1 Matroid Union

Matroid union and matroid intersection are closely related in the sense that one can be derived from the other. However they are from different perspectives and have different applications.

To motivate matroid union theorem we state a well known theorem of Tutte and Nash-Williams on packing disjoint spanning trees in graphs.

Theorem 1 (Nash-Williams and Tutte) *An undirected multi-graph $G = (V, E)$ contains k edge-disjoint spanning trees iff for every partition P of V into ℓ sets, $V_1, V_2, \ldots, V_\ell$, the number of edges crossing the partition P is at least $k(\ell - 1)$.*

It is easy to see that the condition is necessary; if $T_1, \ldots, T_k$ are the edge-disjoint spanning trees then each T_i has to contain at least $\ell - 1$ edges across the partition P to connect them. A useful corollary of the above was observed by Gusfield. It is an easy exercise to derive this from the above theorem.

Corollary 2 *If a multi-graph $G = (V, E)$ is $2k$-edge-connected then G contains k edge-disjoint spanning trees.*

Nash-Williams proved a related theorem on covering the edge-set of a graph by forests.

Theorem 3 (Nash-Williams) *Let $G = (V, E)$ be an undirected multi-graph. Then E can be partitioned into k forests iff for each set $U \subseteq V$,*

$$|E[U]| \leq k(|U| - 1). \tag{1}$$

Again, necessity is easy to see; any forest can contain at most $|U| - 1$ edges from $E[U]$. The above two theorems were first shown via graph theoretica arguments but turn out to be special cases of the matroid union theorem, and hence are properly viewed as matroidal results. We start with a basic result of Nash-Williams that gives a clean proof of the matroid union theorem to follow.

Theorem 4 (Nash-Williams) *Let $\mathcal{M}' = (S', \mathcal{I}')$ be a maroid with rank function r'. Let $f : S' \to S$ be a function mapping S' to S. Let $\mathcal{M} = (S, \mathcal{I})$, where $\mathcal{I} = \{f(I') | I' \in \mathcal{I}'\}$. Then $\mathcal{M}$ is a matroid with rank function r, where*

$$r(U) = \min_{T \subseteq U} (|U \backslash T| + r'(f^{-1}(T))). \tag{2}$$

Proof:

We verify the three axioms.

1. $f(\emptyset) = \emptyset$ and hence $\emptyset \in \mathcal{I}$.

2. Say $A \in \mathcal{I}$ and $B \subseteq A$. Then

$$\begin{aligned} A \in \mathcal{I} \quad &\Rightarrow \quad \exists A' \in \mathcal{I}', \text{ s.t. } f(A') = A \\ &\Rightarrow \quad \forall u \in A, f^{-1}(u) \cap A' \neq \emptyset. \end{aligned}$$

   Let $B' = \{u' \in A' | f(u') \in B\}$, then $B = f(B')$ and since $B' \subseteq A', B' \in \mathcal{I}'$ and hence $B \in \mathcal{I}$.

3. Say $A, B \in \mathcal{I}$ and $|B| > |A|$. Let A' be minimal s.t. $f(A') = A$. Similarly let B' be minimal s.t. $f(B') = B$. Then

$$|f^{-1}(u) \cap A'| = 1, \forall u \in A.$$

   Similarly,

$$|f^{-1}(u) \cap B'| = 1, \forall u \in B.$$

   Therefore,

$$|A'| = |A| \text{ and } |B'| = |B|.$$

   Since

$$A', B' \in \mathcal{I}' \text{ and } |B'| > |A'|,$$

$$\Rightarrow \exists u' \in B' \backslash A', \text{ s.t. } A' + u' \in \mathcal{I}'.$$

   Then

$$A + f(u') \in \mathcal{I} \text{ and } f(u') \in B \backslash A.$$

Therefore $\mathcal{M}$ is a matroid.

