Groups for PKE: A Quick Primer

Discrete Log and DDH Assumptions,
Candidate Trapdoor One-Way Permutations

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  - Examples: $\mathbb{Z}$ (integers, with addition operation; infinite group), $\mathbb{Z}_N$ (integers modulo N), G^n (G group; coordinate-wise op)

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- For collection of groups: GroupGen, an algorithm to select a group

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  - Number of generators of $\mathbb{Z}_N$ =: $\varphi(N)$

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    - (Also cyclic for certain other values of N)

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      - A "key-recovery" attack

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  - e.g.: DLA is widely assumed to hold in $\mathbb{Z}_p^*$ (p prime), but DDH assumption doesn't hold there!

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  - Hence one-wayness will rely on hardness of factoring (and more)

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  - $\Leftrightarrow \exists$ integers b, c s.t. $ab = 1+cN$

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Extended Euclidean algorithm to find (b,d) given (a,N). Used to efficiently invert elements in $\mathbb{Z}_N^*$

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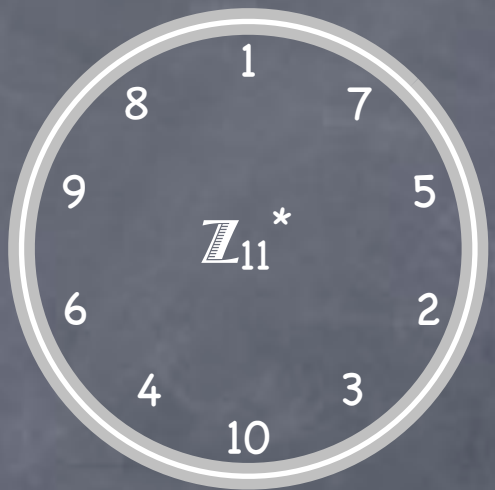

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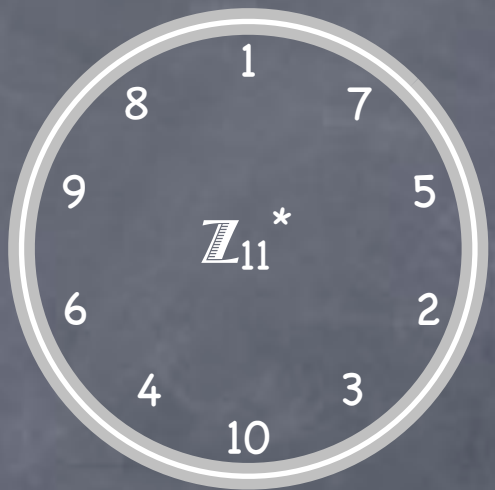

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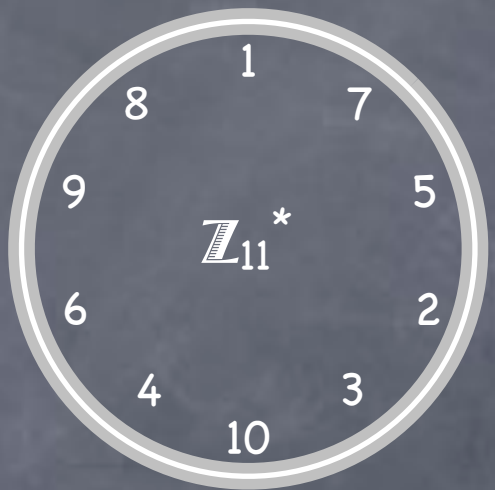

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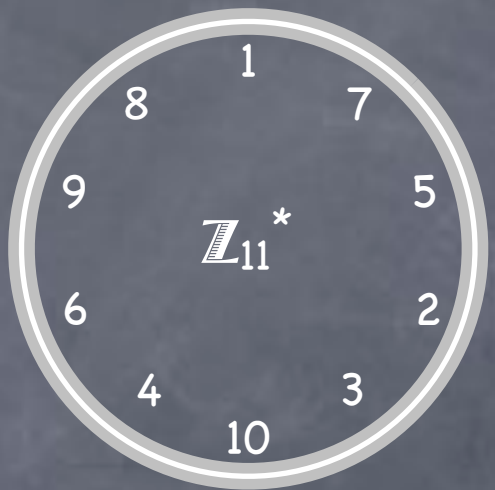

