Game Theory

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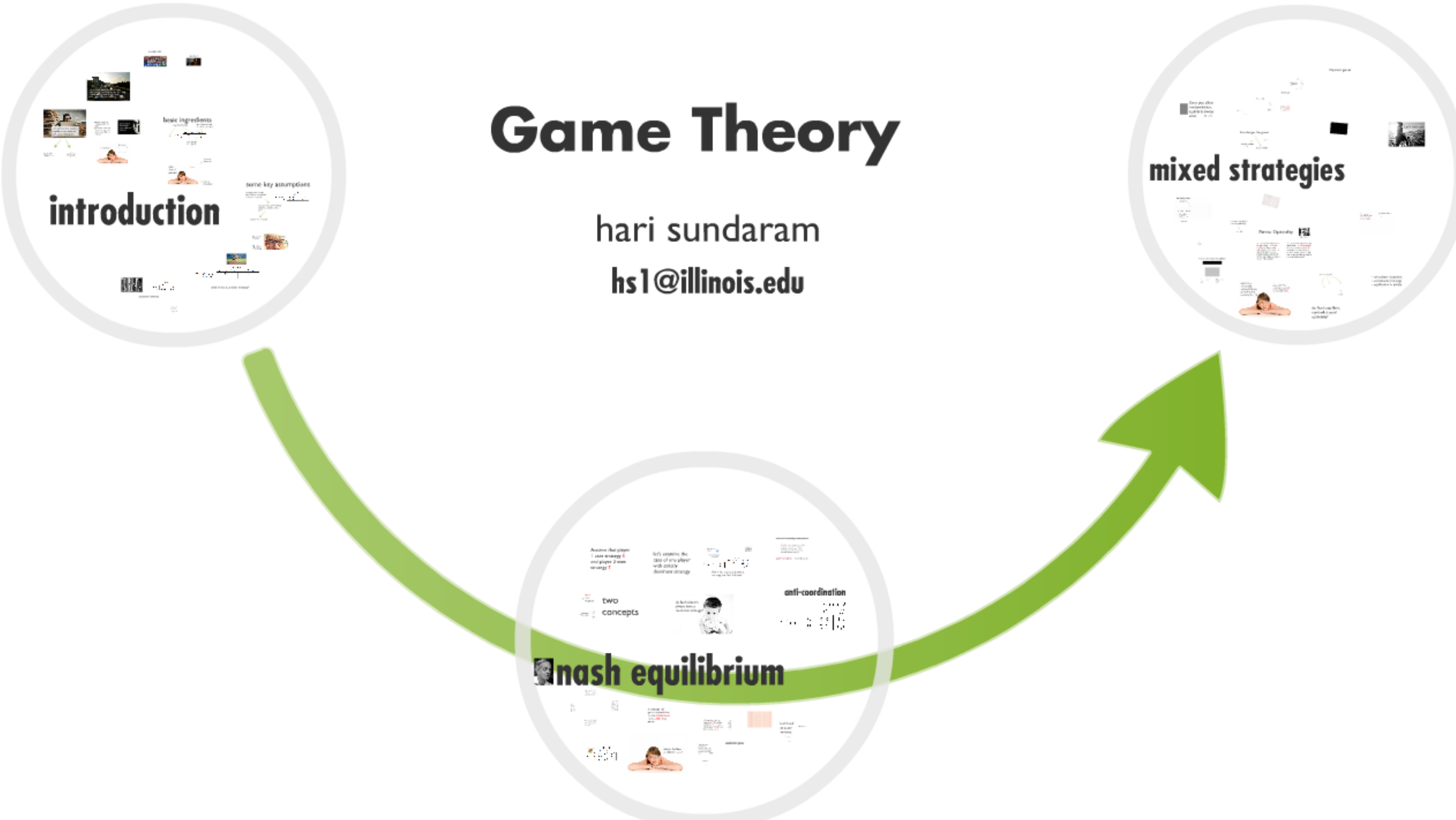