We now derive the rank formula for $\mathcal{M}$. Although one can derive it from elementary methods, it is easy to obtain it from the matroid intersection theorem. Recall that if $\mathcal{M}_1 = (N, \mathcal{I}_1)$ and $\mathcal{M}_2 = (N, \mathcal{I}_2)$ are two matroids on N, then the max cardinality of a common independent set in $\mathcal{I}_1 \wedge \mathcal{I}_2$ is given by

$$\min_{X \subseteq N} r_1(X) + r_2(N \backslash X).$$

Now consider $U \subseteq S$. Let $U' = f^{-1}(U)$. We observe that $A \subseteq U$ is independent in $\mathcal{I}$ iff there is an $A' \subseteq f^{-1}(U)$ such that $|A'| = |A|$, $f(A') = A$ and A' is independent in $\mathcal{I}'$.

Define a matroid $\mathcal{M}'' = (S', \mathcal{I}'')$, where

$$\mathcal{I}'' = \{I \subseteq f^{-1}(U) \mid |I \cap f^{-1}(U)| \leq 1, u \in U\}.$$

Note that $\mathcal{M}''$ is a partition matroid. Let r'' be the rank of $\mathcal{M}''$. We leave the following claim as an exercise.

Claim 5 *$r(U)$ is the size of a maximum cardinality independent set in $\mathcal{M}' \wedge \mathcal{M}''$.*

Therefore, by the matroid intersection theorem we have that

$$r(U) = \min_{T \subseteq U'} (r'(T) + r''(U' \setminus T)) = \min_{T \subseteq U} (r'(f^{-1}(T)) + |U \setminus T|),$$

using the fact that $\mathcal{M}''$ is a partition matroid. We leave it to the reader to verify the second equality in the above. $\square$

From the above we obtain the matroid union theorem before that was formulated by Edmonds. Let $\mathcal{M}_1 = (S_1, \mathcal{I}_1), \ldots, \mathcal{M}_k = (S_k, \mathcal{I}_k)$ be matroids. Define

$$\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2 \vee \cdots \vee \mathcal{M}_k = (S_1 \cup S_2 \cup \cdots \cup S_k, \mathcal{I}),$$

where

$$\mathcal{I} = \mathcal{I}_1 \vee \mathcal{I}_2 \vee \cdots \vee \mathcal{I}_k := \{I_1 \cup I_2 \cup \cdots \cup I_k | I_i \in \mathcal{I}_i, 1 \leq i \leq k\}.$$

Theorem 6 (Matroid Union) *Let $\mathcal{M}_1 = (S_1, \mathcal{I}_1), \ldots, \mathcal{M}_k = (S_k, \mathcal{I}_k)$ be matroids. Then*

$$\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2 \vee \cdots \vee \mathcal{M}_k \tag{3}$$

is a matroid. The rank function of $\mathcal{M}$ is given by r, where

$$r(U) = \min_{T \subseteq U} (|U \backslash T| + r_1(T \cap S_1) + \cdots + r_k(T \cap S_k)). \tag{4}$$

Proof: Let $S'_1, \ldots, S'_k$ be copies of $S_1, \ldots, S_k$, such that

$$S'_i \cap S'_j = \emptyset, i \neq j.$$

Let $\mathcal{M}'_i = (S'_i, \mathcal{I}'_i)$, where $\mathcal{I}'_i$ corresponds to $\mathcal{I}_i$. Let $S' = S'_1 \uplus S'_2 \uplus \cdots \uplus S'_k$ and define $\mathcal{M}' = (S', \mathcal{I}')$, where

$$\mathcal{I}' = \{I'_1 \cup I'_2 \cup \cdots \cup I'_k \mid I'_i \in \mathcal{I}_i\}.$$

Clearly $\mathcal{M}'$ is a matroid since it is disjoint union of matroids.

Now define $f : S' \to S$ where $S = S_1 \cup S_2 \cup \cdots \cup S_k$, and $f(s') = s$ if s' is the copy of s. Then $\mathcal{M}$ is obtained from $\mathcal{M}'$ by f and hence by Theorem 4, $\mathcal{M}$ is a matroid. The rank formula easily follows by applying the formula in Theorem 4 $\mathcal{M}'$ and $\mathcal{M}$. $\square$

The above theorem is also referred to as the matroid *partition* theorem for the following reason. A $U \in S$ is $\mathcal{M}$ independent iff U can be partitioned into $U_1, \ldots, U_k$, such that for $1 \leq i \leq k$, U_i is independent in $\mathcal{I}_i$; note that U_i are allowed to be $\emptyset$.

