Some Basic Primitives

Lecture 3

One-Way Functions, PRG

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- Existence of these notions depends on computational complexity assumptions
- First, some complexity-speak...

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  - Message size?
    - We need security even if sending only one bit!

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- Security guarantees are given asymptotically as a function of the security parameter

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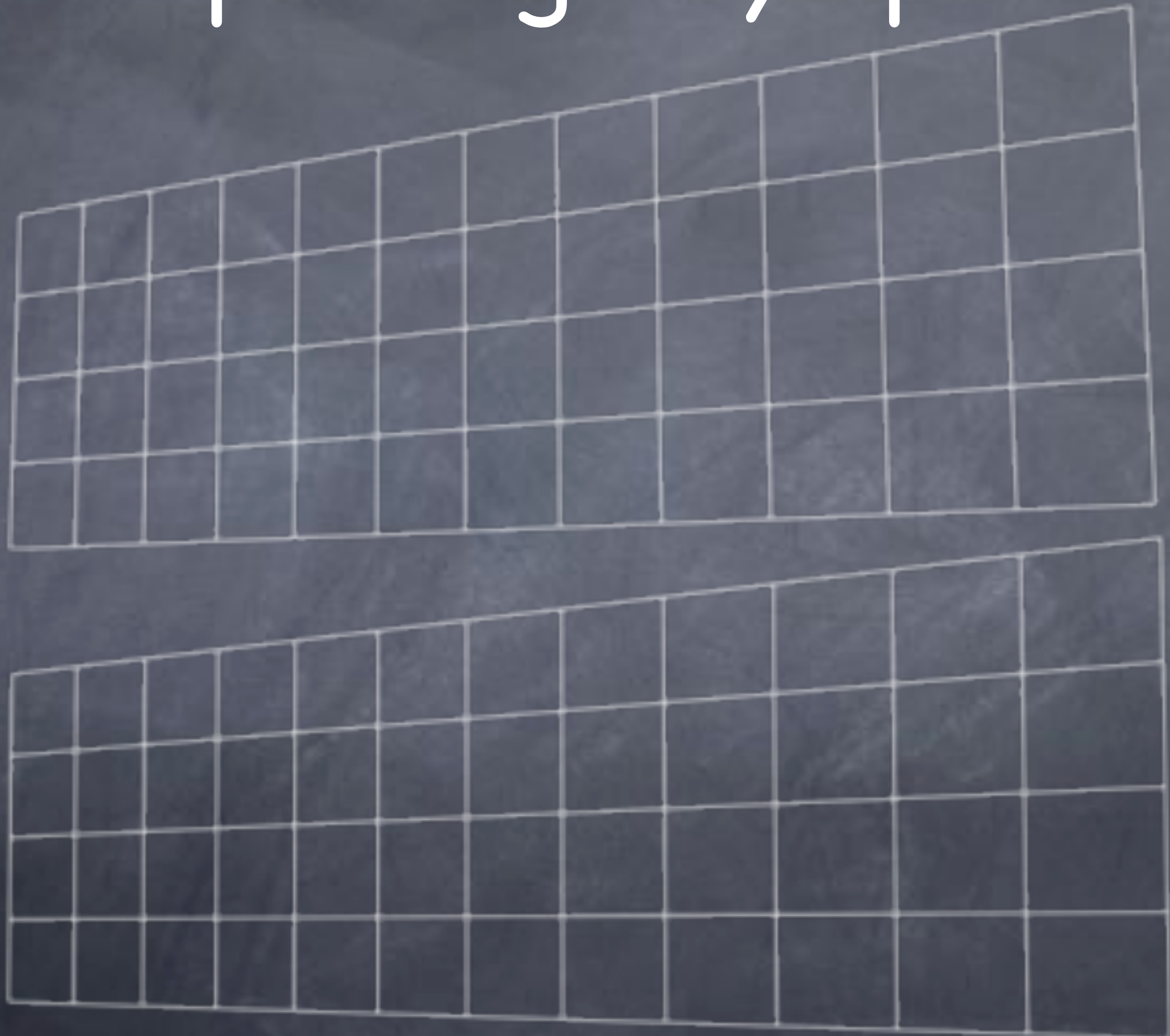

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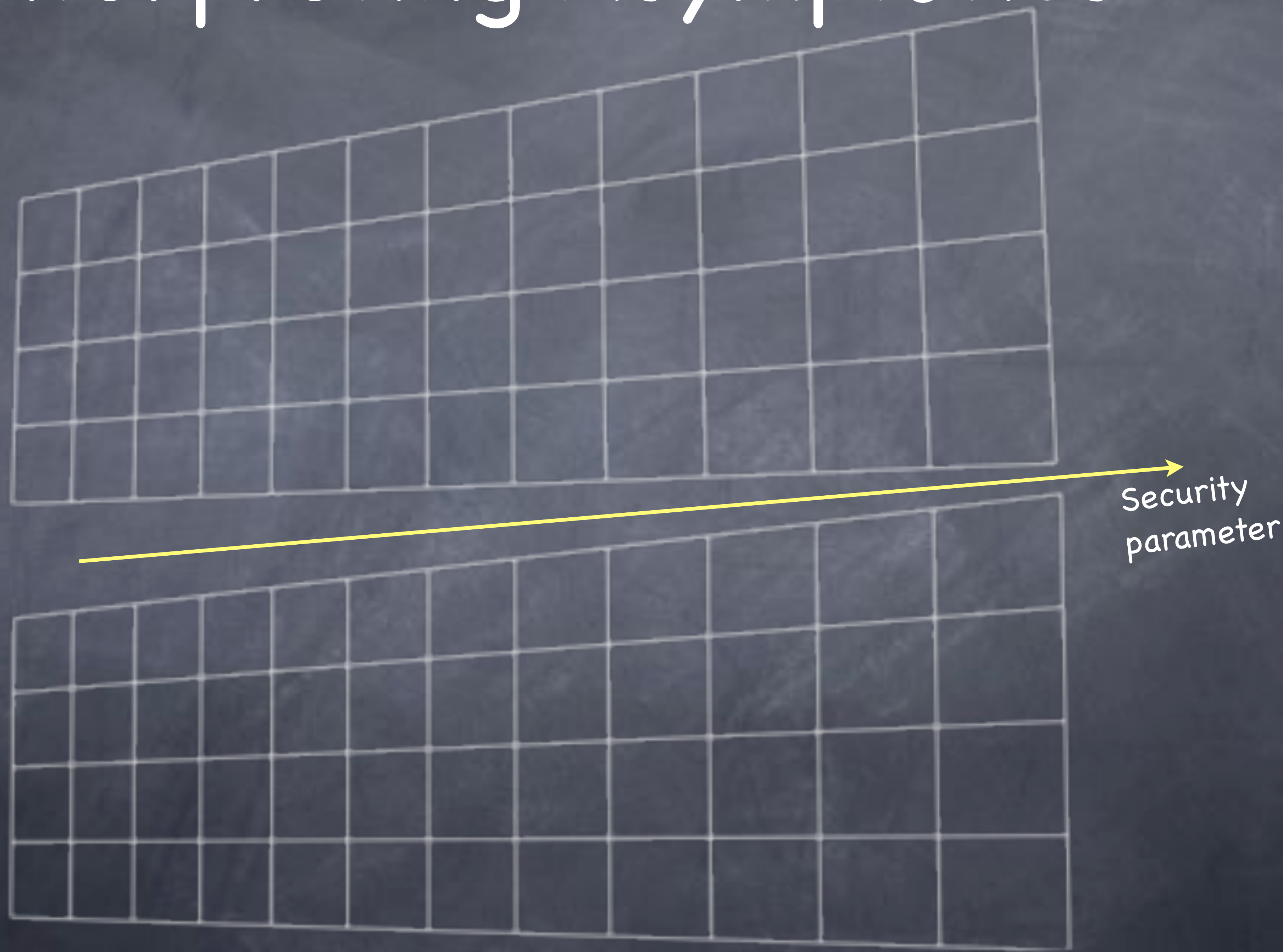