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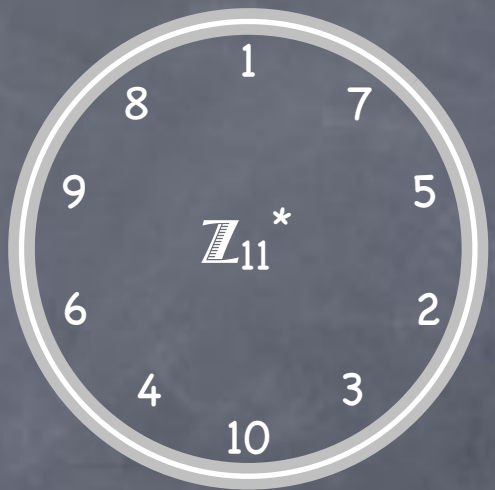

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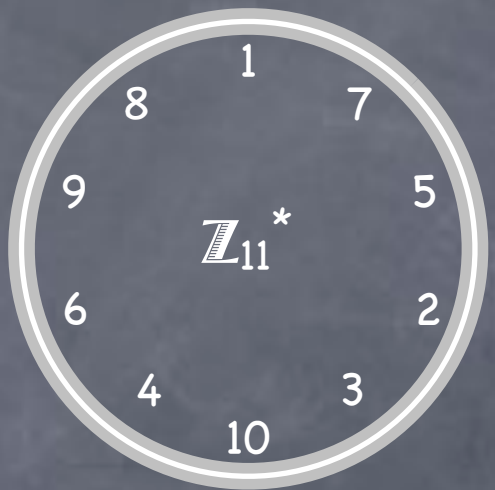

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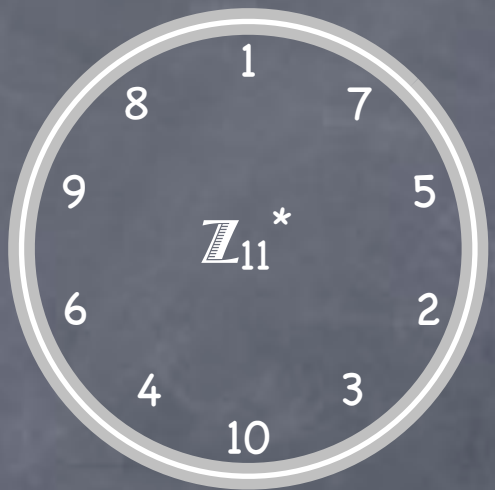

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0	0	0

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  - In fact extends to product of more than two (relatively prime) numbers

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Chinese Remainder Theorem and $\mathbb{Z}_N$

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  - $(a,b) +_{(\text{mod } N)} (a',b') = (a +_{(\text{mod } P)} a', b +_{(\text{mod } Q)} b')$
- CRT: $\mathbb{Z}_N \cong \mathbb{Z}_P \times \mathbb{Z}_Q$ (group isomorphism)

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- Easy to compute the isomorphism (in both directions) if P, Q known [Exercise]

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    - d s.t. $ed=1 \pmod{\varphi(N)}$, $x^{ed} = x \pmod N$

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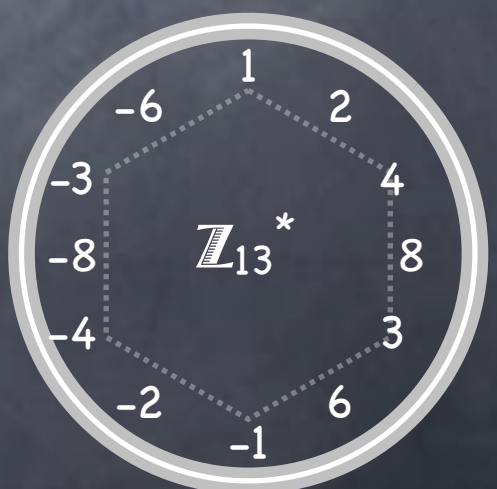