We state a useful corollary.

Corollary 7 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid and k be an integer. Then the maximum rank of the union of k independent sets of $\mathcal{M}$ is equal to*

$$\min_{U \subseteq S} (|S \setminus U| + k \cdot r(U)). \tag{5}$$

Proof: Take $\mathcal{M}'$ to be union of $\mathcal{M}_1 \vee \mathcal{M}_2 \vee \cdots \vee \mathcal{M}_k$, where $\mathcal{M}_i = \mathcal{M}$. Then the union of k independent sets in $\mathcal{M}$ is an independent set in $\mathcal{M}'$. Thus we are asking for the maximum possible rank in $\mathcal{M}'$. S achieves the maximum rank and by the previous theorem

$$\begin{aligned} r'(S) &= \min_{U \subseteq S}(|S \setminus U| + k \cdot r(S \cap U)) &(6)\\ &= \min_{U \subseteq S}(|S \setminus U| + k \cdot r(U)). &(7)\end{aligned}$$

□

We now easily derive two important theorems that were first stated by Edmonds.

Theorem 8 (Matroid base covering theorem) *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. Then S can be covered by k independent sets iff*

$$|U| \leq k \cdot r(U), \forall U \subseteq S. \tag{8}$$

Proof: S can be covered by k independent sets iff the rank of S in the union of $\mathcal{M}_1 \vee \mathcal{M}_2 \vee \cdots \vee \mathcal{M}_k$, where $\mathcal{M}_i = \mathcal{M}$, is equal to $|S|$. By Corollary 7, this is equivalent to

$$|S \setminus U| + k \cdot r(U) \geq |S|, \forall U \subseteq S$$

$$\Rightarrow k \cdot r(U) \geq |U|, \forall U \subseteq S.$$

□

Exercise 9 *Derive Nash-Williams forest-cover theorem (Theorem 3) as a corollary.*

Now we derive the matroid base packing theorem, also formulated by Edmonds.

Theorem 10 (Matroid Base Packing Theorem) *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. Then there are k disjoint bases in $\mathcal{M}$ iff*

$$k(r(S) - r(U)) \leq |S \setminus U|, \forall U \subseteq S. \tag{9}$$

Proof: To see necessity, consider any set $U \subseteq S$. Any base B has the property that $r(B) = r(S)$. And $r(B \cap U) \leq r(U)$. Thus

$$B \cap (S \setminus U) \geq r(S) - r(U).$$

Therefore if there are k disjoint bases then each of these bases requires $r(S) - r(U)$ distinct elements from $S \setminus U$, and hence

$$k(r(S) - r(U)) \leq |S \setminus U|.$$

For sufficiency, we take the k-fold union of $\mathcal{M}$ and there are k disjoint bases if $r'(S)$ in the union matroid $\mathcal{M}'$ satisfies the equation

$$r'(S) = k \cdot r(S)$$

in other words,

$$\min_{U \subseteq S} |S \setminus U| + k \cdot r(U) = k \cdot r(S)$$

$$\Rightarrow |S \setminus U| + k \cdot r(U) \geq k \cdot r(S)|$$

□

Exercise 11 *Derive Nash-Williams-Tutte theorem on packing spanning trees (Theorem 1) as a corollary.*

2 Algorithmic and Polyhedral Aspects

Let $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2 \vee \cdots \vee \mathcal{M}_k$. Algorithmic results for $\mathcal{M}$ follow from an independence oracle or rank oracle for $\mathcal{M}$. Recall that a set $I \in \mathcal{I}$ is independent in $\mathcal{M}$ iff I an be partitioned into $I_1, I_2, \ldots, I_k$ such that for $1 \leq i \leq k$, I_i is independent in $\mathcal{I}_i$. Note that this is non-trivial to solve.