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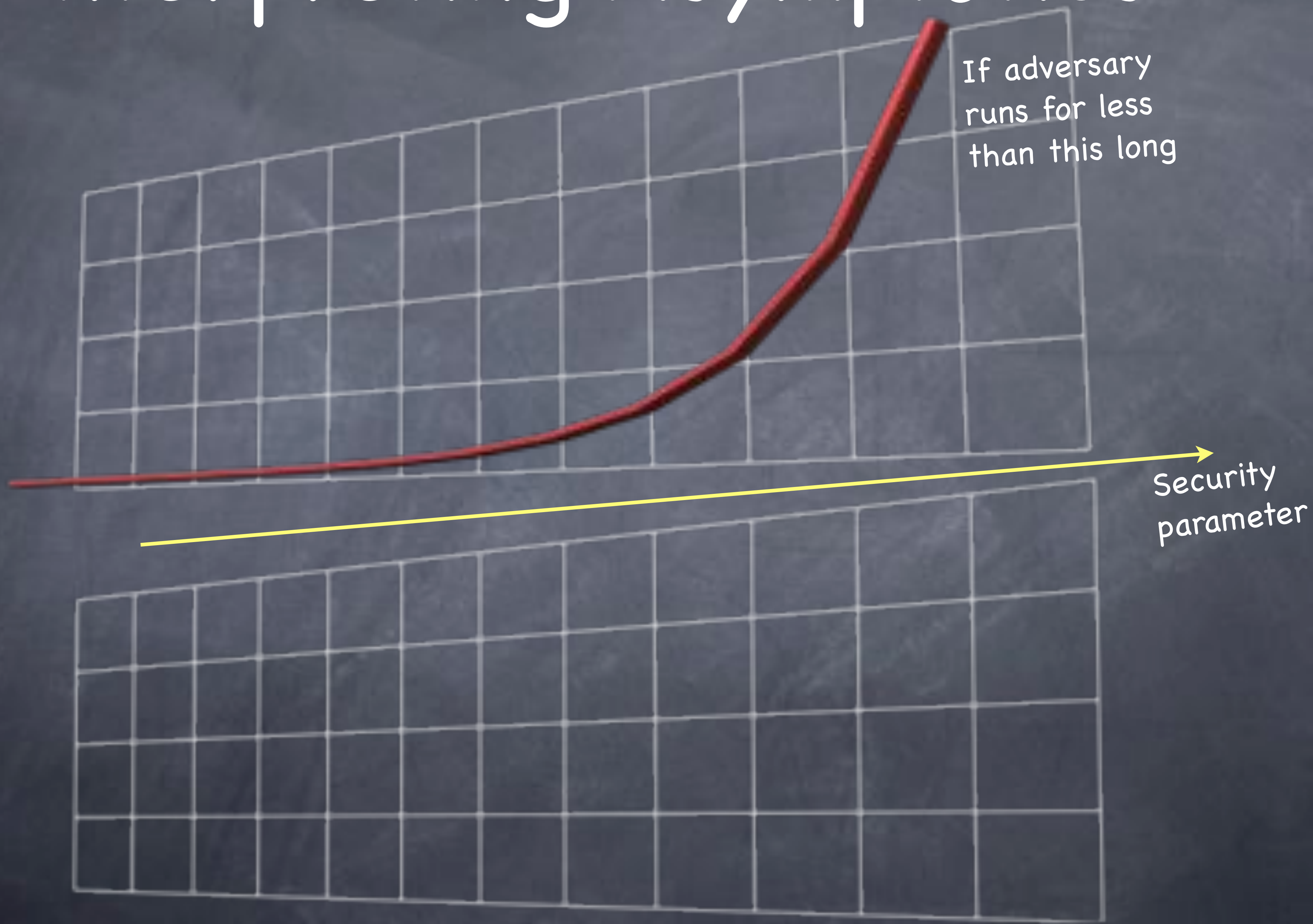

