The Price of Anarchy

Heather Huynh, Vinay Koshy,
Vidushi Ojha

About the Author

Tim Roughgarden

Professor of Computer Science at Columbia University

http://timroughgarden.org/

What is the “price of anarchy”?

“… a worst-case measure of the inefficiency of selfish behaviour.”

(Roughgarden 2009)

Example:

VCG auction

vs.

Auction with no mechanism
(anarchy)

$$\text{Price of anarchy} = \frac{\text{optimal social welfare}}{\text{social welfare in worst equilibrium}}$$

(Wikipedia 2020, “Price of anarchy in auctions”)

Example: Prisoner's Dilemma

Cost matrix:

	Cooperate	**Defect**
Cooperate	1, 1	7, 0
Defect	0, 7	5, 5

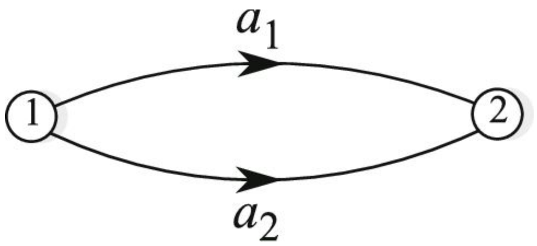
POA =
Equilibrium cost
/
"Optimal" cost

"Optimal" cost =
1 + 1 = 2

Equilibrium cost =
5 + 5 = 10

Price of Anarchy =
10/2 = 5

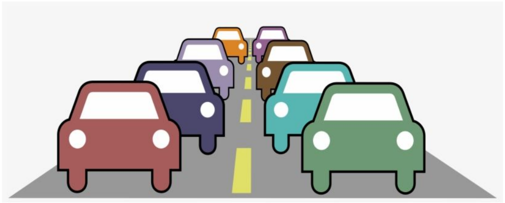
Why do we care?



Generalizes beyond routing

Tells us when we need a mechanism
(POA is not ~1)

Some equilibria are inefficient



Sometimes there's a need for
"centralized" solution

Connections

- Tragedy of the commons
  - Sharing common resources “fairly” is hard
- Price of stability
  - Another related metric; ratio of socially optimal solution to “best” equilibrium
- Bounded rationality
  - People do not always act optimally!
- Mechanism design
  - POA bounds give us an idea of when mechanisms are needed, and how good they are

Related Work

Worst-case Equilibria

Showed upper and lower bound on the **ratio between the worst Nash equilibrium and the overall optimum solution** (coordination ratio)

Mention that Nash equilibrium is an **important indicator of the behavior of non-cooperative agents**

This ratio is what this current paper refers to as the price of anarchy

Koutsoupias, Elias, and Christos Papadimitriou. "Worst-case equilibria." *Annual Symposium on Theoretical Aspects of Computer Science*. Springer, Berlin, Heidelberg, 1999.

Algorithms, Games, and the Internet

Connection of the Internet to Theoretical CS (ex: TCP/IP routing)

Coins the term **price of anarchy** for the coordination ratio

Relates this to Internet architecture where different entities are making different decisions; **how does this compare with the ideal?**

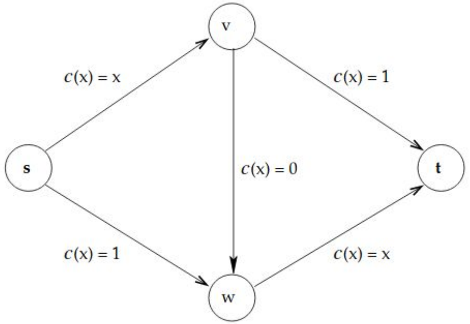
Serves a bit as motivation for why the price of anarchy is important for networks

Papadimitriou, Christos. "Algorithms, games, and the internet." *Proceedings of the thirty-third annual ACM symposium on Theory of computing*. 2001.

