CS 598CSC: Combinatorial Optimization	Lecture date: March 4, 2010
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The presentation here is based on [1] and [2].

1 Introduction to Matroids

Matroids (formally introduced by Whitney in 1935) are combinatorial structures that capture the abstract properties of linear independence defined for vector spaces.

Definition 1 *A matroid $\mathcal{M}$ is a tuple $(S, \mathcal{I})$, where S is a finite ground set and $\mathcal{I} \subseteq 2^S$ (the power set of S) is a collection of* independent sets*, such that:*

1. *$\mathcal{I}$ is nonempty, in particular, $\emptyset \in \mathcal{I}$,*
2. *$\mathcal{I}$ is downward closed; i.e., if $X \in \mathcal{I}$ and $Y \subseteq X$, then $Y \in \mathcal{I}$,*
3. *If $X, Y \in \mathcal{I}$, and $|X| < |Y|$, then $\exists y \in Y \backslash X$ such that $X + y \in \mathcal{I}$.*

Exercise 2 *Show that the third property in Definition 1 can be replaced by the following: if $X, Y \in \mathcal{I}$, $|Y| = |X| + 1$, then $\exists y \in Y \backslash X$ such that $X + y \in \mathcal{I}$.*

Example 3 (Vector Matroid) *Let M be a $m \times n$ matrix with entries in some field $\mathbb{F}$ and v_i be the i^{th} column of M, viewed as a vector in the vector space $\mathbb{F}^m$. Let $S = \{1, 2, \ldots, n\}$ and $\mathcal{I} = \{I : I \subseteq S, \{v_i\}_{i \in I}$ are linearly independent$\}$ (under the usual definition of linear independence in linear algebra). Then $\mathcal{M} = (S, \mathcal{I})$ is a matroid. To see this, notice that properties 1 and 2 of Definition 1 are trivially satisfied. To show property 3, suppose $X, Y \in \mathcal{I}$ and $|X| < |Y|$. If there is no $y \in Y \backslash X$ such that $X + y \in \mathcal{I}$, then Y is in the span of $\{v_x\}_{x \in X}$. Hence, $|Y| \leq |X|$ which is a contradiction.*

Example 4 (Graphic Matroid) *Let $\mathcal{G} = (V, E)$ be an undirected multi-graph (loops allowed). Let $\mathcal{I} = \{I : I \subseteq E,\ I$ induces a forest in $\mathcal{G}\}$. Then $\mathcal{M} = (E, \mathcal{I})$ is a matroid. Again, the first two properties of Definition 1 are easy to verify. To show property 3, suppose $X, Y \in \mathcal{I}$ such that $|X| < |Y|$. Both X and Y induce forests in $\mathcal{G}$. Let $V_1, V_2, \ldots, V_{k(X)}$ be the vertex sets of the connected components in $G[X]$ ($\mathcal{G}$ restricted to the edge set X). Here, $k(X)$ denotes the number of connected components in $G[X]$. Each connected component is a tree. Hence, if there is an edge $y \in Y$ that connects two different components of $\mathcal{G}[X]$ then $\mathcal{G}[X + y]$ is again a forest and we are done. If not, then every edge $y \in Y$ have its both ends in the same component of $\mathcal{G}[X]$. Thus, the number of connected components in $\mathcal{G}[Y]$, denoted by $k(Y)$, is at least $k(X)$. Thus, $|X| = |V| - k(X) \geq |V| - k(Y) = |Y|$, which is a contradiction.*

Example 5 (Uniform Matroid) *Let $\mathcal{M} = (S, \mathcal{I})$, where S is any finite nonempty set, and $\mathcal{I} = \{I : I \subseteq S, |I| \leq k\}$ for some positive integer k. Then $\mathcal{M}$ is a matroid.*

Example 6 (Partition Matroid) *Let $S_1, S_2, \ldots, S_n$ be a partition of S and $k_1, k_2, \ldots, k_n$ be positive integers. Let $\mathcal{I} = \{I : I \subseteq S,\ |I \cap S_i| \leq k_i$ for all $1 \leq i \leq n\}$. Then $\mathcal{M} = (S, \mathcal{I})$ is a matroid.*