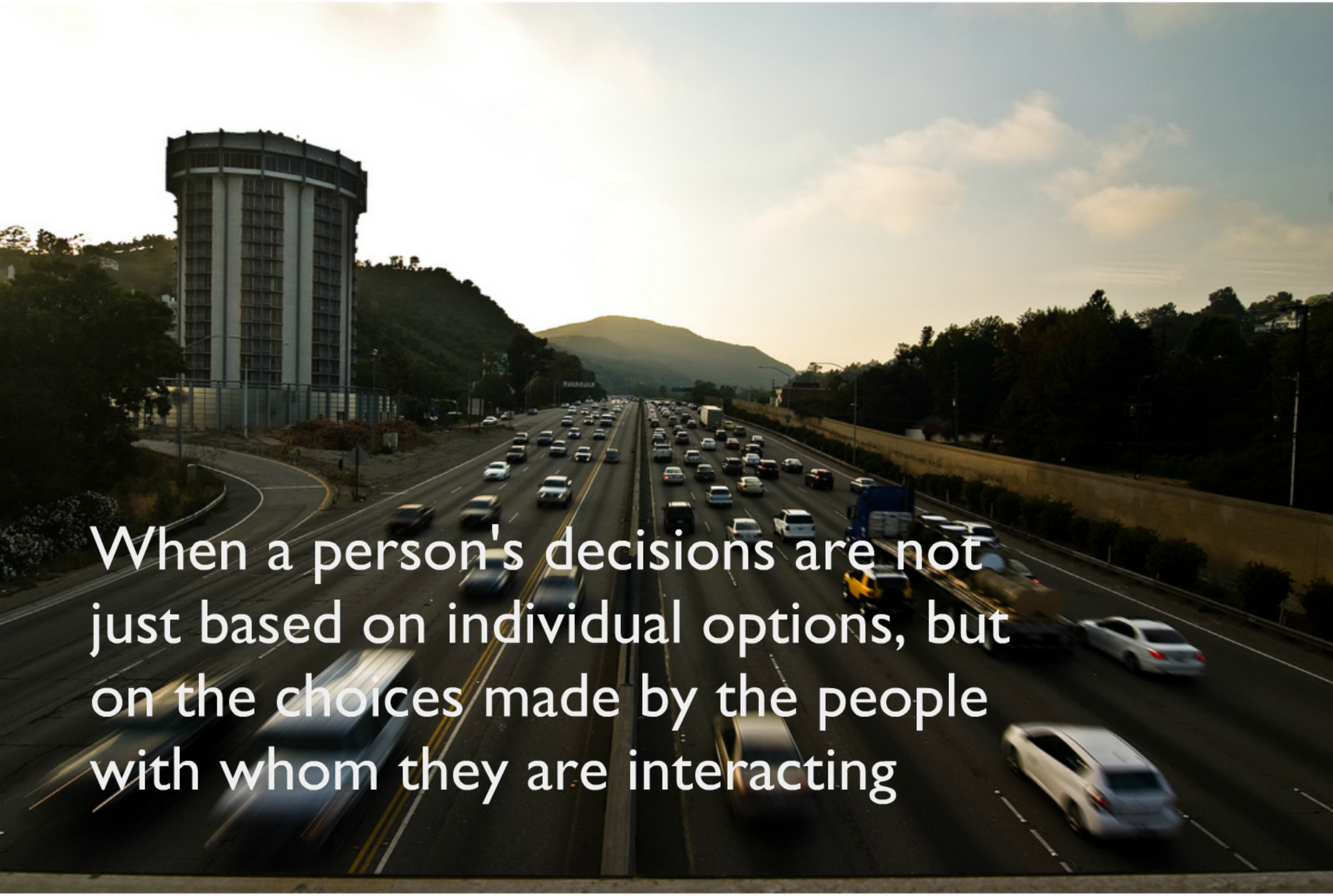
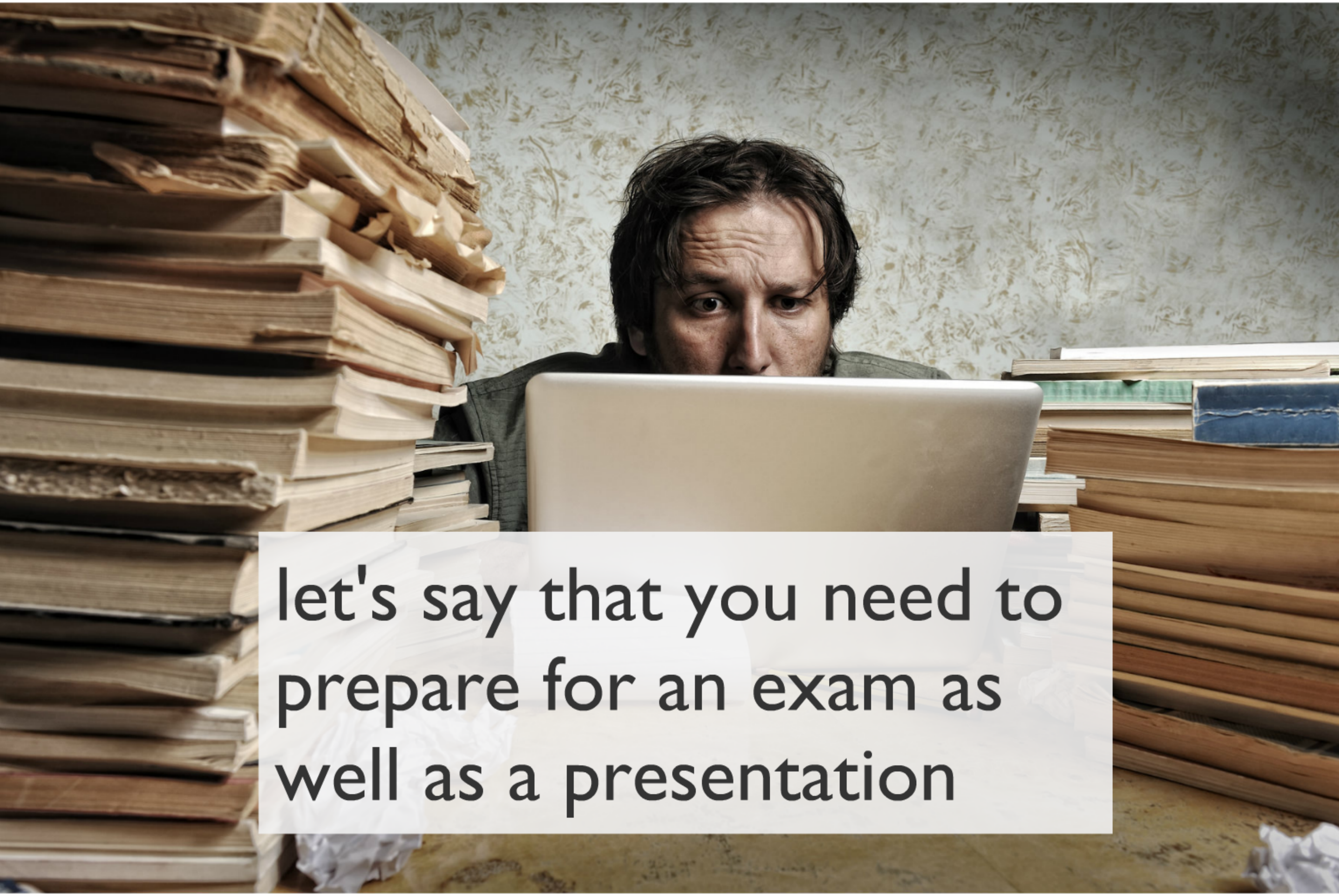