Routing Game

Given:

1. Graph G = (V, E)
2. A cost function for each edge in G*
3. Goal units of traffic, r (will be 1 for all examples)
4. Source s and target t in G



*Assume cost functions are nonnegative, continuous, and non-decreasing

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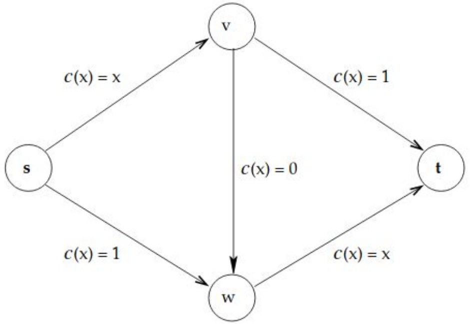
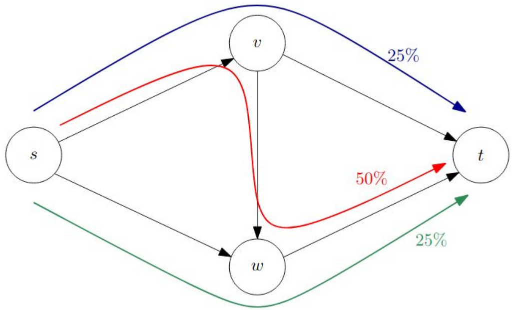


Output:

A *flow* of r units of traffic over the graph.
The cost of a flow can be thought of as the average cost of getting flow from



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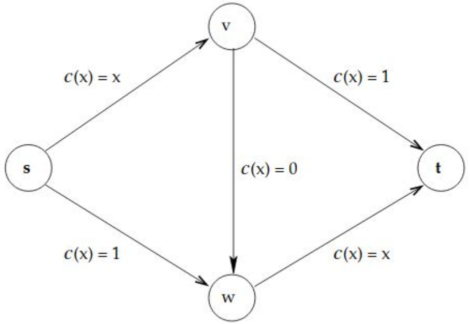
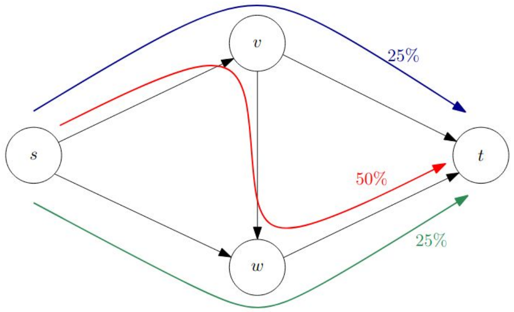
Routing Game

What's the cost of this routing assignment?





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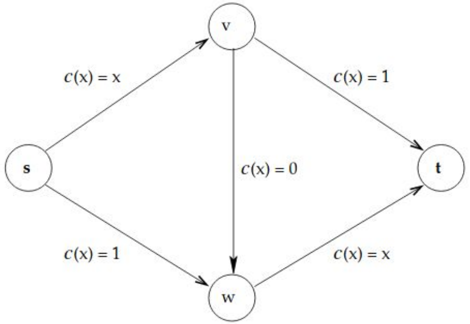
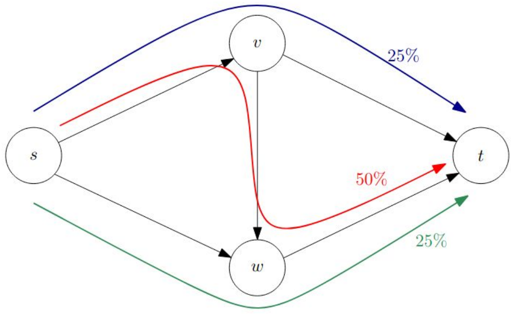
Routing Game

What’s the cost of this routing assignment?

0.25*(.75 + 1)





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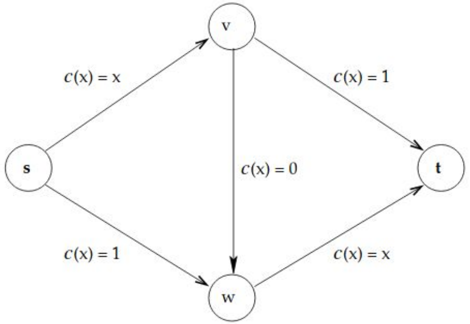
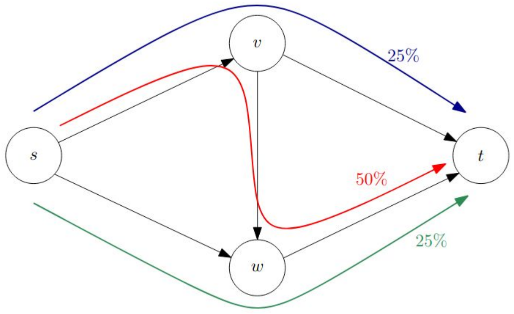
Routing Game

What's the cost of this routing assignment?