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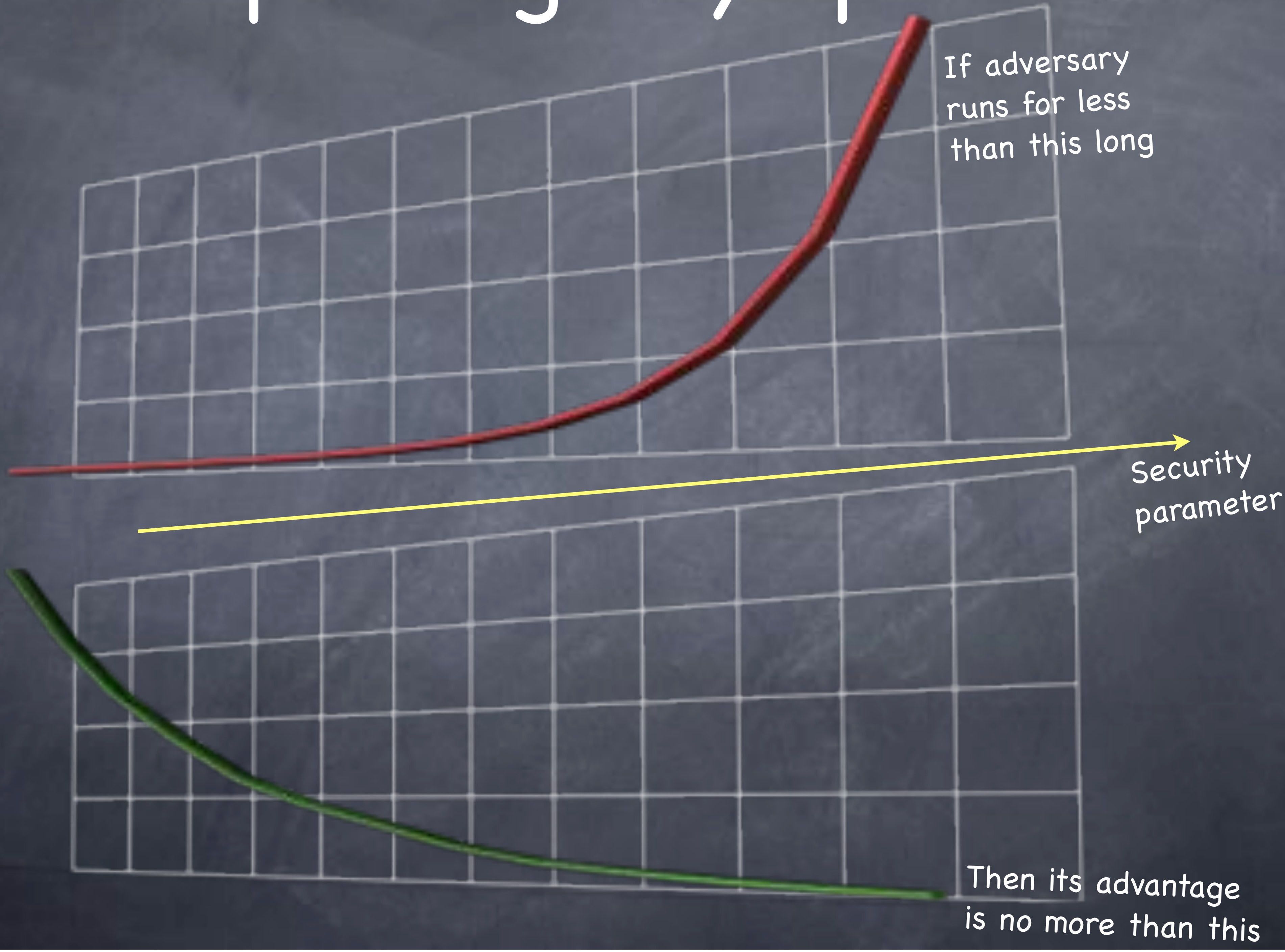

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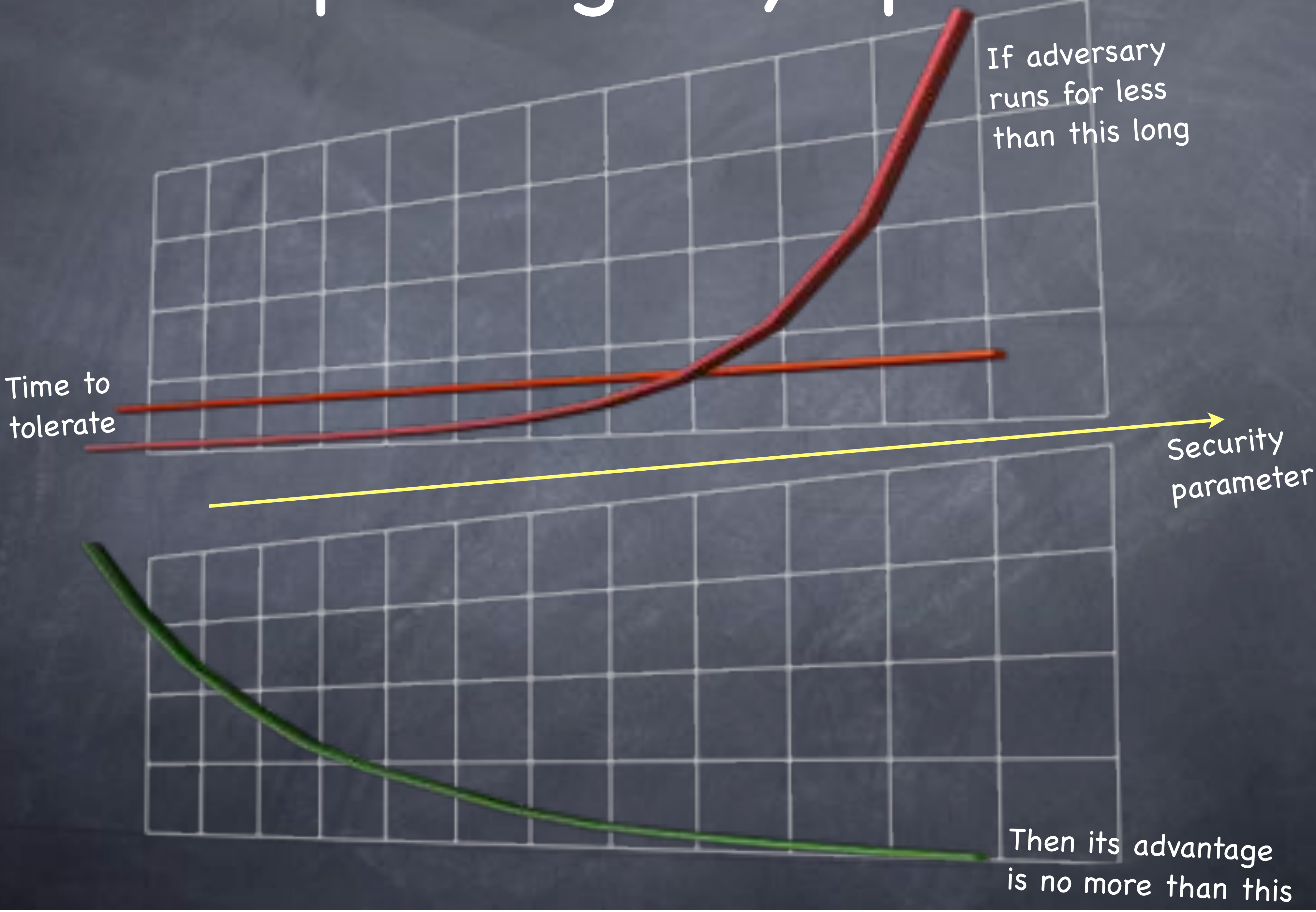

