HW 4

Due on **Wednesday, April 30, 2025 at 9am CST**

598GAA: Geometric Approximation Algorithms, Spring 2025 Version: **1.1**

HW Guidelines: As in previous homework. Solutions must be self contained, and containing all relevant details.

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10 (100 PTS.) Directed net.

You are given a directed graph $\mathsf{G} = (\mathsf{V}, \mathsf{E})$ with (positive) weights on the edges, where $\mathsf{V} = \{v_1, \ldots, v_i\}$. The graph G has n vertices and m edges. Let $d(u, v)$ denote the shortest path length (under the regular weights on the edges) from u to v in G. Observe that $d(u, v)$ might not be equal to $d(v, u)$. Given a parameter $r > 0$, show to compute a weak r-net in G. Your algorithm should be as fast as possible. Prove the bound on the expected running time of our algorithm.

A set of vertex $X \subseteq \mathsf{V}$ is a ***weak r-net*** if

(A) For any $v \in V$, there exists a vertex $x \in X$, such that $d(x, v) < r$ or $d(v, x) < r$.

(B) For all $x, y \in X$, we have $d(x, y) \geq r$ and $d(y, x) \geq r$.

(Hint: First solve the undirected case [think FRT], and then solve the directed case.)

11 SOME METRIC SPACE QUESTIONS.

11.A. Let $(U, \mathcal{d})$ be a finite metric space, where $U = \{p_1, \ldots, p_n\}$. For a point $x \in U$, consider the mapping

$$F(x) = (\mathcal{d}(p_1, x), \mathcal{d}(p_2, x), \ldots, \mathcal{d}(p_n, x)).$$

Prove that for any $x, y \in U$, we have that $\|F(x) - F(y)\|_\infty = \mathcal{d}(x, y)$. Namely, any n-point metric space can be embedded ***isometrically*** (i.e., without distortion) into $\mathbb{R}^n$ with the L_∞-norm.

11.B. Let $(U, \mathcal{d})$ be an HST defined over n points, and let $k > 0$ be an integer. Provide an algorithm that computes the optimal k-median clustering of U in $O(k^2 n)$ time.

Hint: Transform the HST into a tree where every node has only two children. Next, run a dynamic programming algorithm on this tree.

11.C. Prove that the distortion of embedding the star graph with n leaves into $\mathbb{R}^{\mathsf{d}}$ is $\Theta\left(n^{1/\mathsf{d}}\right)$. In addition, show that it can be embedded into $\mathbb{R}^k$ with distortion $\sqrt{2} + \delta$, for any constant $\delta > 0$, for $k = O(\log n)$.

11.D. Let $m(n, \ell)$ be the number of edges in a graph with n vertices and girth at least ℓ.

11.D.i. Show that $m(n, \ell) = O\left(n^{1+2/\ell}\right)$. (Hint: Throw away low degree vertices, and consider a breadth-first search (**BFS**) tree and its depth.)

11.D.ii. Show that there exists a graph with girth $\geq \ell$ and number of edges $m(n, \ell) = \Omega\left(n^{1+1/4\ell}\right)$. (Hint: Generate a random graph by picking every edge with a certain probability. Compute the expected number of bad short cycles, and throw away an edge from each one of them. Argue that the remaining graph still has many edges.)

(By the way, finding the right bounds for $m(n, \ell)$ is still an open problem.)

12 (100 PTS.) Kind of colorful Carathéodory

You are given k disjoint point sets $P_1, \ldots, P_k$ in d dimensions, such that the convex-hull of P_i, denoted by $\mathcal{CH}(P_i)$, contains the origin $o = (0, \ldots, 0)$, for all i. Furthermore, assume that $\mathrm{diameter}(\cup_i P_i) \leq 1$. Let $\varepsilon \in (0,1)$ be a parameter. Prove that there are *disjoint* sets $S_1, \ldots, S_m$, all contained in $\cup_i P_i$, such that:

(I) $m = O(\varepsilon^2 k)$.

(II) For all i, j, we have $|S_i \cap P_j| \leq 1$.

(III) $d(o, \mathcal{CH}(S_i)) = \min_{p \in \mathcal{CH}(S_i)} \|o - p\| \leq \varepsilon$ where o is the origin.

Describe an algorithm as fast as possible to compute these sets $S_1, \ldots, S_m$ – what is the running time of your algorithm?