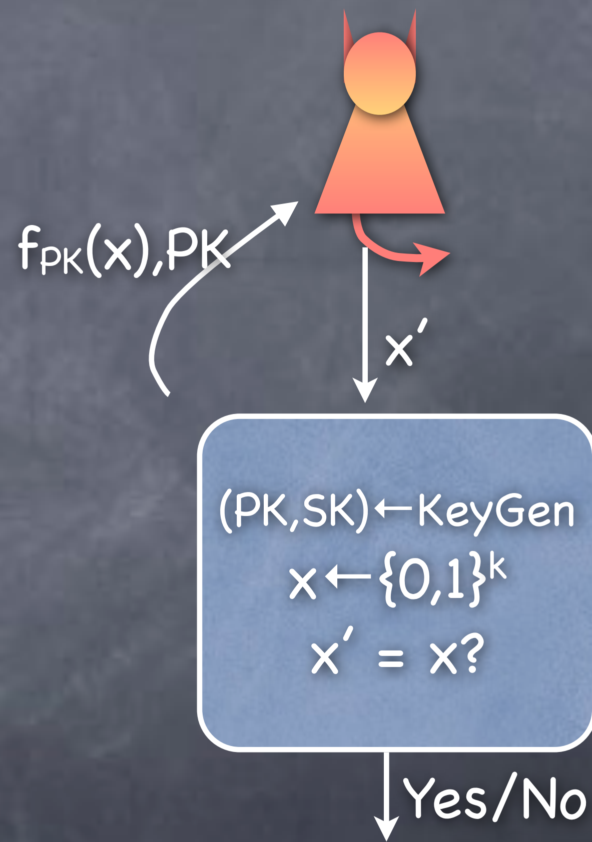

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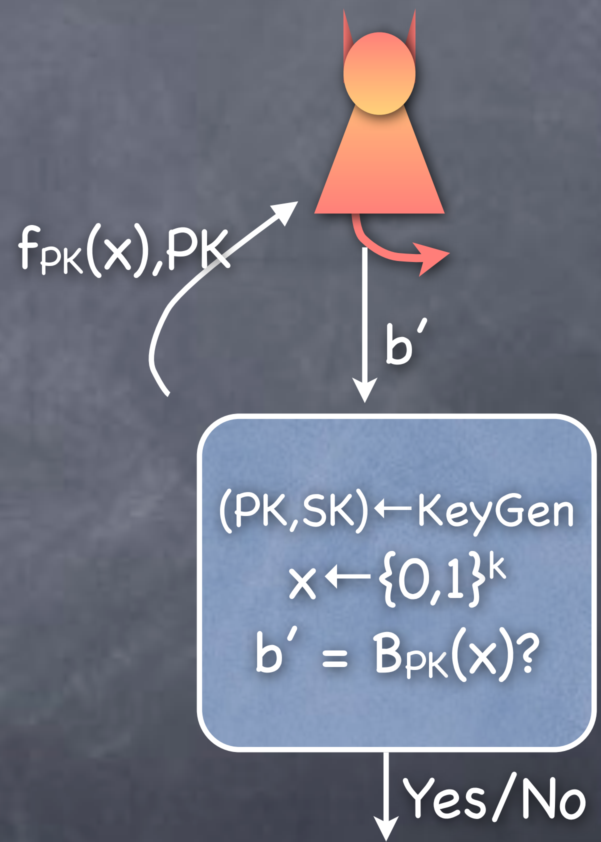
  - Hardcore predicate:
    - B_{PK} s.t. $(PK,f_{PK}(x),B_{PK}(x)) \approx (PK,f_{PK}(x),r)$