Theorem 12 *Given rank functions $r_1, \ldots, r_k$ for $\mathcal{M}_1, \ldots, \mathcal{M}_k$, as polynomial time oracles, there is a polynomial time algorithm to implement the rank function oracle r for $\mathcal{M} = \mathcal{M}_1 \vee \mathcal{M}_2 \vee \cdots \vee \mathcal{M}_k$.*

We sketch the proof of the above theorem. Recall the construction in Theorem 6 that showed $\mathcal{M}$ is a matroid. We first constructed an intermediate matroid $\mathcal{M}'$ by taking copies of $\mathcal{M}_1, \ldots, \mathcal{M}_k$ and then applied Theorem 4 to map $\mathcal{M}'$ to $\mathcal{M}$.

For the matroid $\mathcal{M}'$, one easily obtains an algorithm to implement r' from $r_1, \ldots, r_k$, i.e.

$$r'(U) = \sum_{i=1}^{k} r_i(U \cap S_i').$$

Recall that we obtained the rank function r for $\mathcal{M}$ from r' for $\mathcal{M}'$ using matroid intersection (see proof of Theorem 4). Thus, one can verify that an algorithm for matroid intersection implies an algorithm for r using algorithms for $r_1, \ldots, r_k$. There is also a direct algorithm that avoids using the matroid intersection algorithm — see [1] for details.

Polyhedrally, the base covering and packing theorems imply and are implied by the following

Theorem 13 *Given a matroid $\mathcal{M} = (S, \mathcal{I})$, the independent set polytope and base polytope of $\mathcal{M}$ have the integer decomposition property.*

Exercise 14 *Prove the above theorem using Theorem 8 and 10.*

Capacitated case and algorithmic aspects of packing and covering: The matroid union algorithm allows us to obtain algorithmic versions of the matroid base covering and base packing theorems. As a consequence, for example, there is a polynomial time algorithm that given a multi-graph $G = (V, E)$, outputs the maximum number of edge-disjoint spanning trees in G. It is also possible to solve the capacitated version of the problems in polynomial time. More precisely, let $\mathcal{M} = (S, \mathcal{I})$ and let $c : S \rightarrow \mathcal{Z}_+$ be integer capacities on the elements of S. The capacitated version of the base packing theorem is to ask for the maximum number of bases such that no element $e \in S$ is in more than $c(e)$ bases. Similarly, for the base covering theorem, one seeks a minimum number of independent sets such that each element e is in at least $c(e)$ independent sets. The capacitated case be handled by making $c(e)$ copies of each element e, however, this would give only a pseudo-polynomial time algorithm.

Assuming we have a polynomial time rank oracle for $\mathcal{M}$, the following capaciatated problems can be solved in polynomial time. To solve the capacitated versions, one needs polyhedral methods; see [1] for more details.

1. fractional packing of bases, i.e., let $\mathcal{B}$ denote the set of bases of $\mathcal{M}$,

$$\max_{B \in \mathcal{B}} \lambda_B$$

$$\sum_{B \ni e} \lambda_B \leq c(e), \forall e \in S$$

$$\lambda_B \geq 0$$

2. integer packing of bases, same as above but λ_B are restricted to be integer.

3. fractional covering by independent sets, i.e.

$$\min_{I \in \mathcal{I}} \lambda_I$$

$$\sum_{I \ni e} \lambda_I \geq c(e), \forall e \in S$$

$$\lambda \geq 0$$

4. integer covering by independent sets, same as above but λ_I are constrained to be integer.

Matroid Intersection from Matroid Union: We have seen that the matroid union algorithm follows from an algorithm for matroid intersection. The converse can also be shown. To see this, let $\mathcal{M}_1$ and $\mathcal{M}_2$ be two matroids on the same ground set S. Then, one can find the maximum cardinality common independent set in $\mathcal{M}_1 \wedge \mathcal{M}_2$ be considering $\mathcal{M}_1 \vee \mathcal{M}_2^*$ where $\mathcal{M}_2^*$ is the dual of $\mathcal{M}_2$; See Problem 4 in Homework 3 for details on this.

References

[1] Alexander Schrijver, "Combinatorial Optimization: Polyhedra and Efficiency", Chapter 42, Vol B, Springer-Verlag 2003.