0.25*(.75 + 1) + .5*(.75 + 0 +.75)





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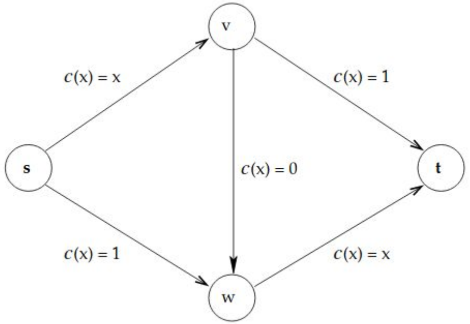
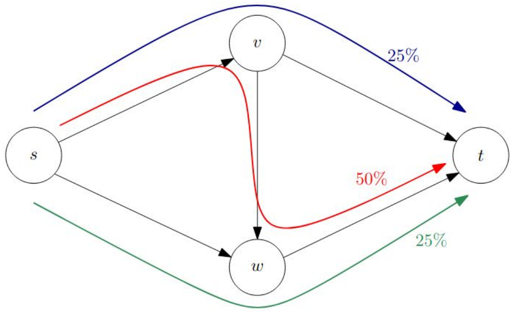
Routing Game

What's the cost of this routing assignment?

0.25*(.75 + 1) + .5*(.75 + 0 +.75) + .25*(1+.75) = 1.625





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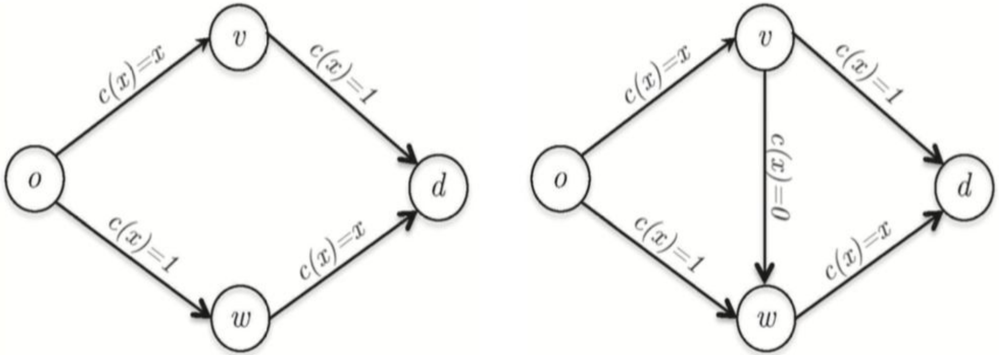
Two types of outcomes

For any routing game, there are many possible flow assignments



(a) Initial network (b) Augmented network

Roughgarden, T. (2016). Introduction and Examples, Twenty lectures on algorithmic game theory. Cambridge University Press.

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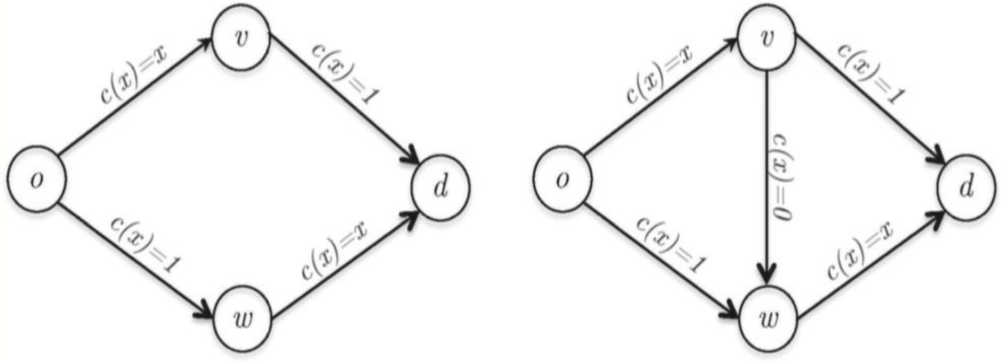
Selfish routing assignment vs **Socially optimal assignment**