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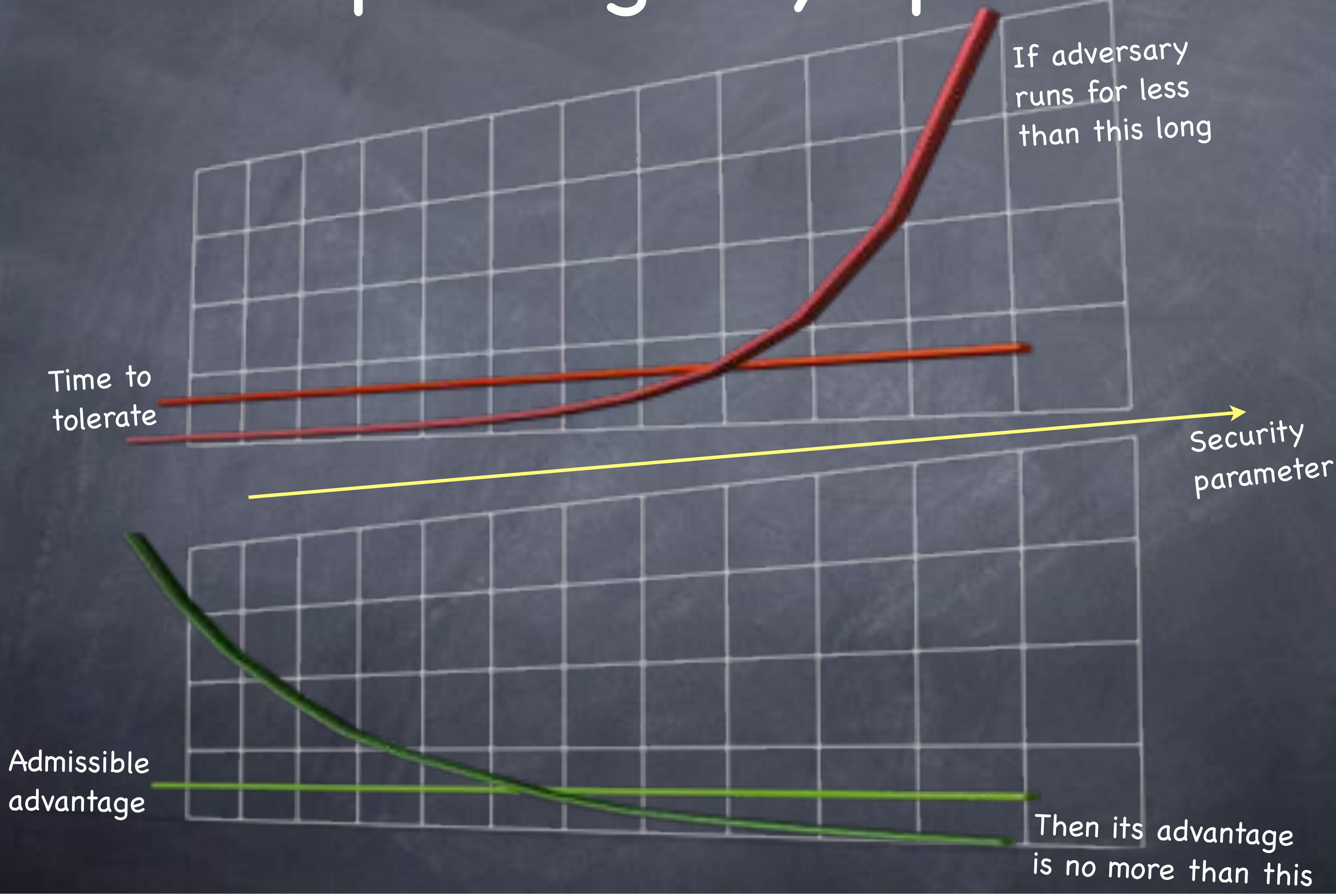

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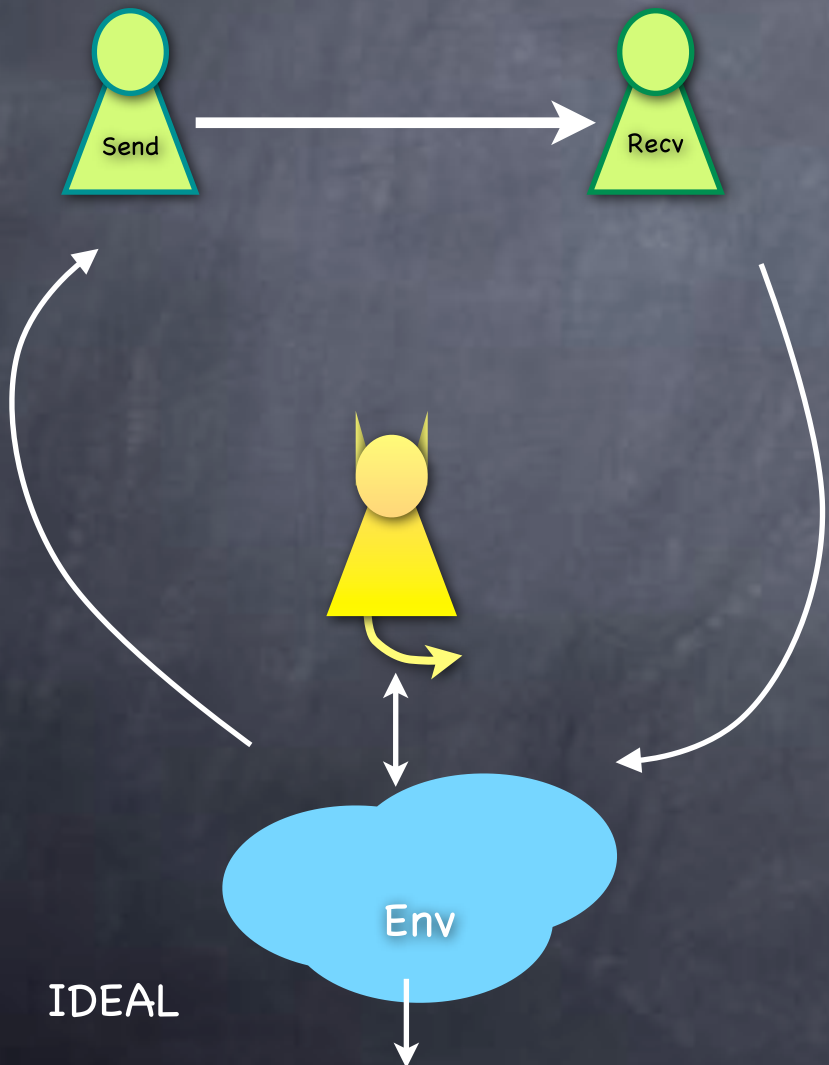
  - So that negl(k) × poly(k) = negl'(k)
    - Needed, because Eve can often increase advantage polynomially by spending that much more time/by seeing that many more messages