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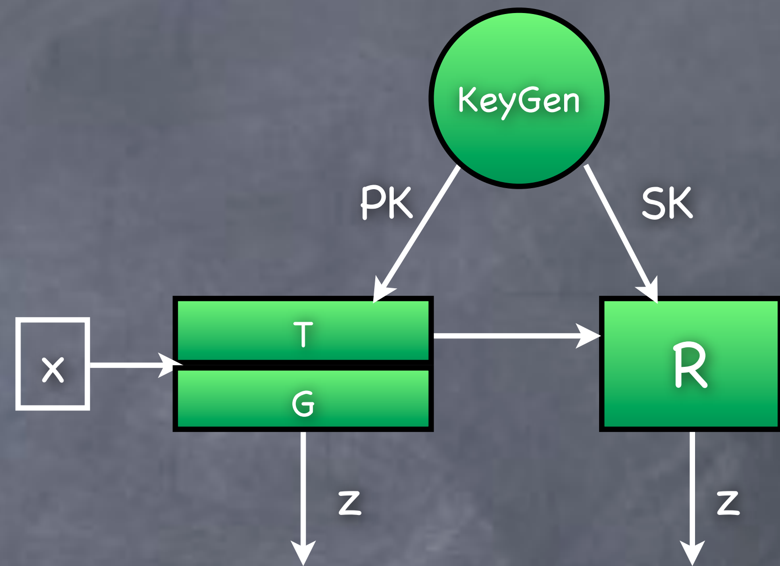

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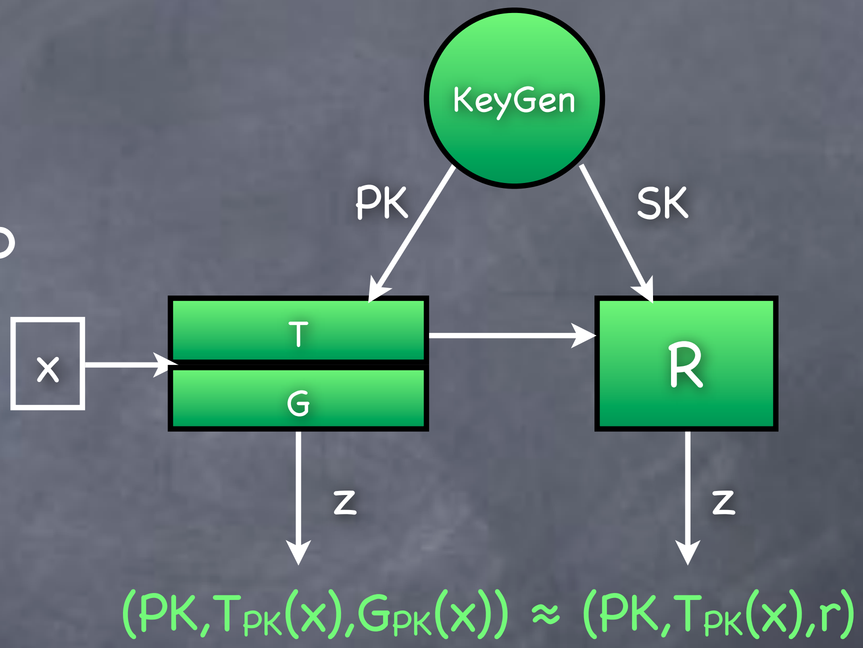

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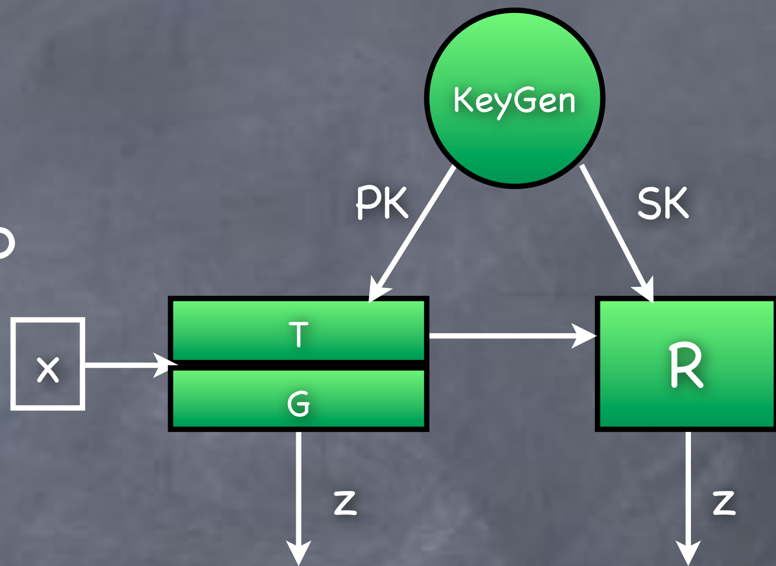

