CS 598CSC: Combinatorial Optimization **Lecture date: February 25, 2010**
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1 Total Dual Integrality

Recall that if A is TUM and b, c are integral vectors, then $\max\{cx : Ax \le b\}$ and $\min\{yb : y \ge 0, yA = c\}$ are attained by integral vectors x and y whenever the optima exist and are finite. This gives rise to a variety of min-max results, for example we derived König's theorem on bipartite graphs. There are many examples where we have integral polyhedra defined by a system $Ax \le b$ but A is not TUM; the polyhedron is integral only for some specific b. We may still ask for the following. Given any c, consider the maximization problem $\max\{cx : Ax \le b\}$; is it the case that the dual minimization problem $\min\{yb : y \ge 0, yA = c\}$ has an integral optimal solution (whenever a finite optimum exists)?

This motivates the following definition:

Definition 1 *A rational system of inequalities $Ax \le b$ is* totally dual integral (TDI) *if, for all integral c, $\min\{yb : y \ge 0, yA = c\}$ is attained by an integral vector y^* whenever the optimum exists and is finite.*

Remark 2 *If A is TUM, $Ax \le b$ is TDI for all b.*

This definition was introduced by Edmonds and Giles[2] to set up the following theorem:

Theorem 3 *If $Ax \le b$ is TDI and b is integral, then $\{x : Ax \le b\}$ is an integral polyhedron.*

This is useful because $Ax \le b$ may be TDI even if A is not TUM; in other words, this is a weaker sufficient condition for integrality of $\{x : Ax \le b\}$ and moreover guarantees that the dual is integral whenever the primal objective vector is integral.

Proof Sketch. Let $P = \{x : Ax \le b\}$. Recall that we had previously shown that the following are equivalent:

(i) P is integral.

(ii) Every face of P contains an integer vector.

(iii) Every minimal face of P contains an integer vector.

(iv) $\max\{cx : x \in P\}$ is achieved by an integral vector whenever the optimum is finite.

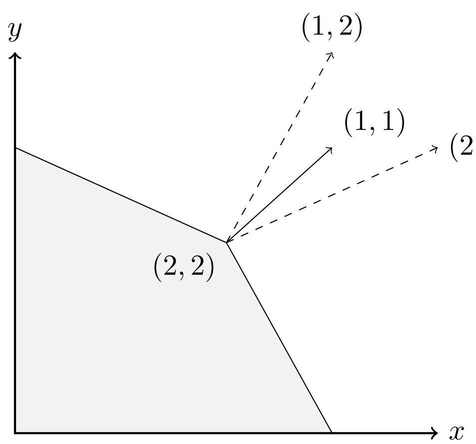
Edmonds and Giles proved two more equivalent conditions:

(v) Every rational supporting hyperplane of P contains an integer vector.

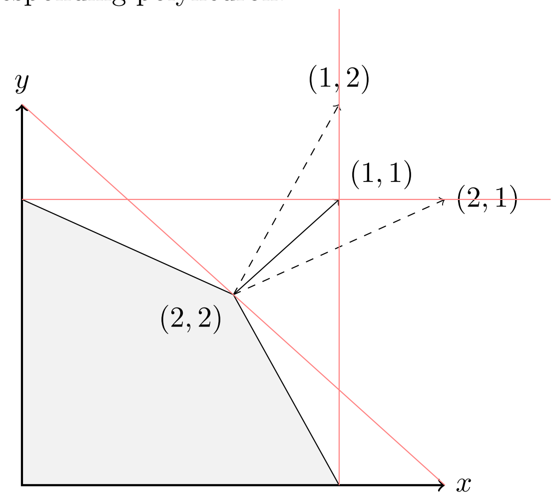
(vi) If c is integral, then $\max\{cx : x \in P\}$ is an integer whenever the optimum exists and is finite.

Condition (vi) implies the theorem as follows. If $Ax \leq b$ is TDI *and* b is integral, $\max\{cx : x \in P\}$ is an integer for all integral c whenever it is finite; this is because the dual optimum is achieved by an integral vector y^* (TDI property) and the objective function by^* is integral because b is integral. This implies that P is integral. □

There's an important subtlety to the definition of total dual integrality: being TDI is a property of a system of inequalities, *not* a property of the corresponding polyhedron.