Symmetric-Key Encryption

SIM-CPA Security



SIM-CPA secure if:

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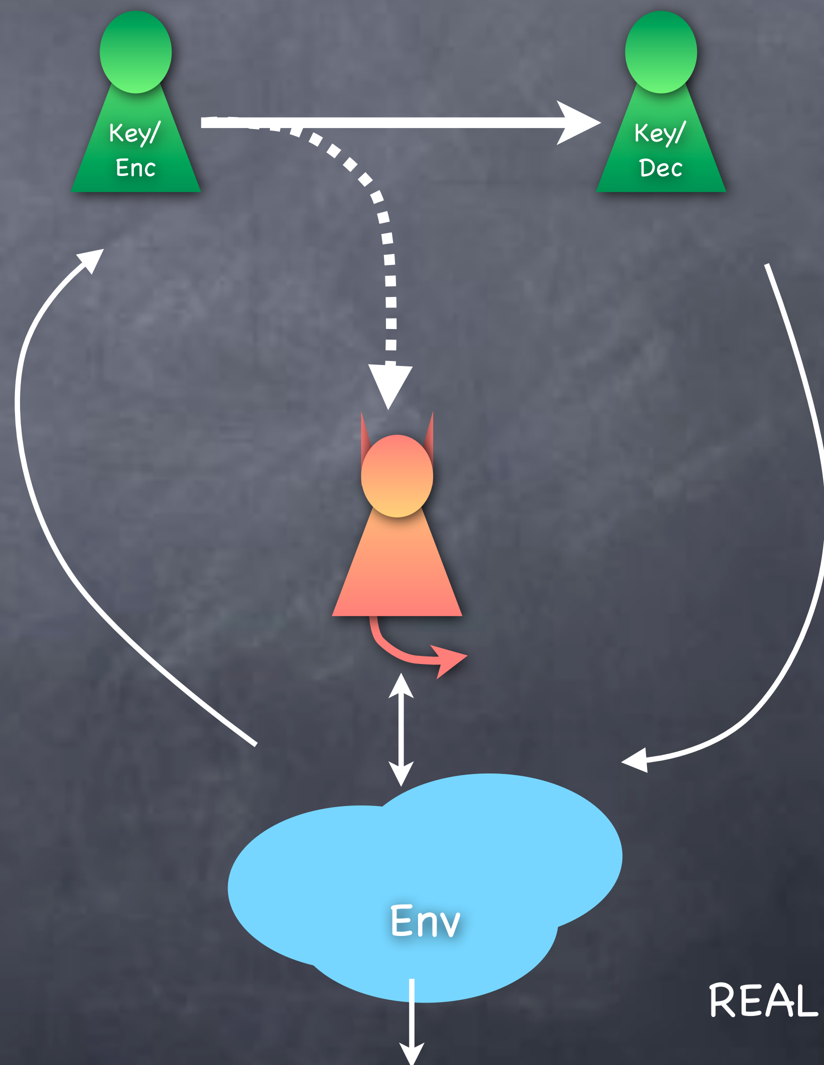

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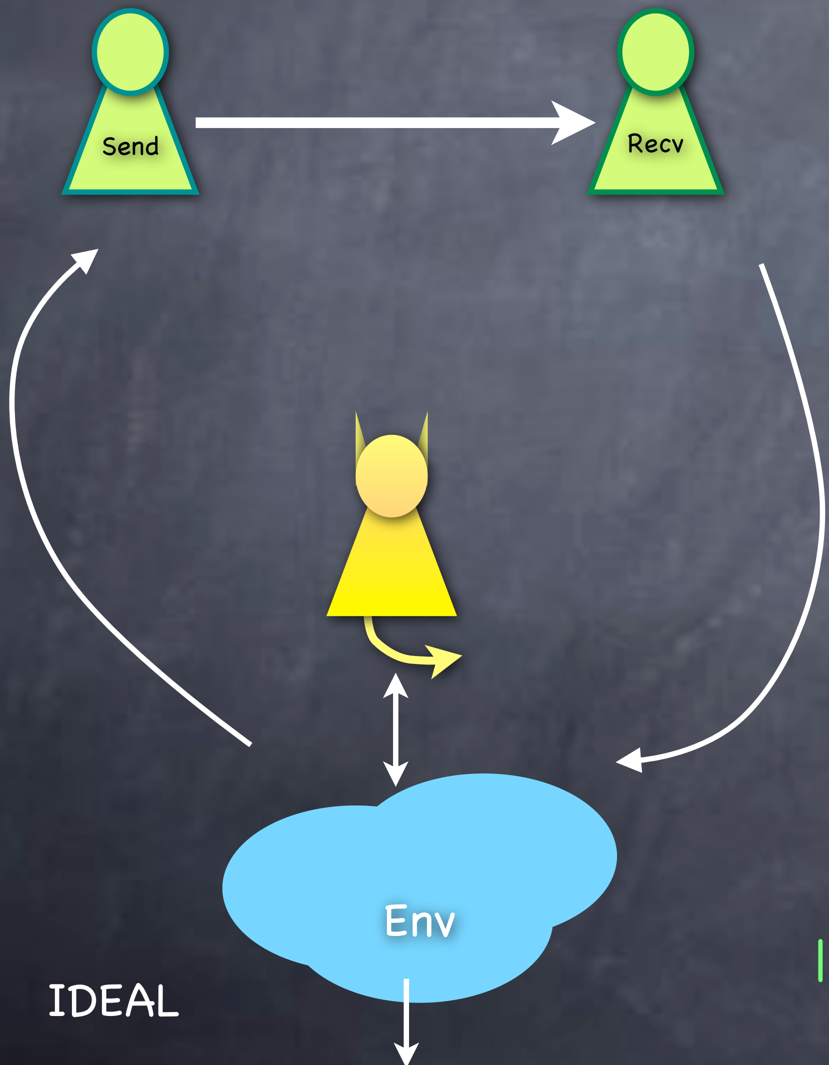