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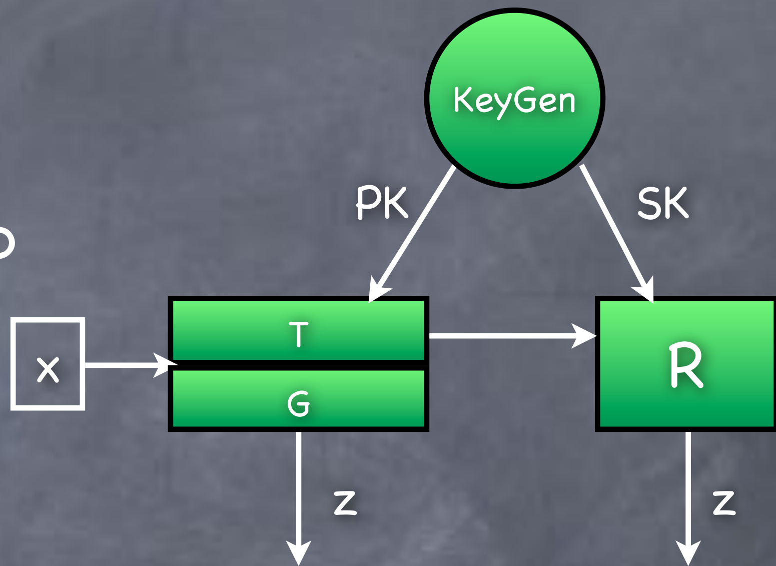

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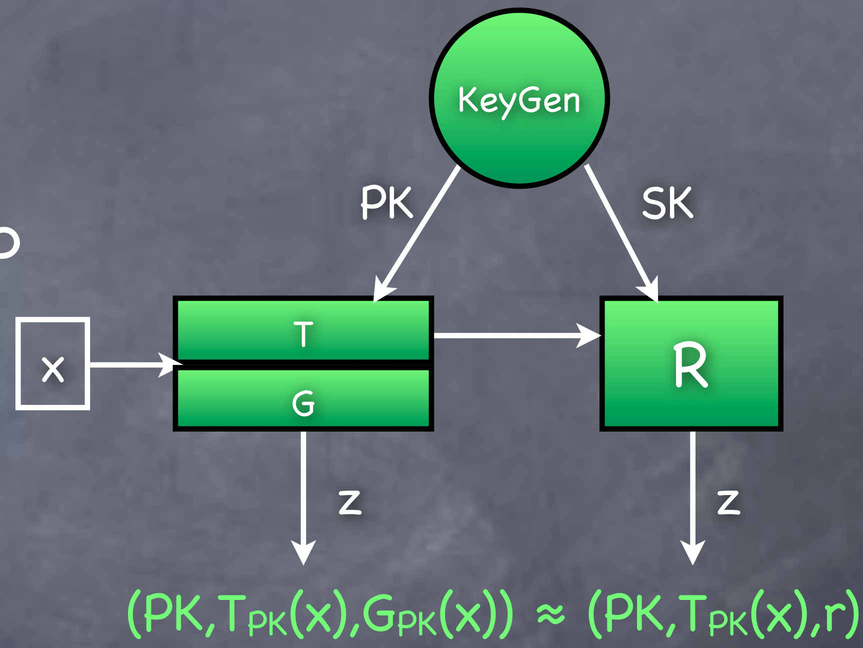