When a person's decisions are not just based on individual options, but on the choices made by the people with whom they are interacting

penalty kicks

auctions


let's say that you need to prepare for an exam as well as a presentation

1 you don't have the time to prepare for both!

2

you have an
accurate idea of
the estimated
grade under
different situations

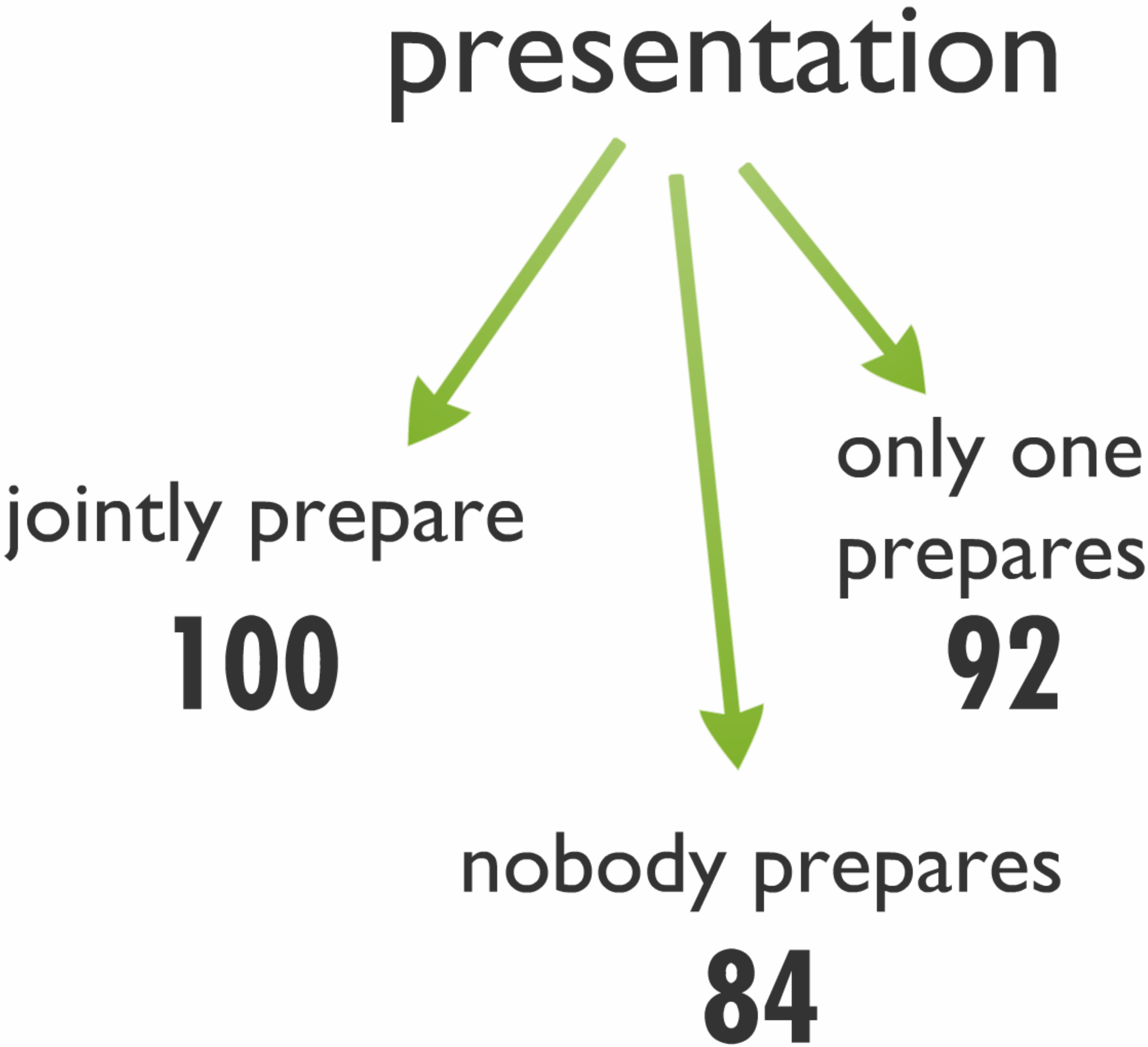
exam

study

92

don't study

80


presentation
jointly prepare
100
only one prepares
92
nobody prepares
84

Payoff matrix

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

completely defines the outcome!