$x \geq 0, y \geq 0$
$x + 2y \leq 6$
$2x + y \leq 6$



$x \geq 0, y \geq 0$
$x + 2y \leq 6$
$2x + y \leq 6$
$x + y \leq 4$
$x \leq 3, y \leq 3$

We will illustrate this with an example from [3]. Consider the system $Ax \leq b$ drawn above on the left. If we take the cost vector c to be $(1, 1)$, then the primal has an optimum at $(2, 2)$ with value 4. The tight constraints at this vertex have normal vectors $(2, 1)$ and $(1, 2)$ (these are rows of A). Therefore, in order for the dual $yA = c$ to have an integer solution, we must be able to express $(1, 1)$ as an *integer* combination of $(2, 1)$ and $(1, 2)$. Since this is impossible, $Ax \leq b$ is not TDI.

However, suppose we add more constraints to obtain the system $A'x \leq b'$ drawn above on the right. Note that this system corresponds to the same polyhedron as $Ax \leq b$. However, now we have an additional normal vector at $(2, 2)$ – namely, $(1, 1)$. Thus $(1, 1)$ is now an integer combination of the normal vectors at $(2, 2)$. The system $A'x \leq b'$ is in fact TDI, even though it corresponds to the same polytope as the (non-TDI) system $Ax \leq b$.

The example demonstrates a necessary for a system to be TDI. We explain this in the general context. Consider the problem $\max\{cx : Ax \leq b\}$ with c integral, and assume it has a finite optimum β. Then it is achieved by some vector x^* in the face F defined by the intersection of $\{x : Ax \leq b\}$ with the hyperplane $cx = \beta$. For simplicity assume that the face F is an extreme point/vertex of the polyhedron and let $A'x^* = b'$ be the set of all inequalities in $Ax \leq b$ that are tight at x^*. The dual is $\min\{yb : y \geq 0, yA = c\}$. By LP duality theory, any dual optimum solution y corresponds to c being expressed a non-negative combination of the row vectors of A', in other words c is in the cone of the row vectors of A'. If $Ax \leq b$ is TDI then we ask for an integral dual optimum solution; this requires that there is an integer solution to $yA' = c, y \geq 0$. This motivates the following definition.

Definition 4 *A set* $\{a_1, \ldots, a_k\}$ *of vectors in* R^n *is a* Hilbert basis *if every integral vector* $x \in Cone(\{a_1, \ldots, a_k\})$ *can be written as* $x = \sum_{i=1}^k \mu_i a_i$, $\mu_i \geq 0$, $\mu_i \in \mathbf{Z}$ *(that is, x is a non-negative integer combination of* $a_1, \ldots, a_k$*). If the* a_i *are themselves integral, we call* $\{a_1, \ldots, a_k\}$ *an* integral Hilbert basis*.*

The following theorem is not difficult to prove with the background that we have developed.

Theorem 5 *The rational system* $Ax \leq b$ *is TDI if and only if the following property is true for each face* F *of* P*; let* $A'x = b'$ *be the set of* all *inequalities in* $Ax \leq b$ *that are tight/active at* F*, then the rows vectors of* A' *form a Hilbert basis.*

Corollary 6 *If the system* $Ax \leq b, \alpha x \leq \beta$ *is TDI then* $Ax \leq b, \alpha x = \beta$ *is also TDI.*

The example above raises the question of whether one can take any rational system $Ax \leq b$ and make it TDI by adding sufficiently many redundant inequalities. Indeed that is possible, and is based on the following theorem.

Theorem 7 *Every rational polyhedral cone has a finite integral Hilbert basis.*

Theorem 8 (Giles-Pulleyblank) *Any rational polyhedron* P *has a representation* $Ax \leq b$ *such that*

(i) $P = \{x : Ax \leq b\}$,

(ii) A *is integral, and*

(iii) $Ax \leq b$ *is TDI.*

Moreover, b *is integral if and only if* P *is integral.*

2 The Cunningham-Marsh Theorem

Suppose we have a graph $G = (V, E)$. Let $P_{odd}(V)$ denote the family of all odd subsets of V with size at least 3. Recall that in our study of matchings, we have examined three different systems of inequalities.