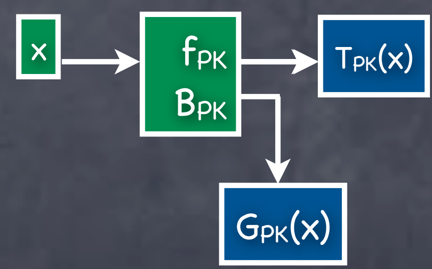

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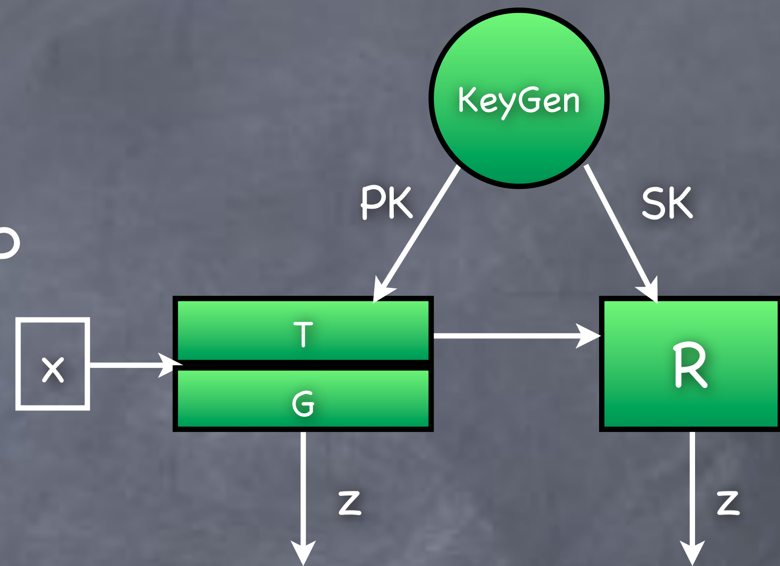

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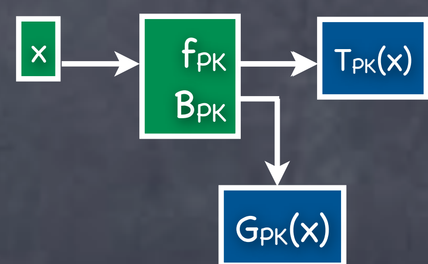

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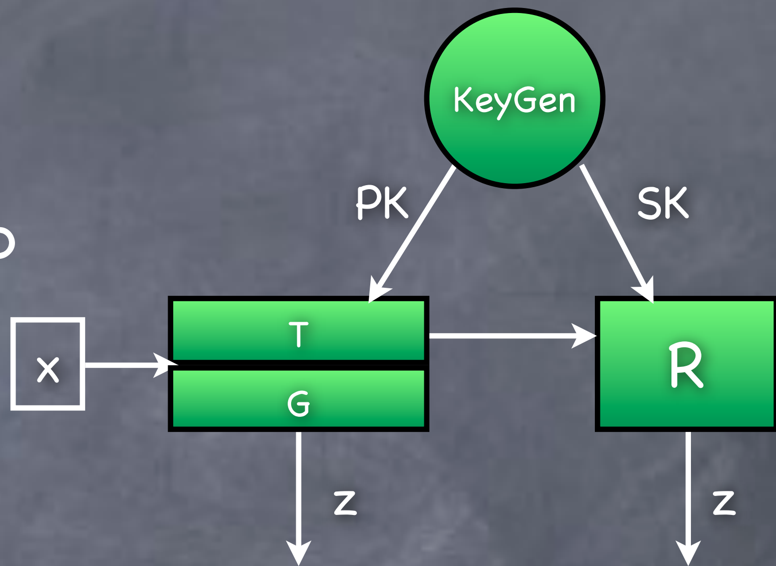

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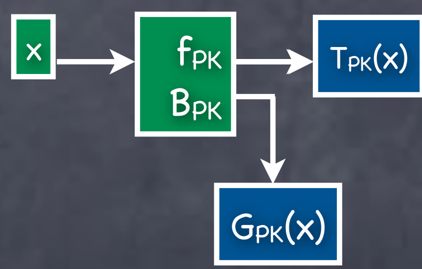