Payoff matrix

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

completely defines the outcome!

what should you do?

basic ingredients

a set of participants

each player has a set of options (strategies)

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

each strategy has a payoff

some key assumptions

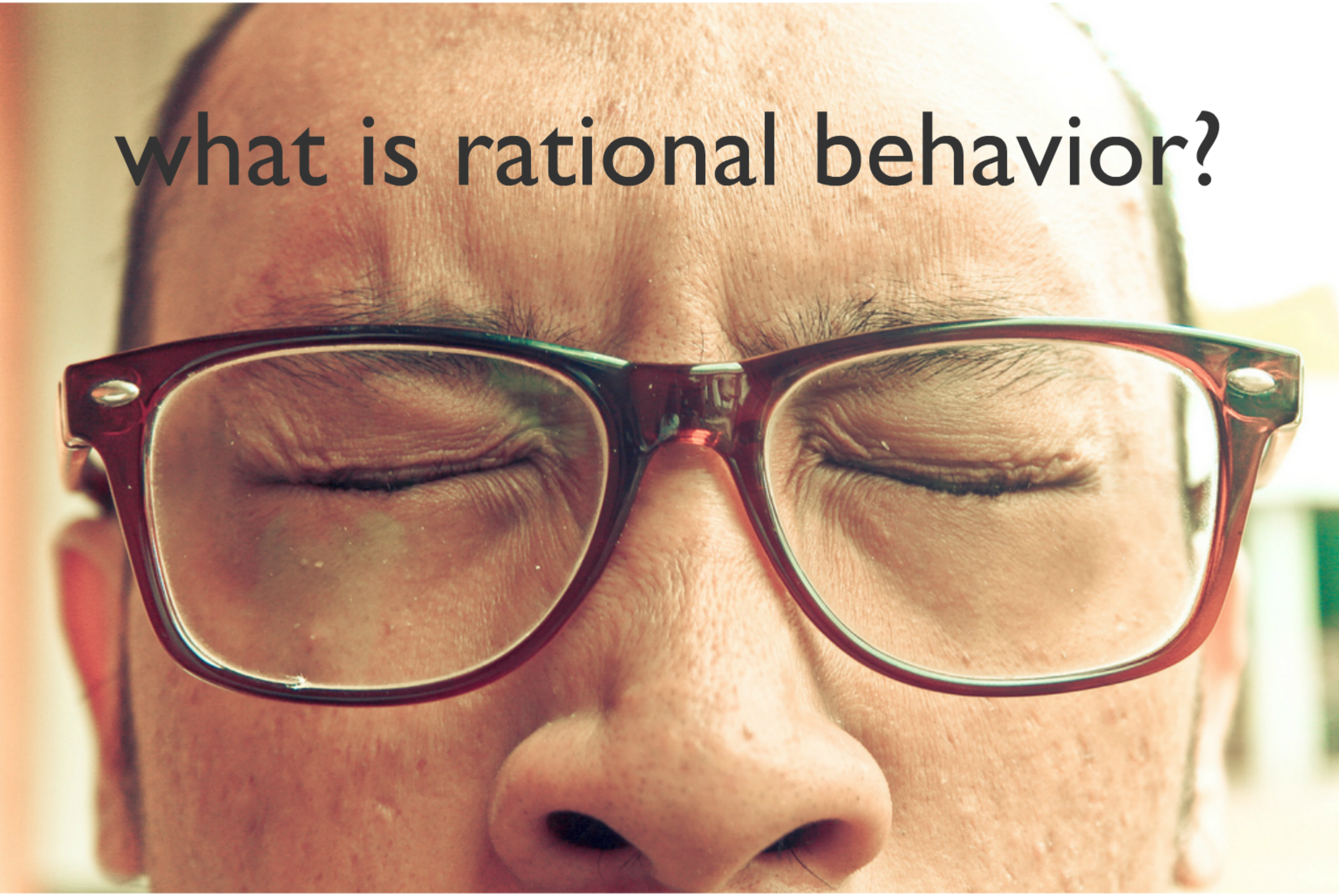
Everything that a player cares about is summarized in the player's payoffs.