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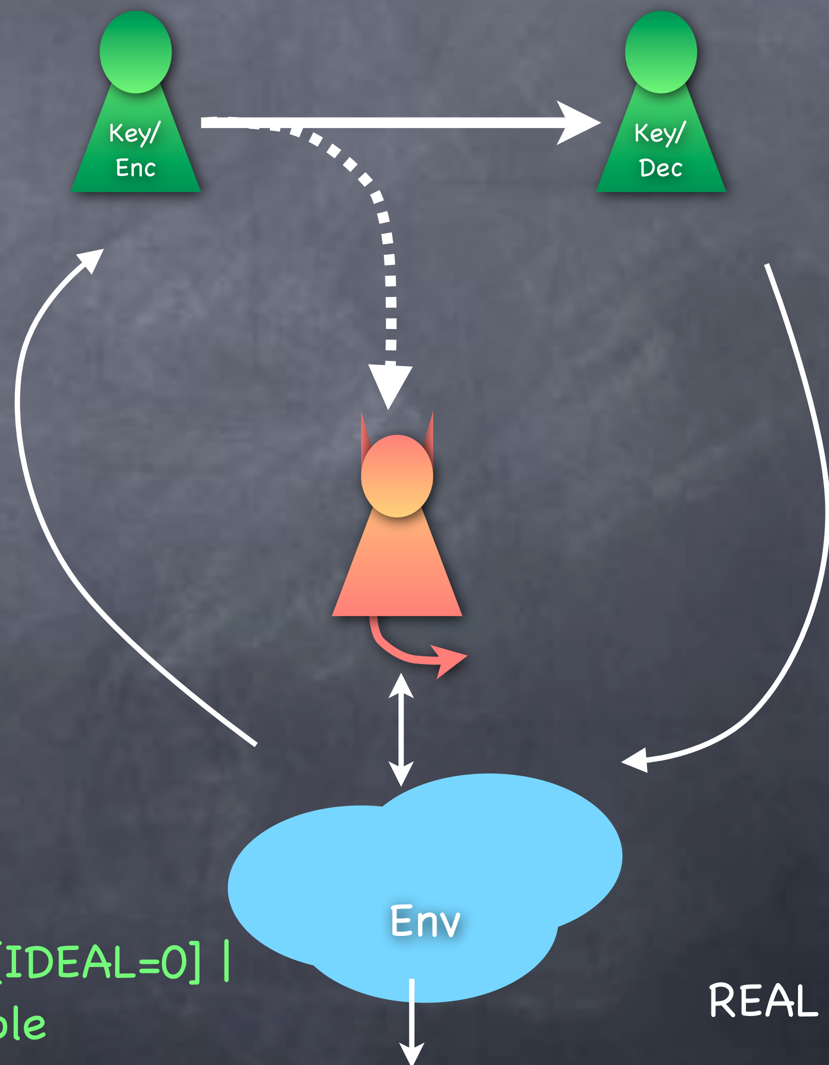

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  - Coming up: One-Way Functions, Hardcore predicates, PRG, ...

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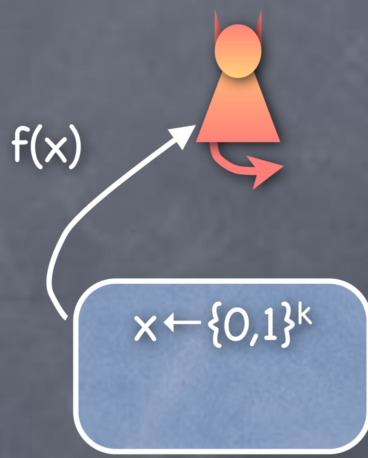

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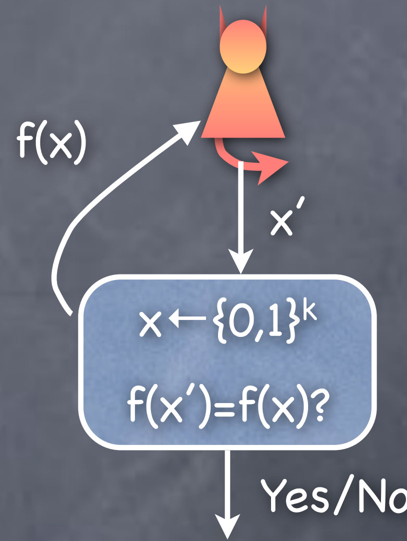

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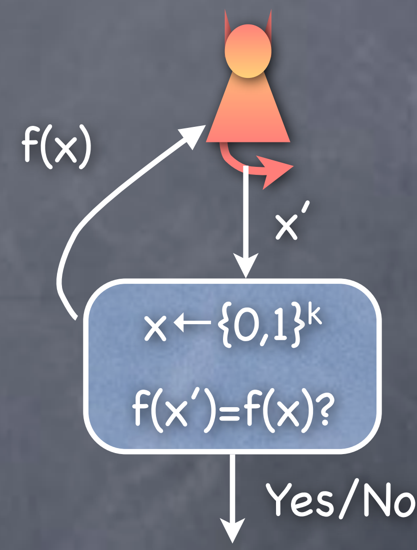

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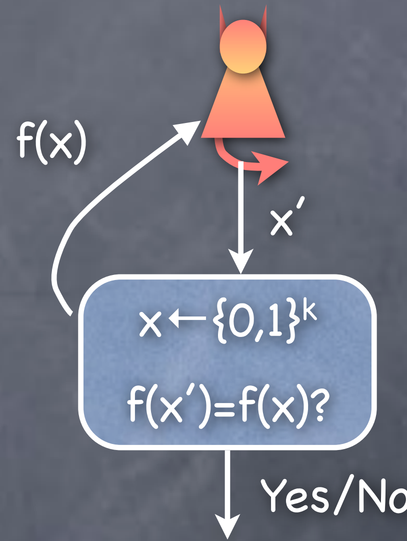

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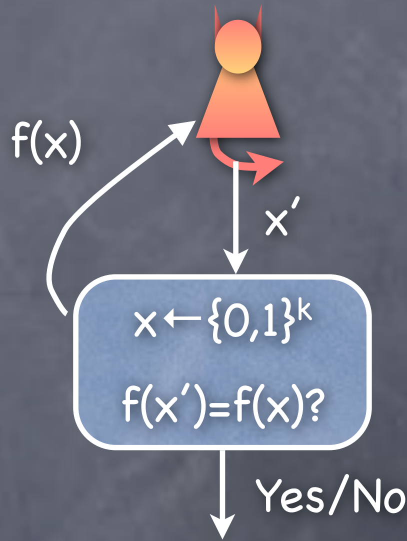