$$P_1 : \quad \begin{aligned} x(\delta(v)) &= 1 && \forall v \in V \\ x(\delta(U)) &\geq 1 && U \in P_{odd}(V) \\ x &\geq 0 \end{aligned}$$

$$P_2 : \quad \begin{aligned} x(\delta(v)) &\leq 1 && \forall v \in V \\ x(E[U]) &\leq \left\lfloor \tfrac{1}{2} |U| \right\rfloor && U \in P_{odd}(V) \\ x &\geq 0 \end{aligned}$$

$$P_3 : \quad \begin{aligned} x(\delta(v)) &= 1 && \forall v \in V \\ x(E[U]) &\leq \left\lfloor \tfrac{1}{2} |U| \right\rfloor && U \in P_{odd}(V) \\ x &\geq 0 \end{aligned}$$

Here P_2 determines the matching polytope for G, while P_1 and P_3 determine the perfect matching polytope.

It is not hard to see that P_1 is not TDI. Consider K_4 with $w(e) = 1$ for every edge e. In this case, the unique optimal dual solution is $y_v = \frac{1}{2}$ for each vertex v.

On the other hand, P_2 and P_3 *are* TDI; this was proven by Cunningham and Marsh[1]. Consider the primal maximization and dual minimzation problems for P_2 below:

maximize wx subject to

$$\begin{aligned} x(\delta(v)) &\le 1 && \forall v \in V \\ x(E[U]) &\le \left\lfloor \tfrac{1}{2}|U| \right\rfloor && \forall U \in P_{odd}(V) \\ x &\ge 0 \end{aligned}$$

minimize $\sum_{v \in V} y_v + \sum_{U \in P_{odd}(V)} z_U \cdot \left\lfloor \frac{1}{2}|U| \right\rfloor$ subject to

$$y_a + y_b + \sum_{\substack{U \in P_{odd}(V) \\ a,b \in U}} z_U \ge w(ab) \ \forall ab \in E$$

$$y \ge 0, \ z \ge 0$$

By integrality of the matching polytope, the maximum value of the primal is the maximum weight of a matching under w; by duality, this equals the minimum value of the dual. The Cunningham-Marsh Theorem tells us that this minimum value is achieved by integral dual vectors y^*, z^* with the additional condition that the sets $\{U : z^*_U > 0\}$ form a laminar family.

Theorem 9 (Cunningham-Marsh) *The system P_2 is TDI (as is P_3). More precisely, for every integral w, there exist integral vectors y and z that are dual feasible such that $\{U : z_U > 0\}$ is laminar and*

$$\sum_{v \in V} y_v + \sum_{U \in P_{odd}(V)} z_U \cdot \left\lfloor \frac{1}{2}|U| \right\rfloor = \nu(w)$$

where $\nu(w)$ is the maximum weight of a matching under w.

Exercise 10 *Show that the Tutte-Berge Formula can be derived from the Cunningham-Marsh Theorem.*

Cunningham and Marsh originally proved this theorem algorithmically, but we present a different proof from [4]; the proof relies on the fact that P_2 is the matching polytope. A different proof is given in [4] that does not assume this and in fact derives that P_2 is the matching polytope as a consequence.

Proof: We will use induction on $|E| + w(E)$ (which is legal because w is integral). Note that if $w(e) \le 0$ for some edge e, we may discard it; hence we may assume that $w(e) \ge 1$ for all $e \in E$.

Case I: Some vertex v belongs to every maximum-weight matching under w. Define $w' : E \to \mathbf{Z}^+$ by

$$\begin{aligned} w'(e) &= w(e) - 1 && \text{if } e \in \delta(v) \\ w'(e) &= w(e) && \text{if } e \notin \delta(v) \end{aligned}$$

Now induct on w'. Let y', z' be an integral optimal dual solution with respect to w' such that $\{U : z'_U > 0\}$ is laminar; the value of this solution is $\nu(w')$. Because v appears in every maximum-weight matching under w, $\nu(w') \le \nu(w) - 1$; by definition of w', $\nu(w') \ge \nu(w) - 1$. Thus $\nu(w') = \nu(w) - 1$.

Let y^* agree with y' everywhere except v, and let $y_v^* = y_v' + 1$. Let $z^* = z'$. Now y^*, z^* is a dual feasible solution with respect to w, the solution is optimal since it has weight $\nu(w') + 1 = \nu(w)$, and $\{U : z_U^* > 0\}$ is laminar since $z^* = z'$.