Each player knows everything about the structure of the game

Each person is rational

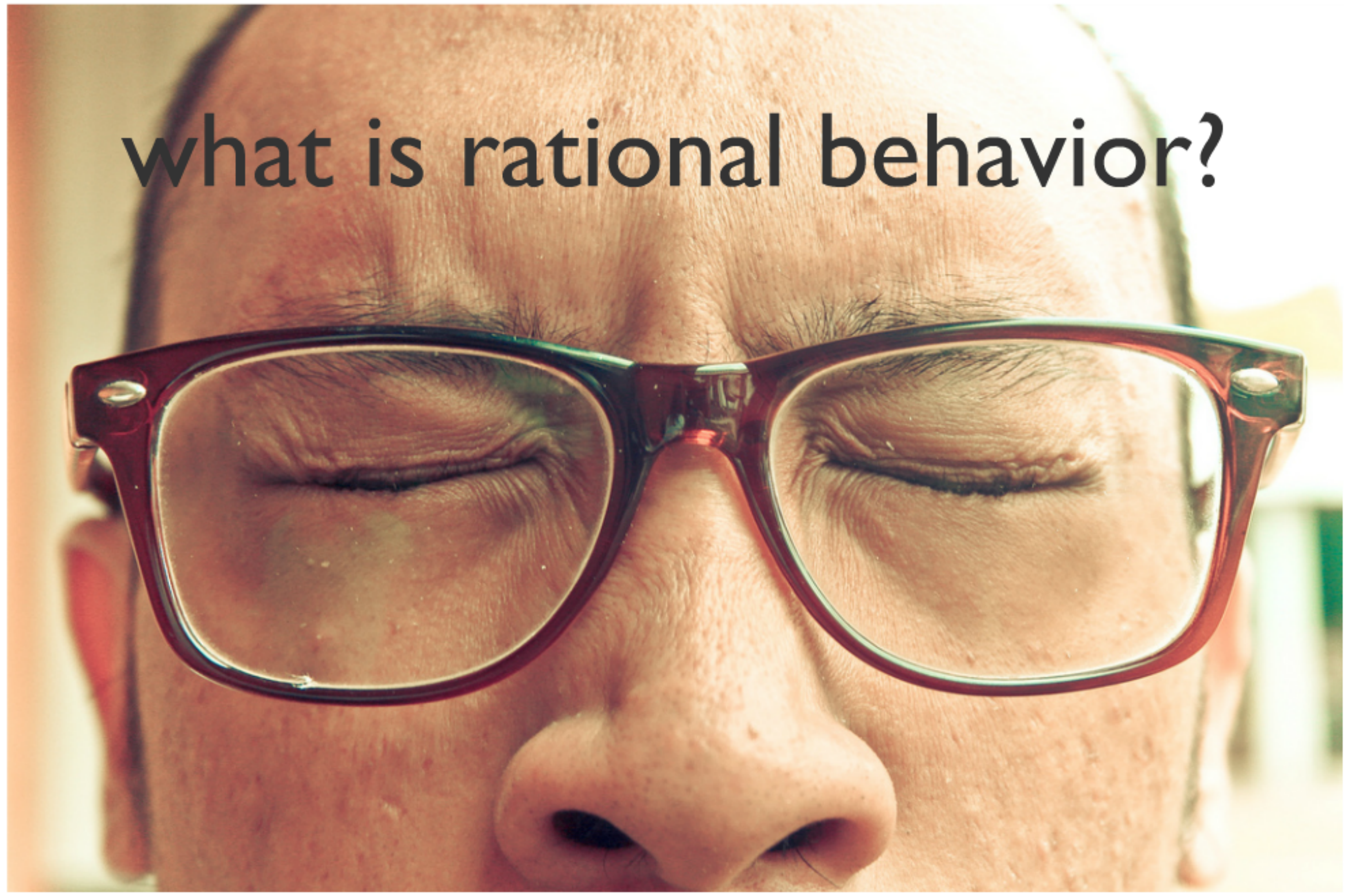
		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

what is rational behavior?



1 each player wants to maximize her own payoff

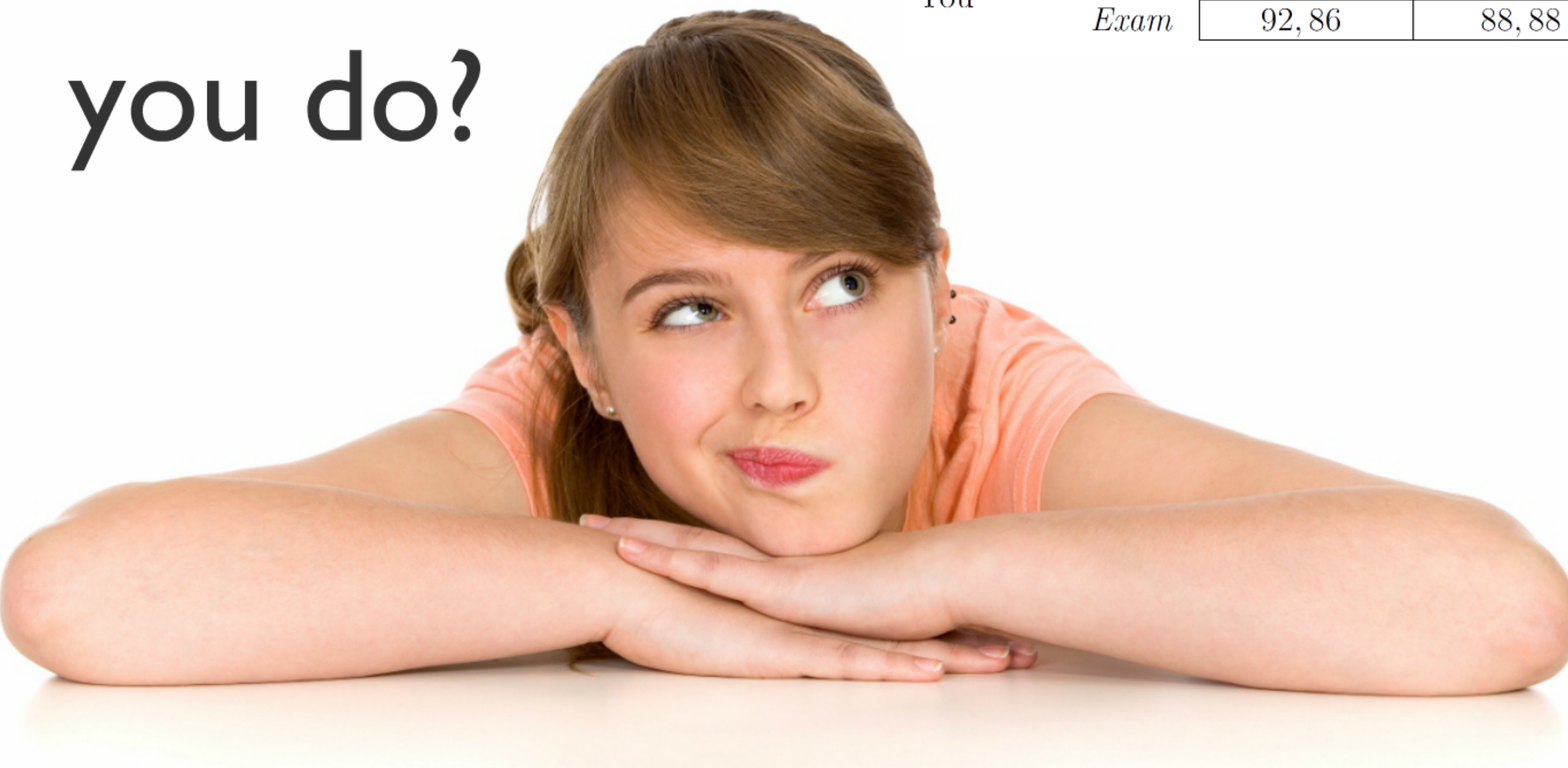
2 each player actually succeeds in selecting the optimal strategy