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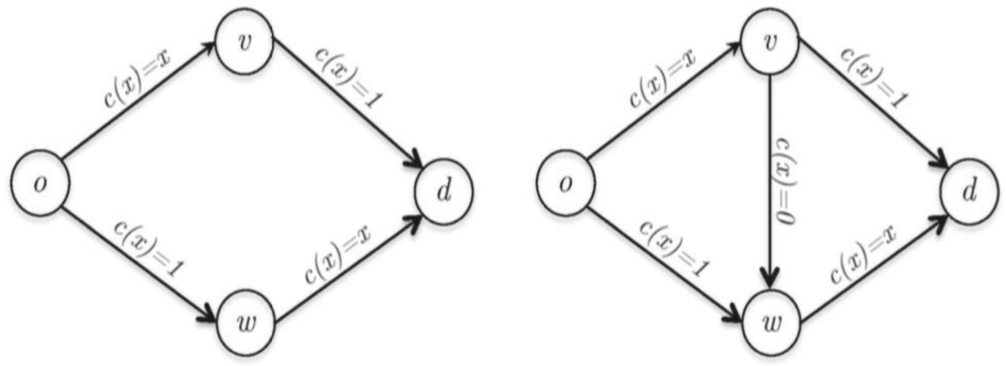

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When is the price of anarchy low?

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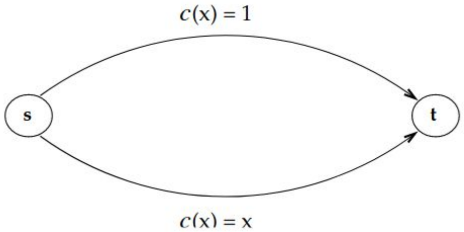
Let's look at a simple example:



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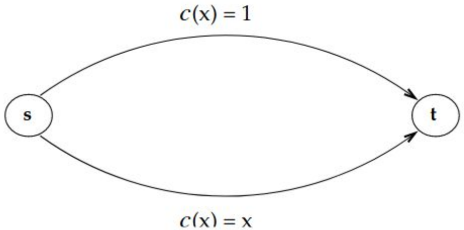

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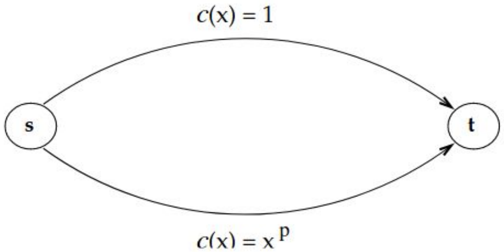

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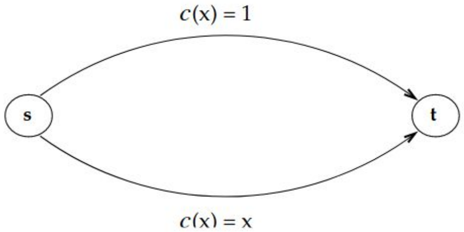

Observation: "Fairness" of the selfish solution

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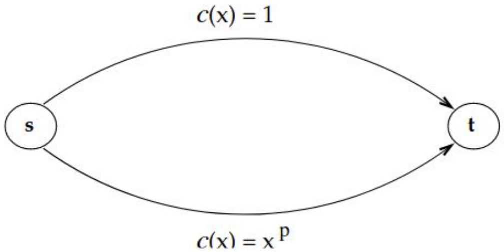

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What did we learn from the previous two examples? So far:

1. Highly nonlinear cost functions can lead to a large POA

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Question: Are nonlinear cost functions the only barrier to a low POA?

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What did we learn from the previous two examples? So far:

1. Highly nonlinear cost functions can lead to a large POA
2. POA is close to 1 when cost functions are linear

Question: Are nonlinear cost functions the only barrier to a low POA?

Answer: Basically…. Yes!