Example 7 (Laminar Matroid) *Let $\mathcal{F}$ be a laminar family on S (i.e., if $X, Y \in \mathcal{F}$ then $X, Y \subseteq S$; and either $X \cap Y = \emptyset$, or $X \subseteq Y$, or $Y \subseteq X$) such that each $x \in S$ is in some set $X \in \mathcal{F}$. For each $X \in \mathcal{F}$, let $k(X)$ be a positive integer associated with it. Let $\mathcal{I} = \{I : I \subseteq S,\ |I \cap X| \leq k(X)\ \forall X \in \mathcal{F}\}$. Then $\mathcal{M} = (S, \mathcal{I})$ is a matroid. Notice that laminar matroids generalize partition matroids, which in turn generalize uniform matroids.*

Exercise 8 *Verify Example 7.*

Example 9 (Transversal Matroid) *Let $\mathcal{G} = (V, E)$ be a bipartite graph with bipartition V_1 and V_2. let $\mathcal{I} = \{I : I \subseteq V_1, \exists$ a matching M in $\mathcal{G}$ that covers $I\}$. Then $\mathcal{M} = (V_1, \mathcal{I})$ is a matroid.*

Example 10 (Matching Matroid) *Let $\mathcal{G} = (V, E)$ be an undirected graph. Let $\mathcal{I} = \{I : I \subseteq V, \exists$ a matching M in $\mathcal{G}$ that covers $I\}$. Then $\mathcal{M} = (V, \mathcal{I})$ is a matroid.*

Exercise 11 *Verify Examples 9 and 10.*

1.1 Base, Circuit, Rank, Span and Flat

Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid.

Definition 12 *A set $X \subseteq S$ such that $X \notin \mathcal{I}$ is called a* dependent set *of $\mathcal{M}$.*

Definition 13 *A* loop *is an element $x \in S$ such that $\{x\}$ is dependent.*

Notice that a loop cannot appear in any sets in $\mathcal{I}$ and can be effectively removed from S.

Definition 14 *A* base *is an inclusion wise maximal set in $\mathcal{I}$.*

Proposition 15 *If B and $\widehat{B}$ are bases of $\mathcal{M}$ then $|B| = |\widehat{B}|$.*

Proof: If $|B| < |\widehat{B}|$ then from Definition 1, $\exists x \in \widehat{B} \backslash B$ such that $B + x \in \mathcal{I}$, contradicting the maximality of B. $\square$

Notice that the notion of base here is similar to that of a *basis* in linear algebra.

Lemma 16 *Let B and $\widehat{B}$ be bases of $\mathcal{M}$ and $x \in \widehat{B} \backslash B$. Then $\exists y \in B \backslash \widehat{B}$ such that $\widehat{B} - x + y$ is a base of $\mathcal{M}$.*

Proof: Since $\widehat{B} - x \in \mathcal{I}$ and $|\widehat{B} - x| < |B|$, $\exists y \in B \backslash \widehat{B}$ such that $\widehat{B} - x + y \in \mathcal{I}$. Then $|\widehat{B} - x + y| = |B|$, implying that $\widehat{B} - x + y$ is a base of $\mathcal{M}$. $\square$

Definition 17 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. Given $\widehat{S} \subseteq S$, let $\widehat{\mathcal{I}} = \{I : I \subseteq \widehat{S}, I \in \mathcal{I}\}$. Then $\widehat{\mathcal{M}} = (\widehat{S}, \widehat{\mathcal{I}})$ is also a matroid and is referred to as the* restriction of $\mathcal{M}$ to $\widehat{S}$.

Definition 18 *Given $\mathcal{M} = (S, \mathcal{I})$ and $\widehat{S} \subseteq S$, $\widehat{B}$ is a* base for $\widehat{S}$ *if $\widehat{B}$ is a base of $\widehat{\mathcal{M}}$, where $\widehat{\mathcal{M}}$ is a restriction of $\mathcal{M}$ to $\widehat{S}$.*

Proposition 19 *Given $\mathcal{M} = (S, \mathcal{I})$, let $B \subseteq X$ be a base for X. Then for any $Y \supseteq X$, there exist a base $\widehat{B}$ for Y that contains B.*

Proof: Notice that B is independent in the restriction of $\mathcal{M}$ to Y (henceforth *independent in* Y). Let $\widehat{B}$ be the maximal independent set in Y that contains B. Since all maximal independent sets have same size, $\widehat{B}$ is a base of Y. □