what should you do?

Payoff matrix

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88



what should you do?

Payoff matrix

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

dominant strategies

limits of rational play

		Suspect 2 NC	Suspect 2 C
Suspect 1	NC	$-1, -1$	$-10, 0$
	C	$0, -10$	$-4, -4$

A prisoner's dilemma

"If you confess, and your partner doesn't confess, then you will be released and your partner will be charged with the crime.

Your confession will be sufficient to convict him of the robbery and he will be sent to prison for 10 years.

If you both confess, then we don't need either of you to testify against the other, and you will both be convicted of the robbery. (Although in this case your sentence will be 4 years only because of your guilty plea.)

	Athlete 2: *Don't Use Drugs*	Athlete 2: *Use Drugs*
Don't Use Drugs	3, 3	1, 4
Use Drugs	4, 1	2, 2

what is the dominant strategy?

nash equilibrium

Assume that player 1 uses strategy S and player 2 uses strategy T

let's examine the case of one player with strictly dominant strategy

Firm 1 has a strictly dominant strategy, but Firm 2 doesn't

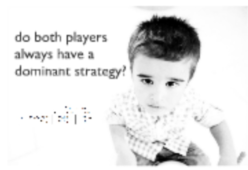
common knowledge assumptions

strict best response

dominant strategies

two concepts

do both players always have a dominant strategy?



anti-coordination

		Animal 2	
		D	H
Animal 1	D	$3, 3$	$1, 5$
	H	$5, 1$	$0, 0$

A reminder: all games analyzed thus far are simultaneous move, single shot games

what is the Nash equilibrium here?

We say that a pair of strategies (S,T) is a Nash equilibrium if S is a best response to T, and T is a best response to S.

coordination games

let's look at some variants

Assume that player 1 uses strategy S and player 2 uses strategy T

two concepts

1 strict best response

$$P_1(S,T) > P_1(S',T)$$

2 dominant strategies

2 dominant strategies

S is the best response for every T

S is the strict best response for every T

do both players always have a dominant strategy?

		Your Partner: *Presentation*	Your Partner: *Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

let's examine the case of one player with strictly dominant strategy

TM

Market Segments Size

Low priced: 60%
High priced: 40%

Same Segment Competition

Firm 1 dominates Firm 2, when they compete in the same segment

Firm 1: 80%
Firm 2: 20%

assume that profit per item is the same in both segments

Market Segments Size

Low priced: 60%
High priced: 40%

Same Segment Competition

Firm 1 dominates Firm 2, when they compete in the same segment

Firm 1: 80%
Firm 2: 20%

		Firm 2: *Low-Priced*	Firm 2: *Upscale*
Firm 1	*Low-Priced*	.48, .12	.60, .40
	Upscale	.40, .60	.32, .08

Firm 1 has a strictly dominant strategy, but Firm 2 doesn't

common knowledge assumptions

The Firms know the structure of the game, they know that each of them know the structure of the game, they know that each of them know that each of them know etc.

John C. Harsanyi. Game with incomplete information played by "Bayesian" players, I–III. Part I: The basic model. Management Science, 14(3):159–182, November 1967.

how realistic is this?

Let's now consider the case when two firms are competing to do business with one of three clients (A,B and C)

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		Firm 2: A	Firm 2: B	Firm 2: C
Firm 1	A	$4, 4$	$0, 2$	$0, 2$
	B	$0, 0$	$1, 1$	$0, 2$
	C	$0, 0$	$0, 2$	$1, 1$

A reminder: all games analyzed thus far are simultaneous move, single shot games

We say that a pair of strategies (S,T) is a Nash equilibrium if S is a best response to T, and T is a best response to S.

		Firm 2		
		A	B	C
	A	$4,4$	$0,2$	$0,2$
Firm 1	B	$0,0$	$1,1$	$0,2$
	C	$0,0$	$0,2$	$1,1$

what is the Nash equilibrium here?

coordination games

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	1, 1

Games can have multiple Nash equilibria; it becomes hard to predict how rational players would behave occurs in many scenarios: two manufacturers, attack maneuvers, waiting for someone at the mall etc..