Theorem

Theorem 3.1 (Tight POA Bounds for Selfish Routing (Informal) [5]) *Among all networks with cost functions in a set $\mathcal{C}$, the largest POA is achieved in a Pigou-like network.*

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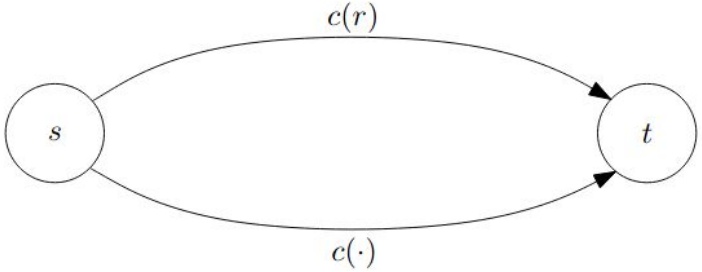
Pigou-like network:



Notice: Selfish-routing outcome = All traffic on bottom edge

Proof Sketch

Theorem 3.1 (Tight POA Bounds for Selfish Routing (Informal) [5]) *Among all networks with cost functions in a set $\mathcal{C}$, the largest POA is achieved in a Pigou-like network.*

1. For a cost function family C, what’s the worst POA on a Pigou-like network?
2. Edge by edge, the gap in costs between the the cost of the selfish solution and the socially optimal solution is no worse than the Pigou bound plus some error term
3. If we sum over all the edges in the graph, the error terms become manageable

Critiques

Strengths

- Takes into account strategic behavior of actors
- Able to pinpoint what causes the inefficiency (cost functions)
- Usefulness of POA as a metric for how good a mechanism is

Weaknesses and Future Work

- Actors not always rational
- Assumption of perfect information → future work could be a network where actors have incomplete knowledge
- Socially optimal solutions are not necessarily “fair” -- how do we reduce the unfairness?

What are some other strengths and weaknesses you see?

Discussion

- The author mentions a “benevolent dictator” in his examples. How do you democratize something like this?
- Is there a way to have something in between being forced to share or being able to behave selfishly?
  - Related to the tragedy of the commons
  - How can we encourage people to take the socially optimal solution?
- Situations where people value the socially optimal solution (“do the right thing”)?

References

- Koutsoupias, Elias, and Christos Papadimitriou. "Worst-case equilibria." *Annual Symposium on Theoretical Aspects of Computer Science*. Springer, Berlin, Heidelberg, 1999.
- Papadimitriou, Christos. "Algorithms, games, and the internet." *Proceedings of the thirty-third annual ACM symposium on Theory of computing*. 2001.
- Roughgarden, Tim. “Intrinsic robustness of the price of anarchy.” In Proceedings of the forty-first annual ACM symposium on Theory of computing, pp. 513-522. 2009.
- Roughgarden, T. (2016). Introduction and Examples, Twenty lectures on algorithmic game theory. Cambridge University Press.
- Wikipedia. “Price of anarchy in auctions.” 25 January 2020. https://en.wikipedia.org/wiki/Price_of_anarchy_in_auctions.
- D. Braess. “Uber ein Paradoxon aus der Verkehrsplanung.” *Unternehmensforschung*,12:258–268, 1968.
- J. R. Correa, A. S. Schulz, and N. E. Stier Moses. Selfish routing in capacitated networks. *Mathematics of Operations Research*, 29(4):961–976, 2004.
- E. Koutsoupias and C. H. Papadimitriou. Worst-case equilibria. In *Proceedings of the16th Annual Symposium on Theoretical Aspects of Computer Science (STACS)*, volume 1563 of *Lecture Notes in Computer Science*, pages 404–413, 1999.
- A. C. Pigou. *The Economics of Welfare*. Macmillan, 1920.
- T. Roughgarden. The price of anarchy is independent of the network topology. *Journal of Computer and System Sciences*, 67(2):341–364, 2003.
- T. Roughgarden and E. Tardos. How bad is selfish routing? *Journal of the ACM*, 49(2):236–259, 2002.
- Y. Sheffi. *Urban Transportation Networks: Equilibrium Analysis with Mathematical Programming Methods*. Prentice-Hall, 1985.

Questions for Hari

- Do we need to go into the proof in more detail than this proof sketch?

Who’s doing what

- What is POA? Why do we care? Examples (Vidushi)
- Related work (Heather)
- Setup of problem (Vinay)
- Proof sketch (Vinay)
- Critiques/strengths (all)
- Applications (Vidushi)