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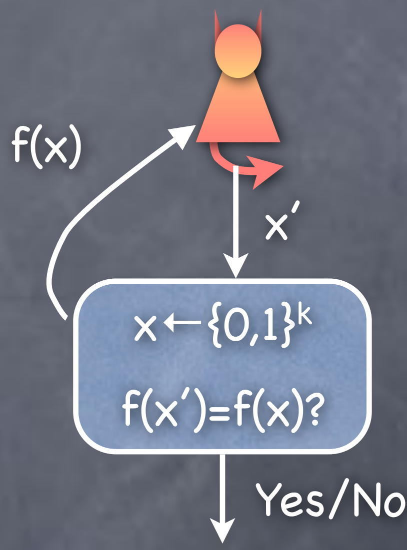

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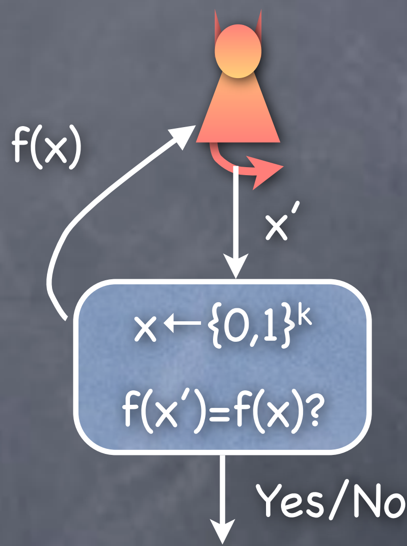

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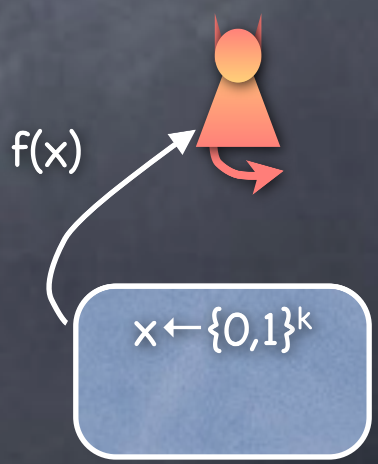

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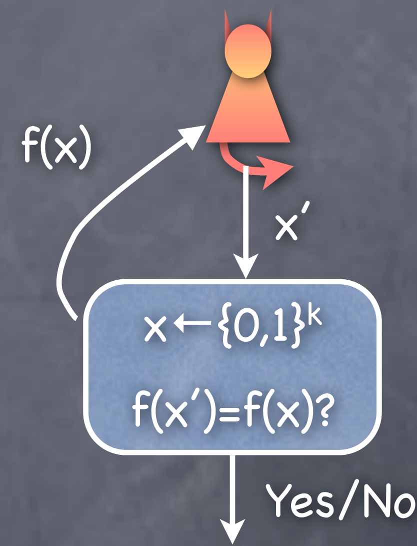

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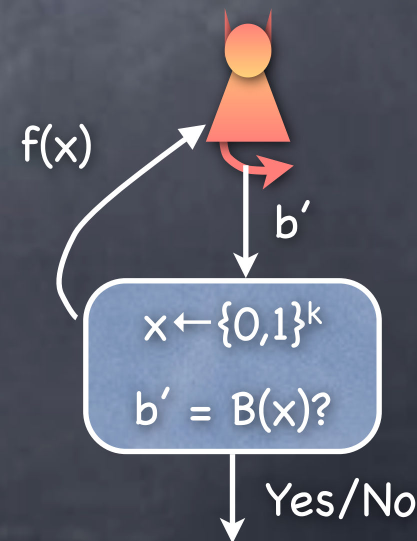

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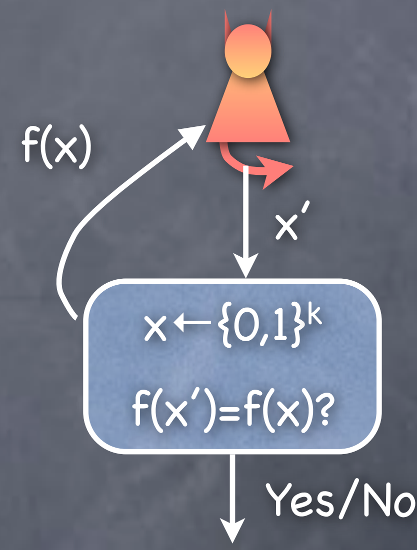

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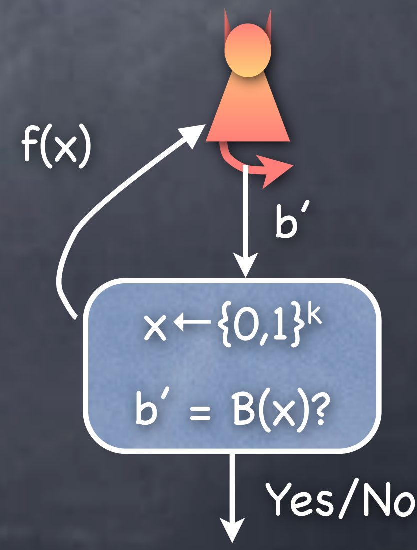

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  - Fact: input distribution (x,y) random k-bit integers will also work (if k-bit primes distribution works)
    - Important that we require $|x|=|y|=k$, not $|x.y|=k$ (otherwise, 2 is a factor of x.y with 3/4 probability)

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  - Inverting f_{subsum} known to be NP-complete, but assuming that it is a OWF is "stronger" than assuming P≠NP

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    - Reduction: Given an algorithm for finding LSB(x) from $f_{Rabin}(x;n)$ for random x, show how to invert f_{Rabin}

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- $g_f(x,r) = (f(x), r)$, where $|r|=|x|$

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  - Can show that a predictor of B(x,r) with non-negligible advantage can be turned into an inversion algorithm for f

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  - Turns out they are equivalent!