Games can have multiple Nash equilibria; it becomes hard to predict how rational players would behave

unbalanced coordination

		Your Partner	
		PowerPoint	*Keynote*
You	*PowerPoint*	1, 1	0, 0
	Keynote	0, 0	2, 2

	Your Partner	
	PowerPoint	*Keynote*
oint	1, 1	0, 0
ote	0, 0	2, 2

still Nash equilibria!

Dynamic games
Once you allow randomization, equilibria always exist.
this changes the game!
mixed strategies
pure strategies
Let's exercise
mixed strategies
therefore, equality is the only possibility!
q = 1/2
Pareto Optimality
A choice of strategies—one by each player—is Pareto-inefficient if there is one other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.
A choice of strategies—one by each player—is Pareto-optimal if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.
what is the relationship between Pareto optimality and social optimality?
pareto optimal
nash
do Nash equilibria contradict social optimality?
• two player situations
• evolutionary biology
• equilibrium in beliefs

matching pennies

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

No Nash
equilibrium

Once you allow randomization, equilibria always exist.

matching pennies

		Player 2 H	Player 2 T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

No Nash equilibrium

		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

Say player 2 commits to H with probability q and to T with probability (1-q).

this changes the game!

Player 1 \ Player 2	H	T
H	$-1, +1$	$+1, -1$
T	$+1, -1$	$-1, +1$

Say player 2 commits to H with probability q and to T with probability (1-q).

mixed strategies

$$q \in (0, 1)$$

pure strategies

Expected Payoffs for Player 1

H: $(-1)q + (1)(1-q) = 1 - 2q$

T: $(1)q + (-1)(1-q) = 2q - 1$

Player 2

		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

Pure strategies cannot be part of the matching pennies Nash equilibrium*

Expected Payoffs for Player 1

H: $(-1)q + (1)(1-q) = 1 - 2q$

T: $(1)q + (-1)(1-q) = 2q - 1$

		Player 2: H	Player 2: T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

but in general, you can have pure strategies as a solution when you randomize

- $(1)q + (-1)(1-q) = 2q - 1$

		Player 2: H	Player 2: T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

Expected Payoffs for Player 1

H: $(-1)q + (1)(1-q) = 1 - 2q$
T: $(1)q + (-1)(1-q) = 2q - 1$

Player 2

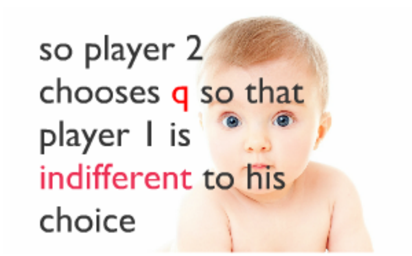
Player 1	H	T
H	−1, +1	+1, −1
T	+1, −1	−1, +1

$$\text{if } 1 - 2q \neq 2q - 1$$

then a clear dominant strategy exists!

$$\therefore q = 1/2$$

so player 2 chooses q so that player 1 is indifferent to his choice



		Player 2	
		H	T
Player 1	H	$-1, +1$	$+1, -1$
	T	$+1, -1$	$-1, +1$

What is the mixed-strategy Nash Equilibrium?

		Defense	
		Defend Pass	Defend Run
Offense	Pass	0, 0	10, −10
	Run	5, −5	0, 0

What is the mixed-strategy Nash Equilibrium?

		Defense: Defend Pass	Defense: Defend Run
Offense	Pass	0, 0	10, −10
	Run	5, −5	0, 0

$P: 0 \times q + 10 \times (1-q)$

$R: 5 \times q + 0 \times (1-q)$

q=2/3

$DP: 0 \times p + -5 \times (1-p)$

$DR: -10 \times p + 0 \times (1-p)$

p=1/3


is any of this selfishness good for society as a whole?

A choice of strategies—one by each player—is Pareto-inefficient if there is one other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.

A choice of strategies—one by each player—is Pareto-optimal if there is no other choice of strategies in which all players receive payoffs at least as high, and at least one player receives a strictly higher payoff.

Pareto Optimality

Vilfredo Pareto

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pareto optimal

You \ Your Partner	*Presentation*	*Exam*
Presentation	90, 90	86, 92
Exam	92, 86	88, 88

nash

A choice of strategies—one by each player—is a social welfare maximizer (or socially optimal) if it maximizes the sum of the players' payoffs.

what is the relationship between Pareto optimality and social optimality?

A choice of strategies—one by each player—is a social welfare maximizer (or socially optimal) if it maximizes the sum of the players' payoffs.

		Your Partner	
		Presentation	*Exam*
You	*Presentation*	90, 90	86, 92
	Exam	92, 86	88, 88

the set of strategies
adopted by n players

$$(S_1, S_2, \ldots, S_n)$$

$$P_i(S_1, S_2, \ldots, S_n)$$

payoff to i

best response

$$P_i(S_1, S_2, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots, S_n) \geq P_i(S_1, S_2, \ldots, S_{i-1}, S_i', S_{i+1}, \ldots, S_n)$$

although dominant and strictly dominant strategies can exist in games with many players and many strategies, they are rare

A strategy is strictly dominated if there is some other strategy available to the same player that produces a strictly higher payoffs in response to every choice of strategies by the other players.

$$P_i(S_1, S_2, \ldots, S_{i-1}, S_i', S_{i+1}, \ldots, S_n) > P_i(S_1, S_2, \ldots, S_{i-1}, S_i, S_{i+1}, \ldots, S_n)$$

There are six towns and two Firms want to open stores. Firm 1 has the option of opening its store in any of towns A, C, or E, while Firm 2 has the option of opening its store in any of towns B, D, or F. These decisions will be executed simultaneously.