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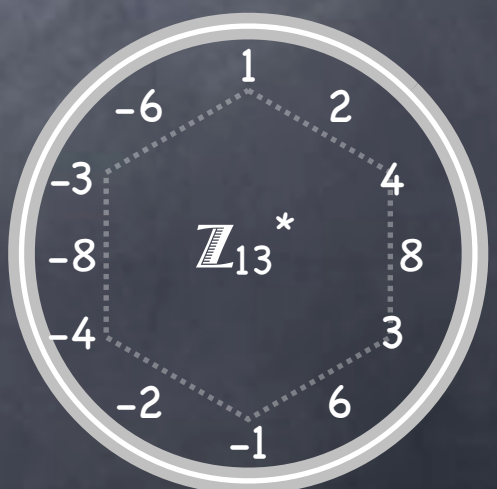

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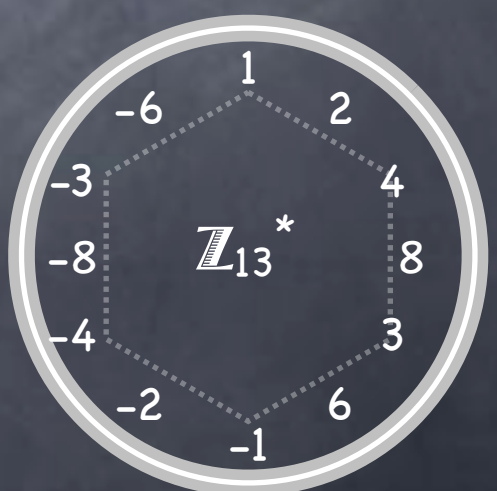

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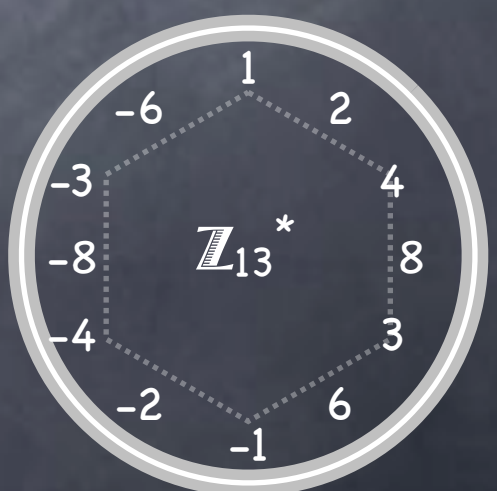

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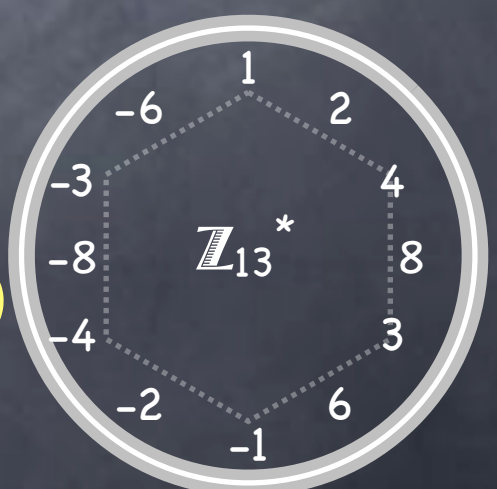

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    - $\sqrt{(x^2,y^2)} = (\pm x, \pm y)$ in $\mathbb{Z}_P^* \times \mathbb{Z}_Q^*$ but exactly one in $\mathbb{QR}_P^* \times \mathbb{QR}_Q^*$
    - Can efficiently do this, if can compute (and invert) the isomorphism from $\mathbb{QR}_N^*$ to $\mathbb{QR}_P^* \times \mathbb{QR}_Q^*$
      - (P,Q) is a trapdoor
    - Without trapdoor, OWF candidate ($\mathbb{QR}_N^*$ forms $1/4^{th}$ of $\mathbb{Z}_N^*$)

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