Definition 20 *Given $\mathcal{M} = (S, \mathcal{I})$, a circuit is a minimal dependent set (i.e., an inclusion wise minimal set in $2^S \backslash \mathcal{I}$). Thus, if C is a circuit then $\forall x \in C$, $C - x \in \mathcal{I}$.*

The definition of a circuit is related to graph theory in the following sense: if $\mathcal{M}$ is the graphic matroid of a graph $\mathcal{G}$, then the circuits of $\mathcal{M}$ are the cycles of $\mathcal{G}$. Single element circuits of a matroid are loops; if $\mathcal{M}$ is a graphic matroid of a graph $\mathcal{G}$, then the set of loops of $\mathcal{M}$ is precisely the set of loops of $\mathcal{G}$.

Lemma 21 *Let C_1 and C_2 be two circuits such that $C_1 \neq C_2$ and $x \in C_1 \cap C_2$. Then for every $x_1 \in C_1 \backslash C_2$ there is a circuit C such that $x_1 \in C$ and $C \subseteq C_1 \cup C_2 - x$. In particular, $C_1 \cup C_2 - x$ contains a circuit.*

Proof: Notice that $C_1 \backslash C_2$ is nonempty (and so is $C_2 \backslash C_1$), otherwise, $C_1 \subseteq C_2$. Since $C_1 \neq C_2$, C_1 is a strict subset of C_2, contradicting the minimality of C_2.

Let $C_1 \cup C_2 - x$ contain no circuits. Then $B = C_1 \cup C_2 - x$ is independent, and hence, a base for $C_1 \cup C_2$ (since it is maximal). Also, $|B| = |C_1 \cup C_2| - 1$. Since $C_1 \cap C_2$ is an independent set, we can find a base $\widehat{B}$ for $C_1 \cup C_2$ that contains $C_1 \cap C_2$. Then $|\widehat{B}| = |B| = |C_1 \cup C_2| - 1$. Since $C_1 \setminus C_2$ and $C_2 \setminus C_1$ are both non-empty, this is possible only if either $C_1 \subseteq \widehat{B}$ or $C_2 \subseteq \widehat{B}$, contradicting that $\widehat{B}$ is a base. Hence, $C_1 \cup C_2 - x$ must contain a circuit.

Now let $x_1 \in C_1 \backslash C_2$. Let B_1 be a base for $C_1 \cup C_2$ that contains $C_1 - x_1$, and B_2 be a base for $C_1 \cup C_2$ that contains $C_2 - x$. Clearly, $x_1 \notin B_1$ and $x \notin B_2$. If $x_1 \notin B_2$ then $B_2 + x_1$ must have a circuit and we are done. If $x_1 \in B_2$, then from Lemma 16, there exists $\widehat{x} \in B_1 \backslash B_2$ such that $\widehat{B} = B_2 - x_1 + \widehat{x}$ is a base for $C_1 \cup C_2$. Notice that $\widehat{x} \neq x$, otherwise $C_2 \subseteq \widehat{B}$. Thus, $x_1 \notin \widehat{B}$ and $\widehat{B} + x_1$ contains the circuit satisfying the condition of Lemma 21. □

Corollary 22 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. If $X \in \mathcal{I}$ and $y \notin X$ then either $X + y \in \mathcal{I}$ or there is a unique circuit C in $X + y$. Moreover, for each $\widehat{y} \in C$, $X + y - \widehat{y} \in \mathcal{I}$.*

Proof: If $X + y \notin \mathcal{I}$, then it must contain a circuit C_1. Assume there is another circuit $C_2 \subseteq X + y$, and $C_1 \neq C_2$. Since $X \in \mathcal{I}$, both C_1 and C_2 must contain y. From Lemma 21, $C_1 \cup C_2 - y$ contains a circuit. But this is a contradiction since $C_1 \cup C_2 - y \subseteq X$. Hence, $X + y$ contains a unique circuit, call it C. Now, if for some $\widehat{y} \in C$, $X + y - \widehat{y} \notin \mathcal{I}$, then $X + y - \widehat{y}$ is dependent and contains a circuit $\widehat{C}$. However, $\widehat{C} \neq C$ since $\widehat{y} \notin \widehat{C}$, contradicting that C is unique. □

Corollary 23 *If B and $\widehat{B}$ are bases. Let $\widehat{x} \in \widehat{B} \backslash B$, then $\exists x \in B \backslash \widehat{B}$ such that $B + \widehat{x} - x$ is a base.*

Proof Sketch. Follows from Corollary 22. □

Definition 24 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. The* rank function*, denoted by $r_{\mathcal{M}}$, of $\mathcal{M}$ is $r_{\mathcal{M}} : 2^S \mapsto \mathbb{Z}_+$, where for $X \subseteq S$, $r_{\mathcal{M}}(X)$ is the size of a maximum independent set contained in X.*

Note that the above definition assigns a unique number to each set X since all maximal independent sets contained in X have the same cardinality.