		Firm 2		
		B	*D*	*F*
Firm 1	*A*	1, 5	2, 4	3, 3
	C	4, 2	3, 3	4, 2
	E	3, 3	2, 4	5, 1

payoff matrix

→

		Firm 2	
		B	*D*
Firm 1	*C*	4, 2	3, 3
	E	3, 3	2, 4

Nash equilibria survive iterated deletion

1 We start with any *n*-player game, find all the strictly dominated strategies, and delete them.

2 We then consider the reduced game in which these strategies have been removed. In this reduced game there may be strategies that are now strictly dominated, despite not having been strictly dominated in the full game. We find these strategies and delete them.

3 We continue this process, repeatedly finding and removing strictly dominated strategies until none can be found.

1

We start with any n-player game, find all the strictly dominated strategies, and delete them.

2

We then consider the reduced game in which these strategies have been removed. In this reduced game there may be strategies that are now strictly dominated, despite not having been strictly dominated in the full game. We find these strategies and delete them.

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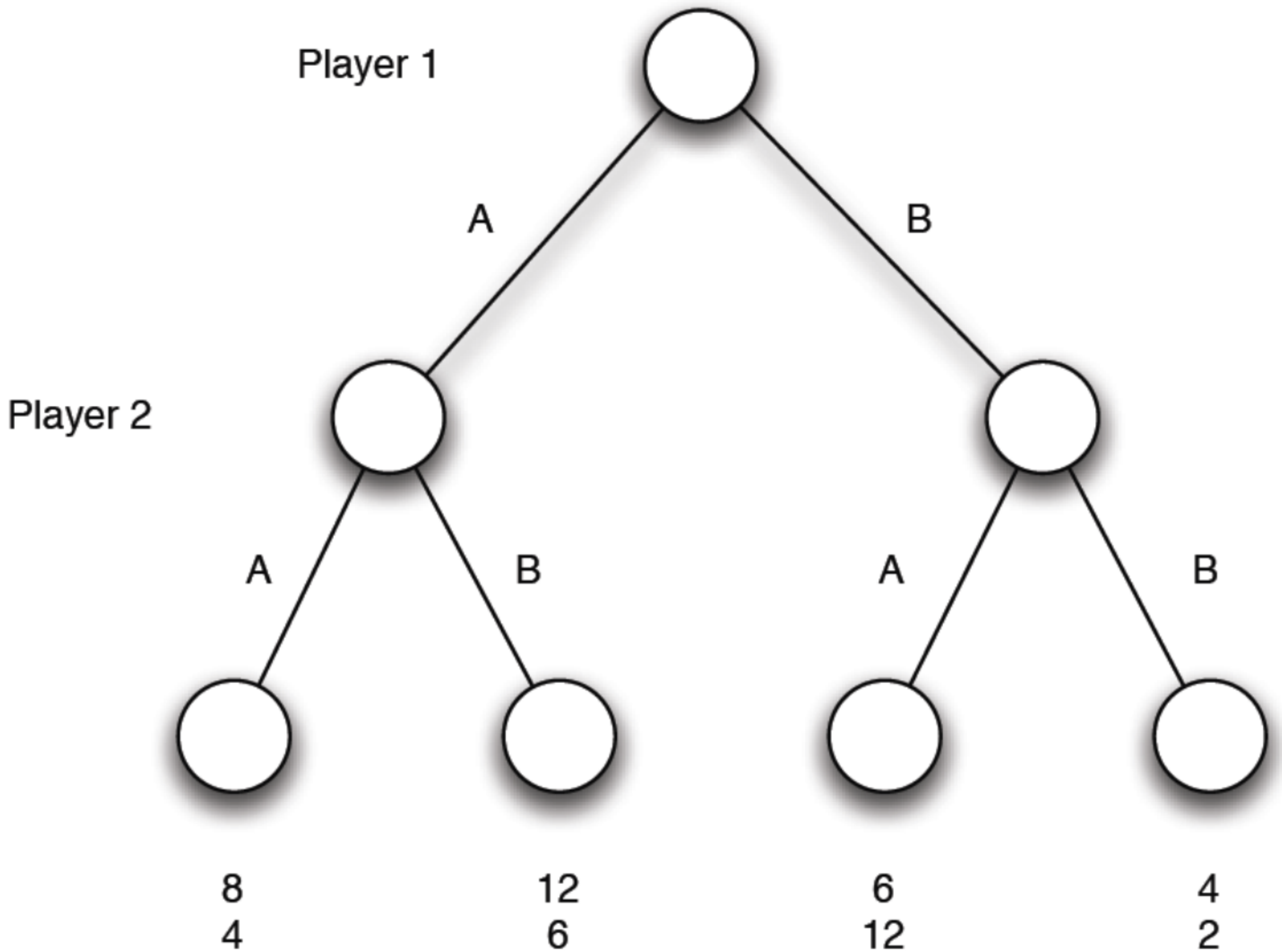

Dynamic games