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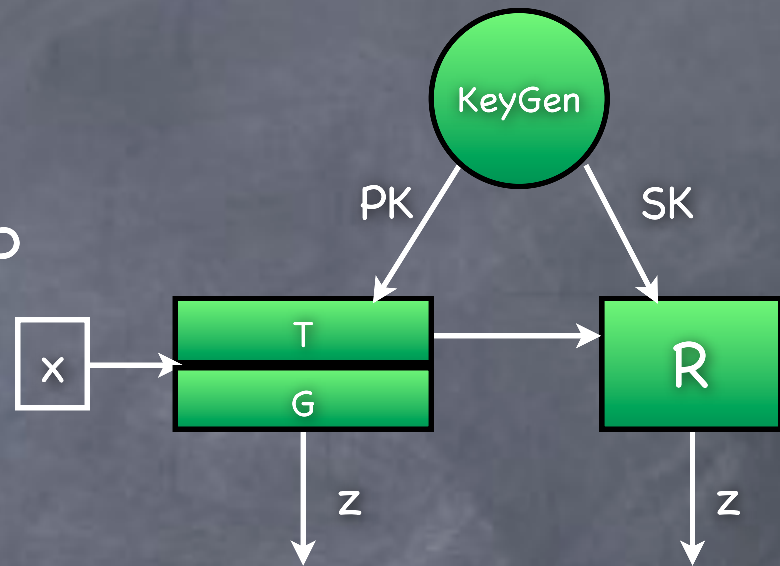

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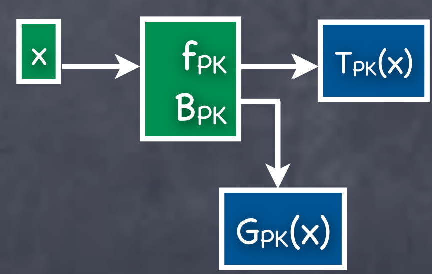

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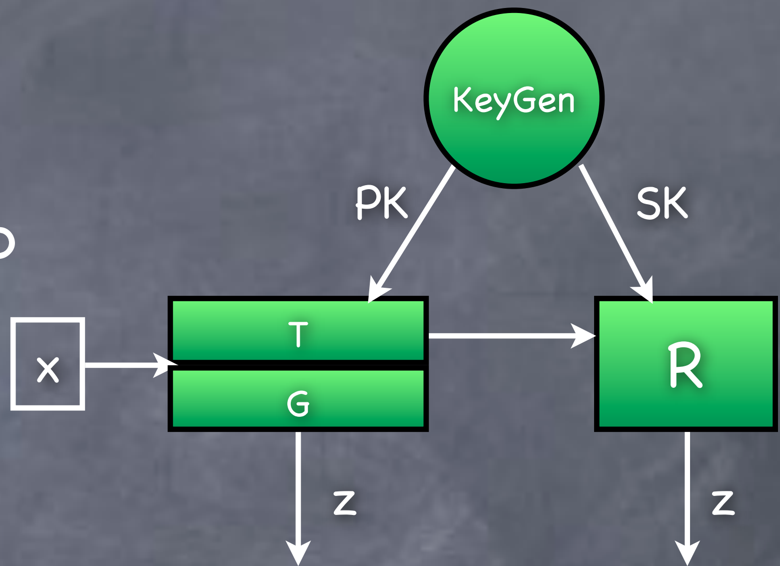

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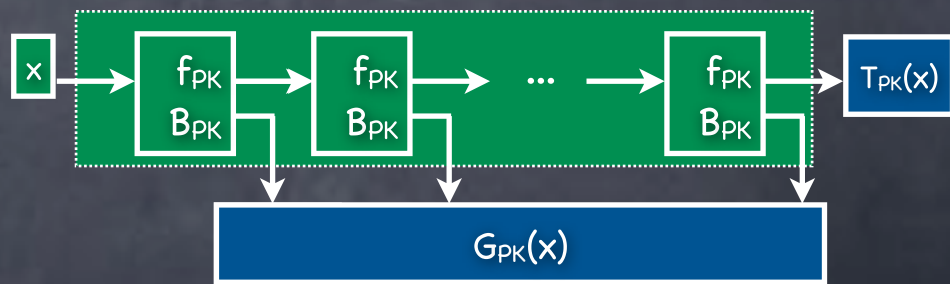

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  - RSA function: $f_{RSA}(x; N,e) = x^e \bmod N$ where $N=PQ$, P,Q k-bit primes, e s.t. $\gcd(e,\varphi(N)) = 1$ (and x uniform from {0...N})

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see handout

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- Next: CCA secure PKE