Proposition 25 *Given a matroid $\mathcal{M} = (S, \mathcal{I})$, the rank function $r_{\mathcal{M}}$ has the following properties:*

1. $0 \le r_{\mathcal{M}}(X) \le |X|$ *for all* $X \subseteq S$.
2. $r_{\mathcal{M}}$ *is submodular; i.e., for any* $X, Y \subseteq S$, $r_{\mathcal{M}}(X \cup Y) + r_{\mathcal{M}}(X \cap Y) \le r_{\mathcal{M}}(X) + r_{\mathcal{M}}(Y)$.

Proof: Property 1 is by the definition of $r_{\mathcal{M}}$. To show the second property, we use the equivalent definition of submodularity; i.e., we show that if $X \subseteq Y$ and $z \in S$, then $r_{\mathcal{M}}(X+z) - r_{\mathcal{M}}(X) \ge r_{\mathcal{M}}(Y+z) - r_{\mathcal{M}}(Y)$. First notice that for any $X \subseteq S$ and $z \in S$, $r_{\mathcal{M}}(X+z) \le r_{\mathcal{M}}(X) + 1$. Thus, we only need to show that if $r_{\mathcal{M}}(Y+z) - r_{\mathcal{M}}(Y) = 1$ then $r_{\mathcal{M}}(X+z) - r_{\mathcal{M}}(X) = 1$ for any $X \subseteq Y$.

If $r_{\mathcal{M}}(Y+z) - r_{\mathcal{M}}(Y) = 1$, then every base B of $Y+z$ contains z. Let $\widehat{B}$ be a base of X. Since $X \subseteq Y+z$, from Proposition 19, there exists a base $\bar{B}$ of $Y+z$ such that $\bar{B} \supseteq \widehat{B}$. Then $\widehat{B}+z$ is independent, implying $r_{\mathcal{M}}(X+z) - r_{\mathcal{M}}(X) = 1$ as $\widehat{B}+z$ is a base in $X+z$. □

Definition 26 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. For any $X \subseteq S$, the* span *of X, denoted by $span_M(X)$, is defined as $span_M(X) = \{y : y \in S, r_{\mathcal{M}}(X+y) = r_{\mathcal{M}}(X)\}$. A set $X \subseteq S$ is* spanning *if $span_M(X) = S$.*

Exercise 27 *Prove the following properties about the span function $span_M : 2^S \to 2^S$.*

- *If $T, U \subseteq S$ and $U \subseteq span_M(T)$ then $span_M(U) \subseteq span_M(T)$.*
- *If $T \subseteq S$, $t \in S \setminus T$ and $s \in span_M(T+t) \setminus span_M(T)$ then $t \in span_M(T+s)$.*

Definition 28 *Let $\mathcal{M} = (S, \mathcal{I})$ be a matroid. A subset $X \subseteq S$ is a* flat *of $\mathcal{M}$ iff $span_M(X) = X$.*

Exercise 29 *Prove the following properties about flats.*

- *If F_1 and F_2 are flats then $F_1 \cap F_2$ is a flat.*
- *If F is a flat and $t \in S \setminus F$ and F' is a smallest flat containing $F+t$ then there is no flat F'' with $F \subset F'' \subset F'$.*

Remark 30 *We showed basic properties of bases, circuits, rank, span and flats of a matroid. One can show that a matroid can alternatively be specified by defining its bases or circuits or rank or span or flats that satisfy these properties. We refer the reader to [2].*

References

[1] J. Lee. "Matroids and the greedy algorithm". In *A First Course in Combinatorial Optimization*, Ch. 1, 49–74, Cambridge University Press, 2004.

[2] A. Schrijver. *Combinatorial optimization: polyhedra and efficiency*, Chapter 39 (Vol B), Springer, 2003.