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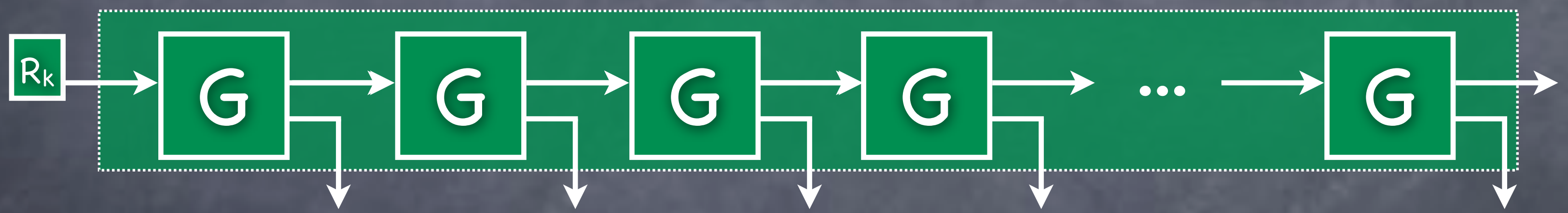

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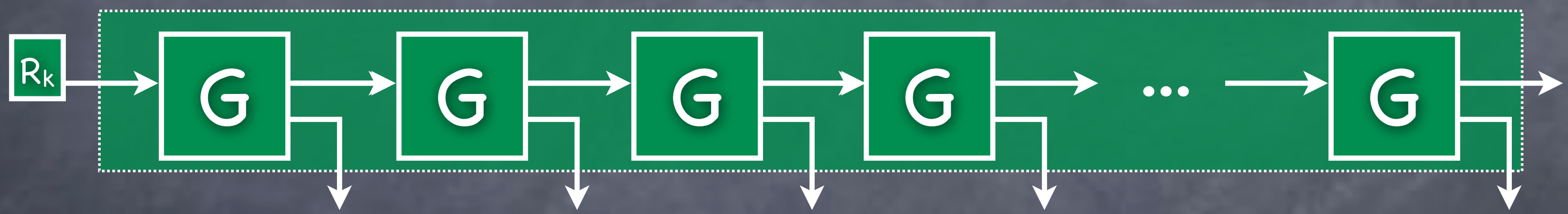

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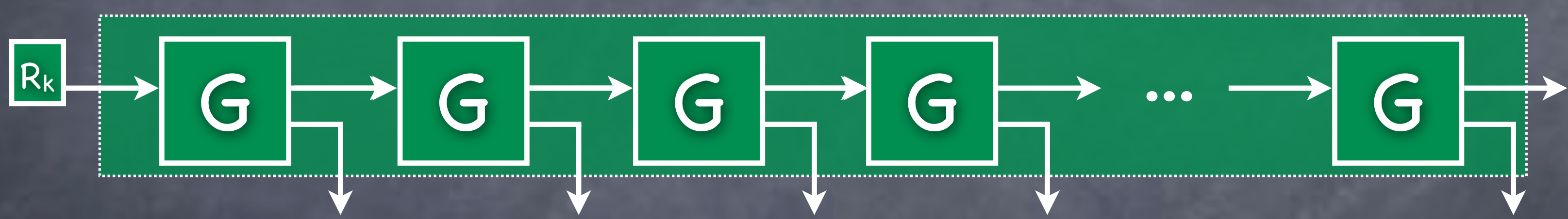

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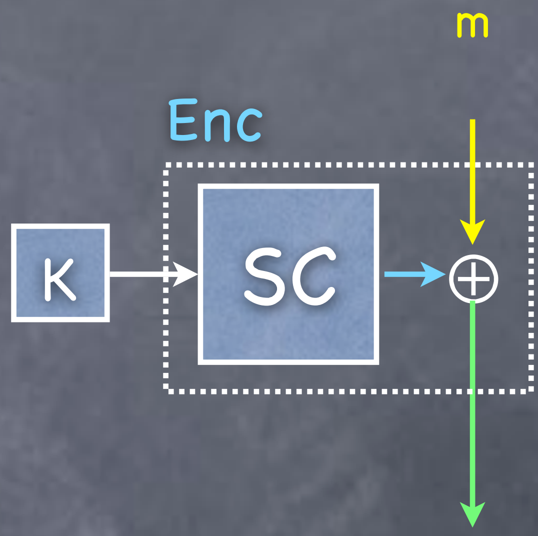

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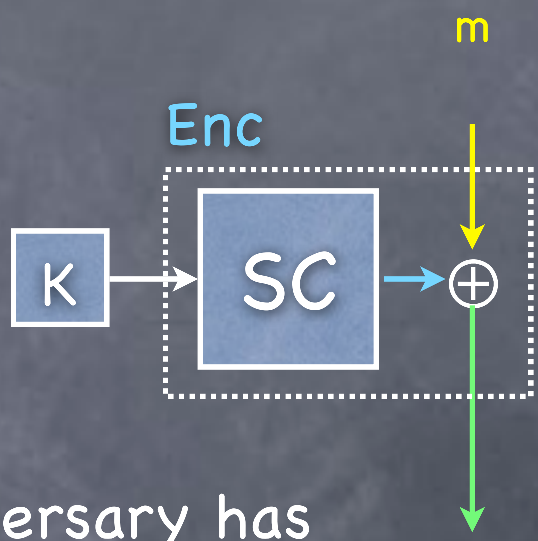

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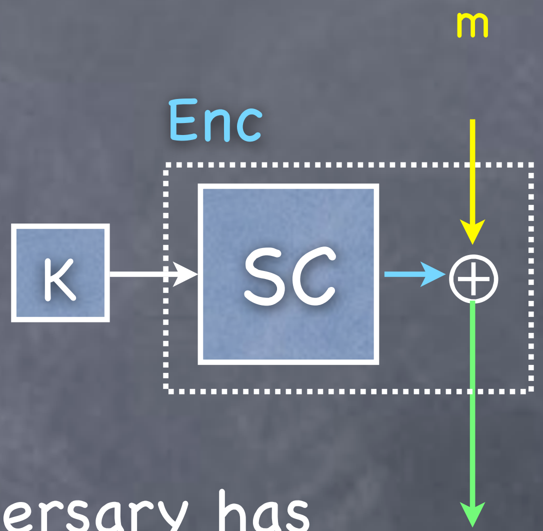

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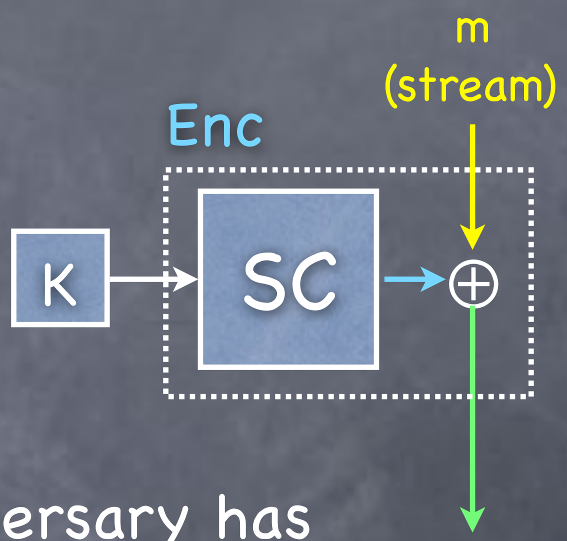

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  - Next: Constructing a proper (multi-message) SKE scheme