Case II: No vertex belongs to every maximum-weight matching under w. Let y, z be a *fractional* optimal dual solution. Observe that $y = 0$, since $y_v > 0$ for some vertex v, together with complementary slackness, would imply that every optimal primal solution covers v, i.e. v belongs to every maximum-weight matching under w. Among all optimal dual solutions y, z (with $y = 0$) choose the one that maximizes $\sum_{U \in P_{odd}(V)} z_U \left\lfloor \frac{1}{2} |U| \right\rfloor^2$. To complete the proof, we just need to show that z is integral and $\{U : z_U > 0\}$ is laminar.

Suppose $\{U : z_U > 0\}$ is not laminar; choose $W, X \in P_{odd}(V)$ with $z_W > 0, z_X > 0$, and $W \cap X \neq \emptyset$. We claim that $|W \cap X|$ is odd. Choose $v \in W \cap X$, and let M be a maxmimum-weight matching under w that misses v. Since $z_W > 0$, by complementary slackness, M contains $\left\lfloor \frac{1}{2} |W| \right\rfloor$ edges inside W; thus v is the *only* vertex in W missed by M. Similarly, v is the only vertex in X missed by M. Thus M covers $W \cap X - \{v\}$ using only edges inside $W \cap X - \{v\}$, so $|W \cap X - \{v\}|$ is even, and so $|W \cap X|$ is odd. Let ϵ be the smaller of z_W and z_X; form a new dual solution by decreasing z_W and z_X by ϵ and increasing $z_{W \cap X}$ and $z_{W \cup X}$ by ϵ (this is an uncrossing step).

We claim that this change maintains dual feasibility and optimality. Clearly z_W and z_X are still nonnegative. If an edge e is contained in W and X, then the sum in e's dual constraint loses 2ϵ from z_W and z_X, but gains 2ϵ from $z_{W \cap X}$ and $z_{W \cup X}$, and hence still holds. Likewise, if e is contained in W but not X (or vice-versa), the sum loses ϵ from z_W but gains ϵ from $z_{W \cup X}$. Thus these changes maintained dual feasibility and did not change the value of the solution, so we still have an optimal solution. However, we have increased $\sum_{U \in P_{odd}(V)} z_U \left\lfloor \frac{1}{2} |U| \right\rfloor^2$ (the reader should verify this), which contradicts the choice of z. Thus $\{U : z_U > 0\}$ is laminar.

Suppose instead that z is not integral. Choose a maximal $U \in P_{odd}(V)$ such that z_U is not an integer. Let $U_1, \ldots, U_k$ be maximal odd sets contained in U such that each $z_{U_i} > 0$. (Note that we may have $k = 0$.) By laminarity, $U_1, \ldots, U_k$ are disjoint. Let $\alpha = z_U - \lfloor z_U \rfloor$. Form a new dual solution by decreasing z_U by α and increasing each z_{U_i} by α.

We claim that the resulting solution is dual feasible. Clearly we still have $z_U \geq 0$, and no other dual variable was decreased. Thus we need only consider the edge constraints; moreover, the only constraints affected are those corresponding to edges contained within U. Let e be an edge contained in U. If e is contained in some U_i, then the sum in e's constraint loses α from z_U but gains α from z_{U_i}, so the sum does not change. On the other hand, suppose e is not contained in any U_i. By maximality of U and the U_i, U is the only set in P_{odd} containing e. Thus before we changed z_U we had $z_U \geq w(e)$; because $w(e)$ is integral, we must still have $z_U \geq w(e)$. Hence our new solution is dual feasible.

Since the U_i are disjoint, contained in U, and odd sets, $\left\lfloor \frac{1}{2} |U| \right\rfloor > \sum_{i=1}^{k} \left\lfloor \frac{1}{2} |U_i| \right\rfloor$. Thus our new solution has a smaller dual value than the old solution, which contradicts the optimality of z. It follows that z was integral, which completes the proof.

To show that the system P_3 is TDI, we use Corollary 6 and the fact that system P_2 is TDI. □

References

[1] W. H. Cunningham and A. B. Marsh, III. *A primal algorithm for optimal matching.* Mathematical Programming Study 8 (1978), 50–72.

[2] J. Edmonds and R. Giles. *Total dual integrality of linear inequality systems.* Progress in combinatorial optimization (Waterloo, Ont., 1982), 117–129, Academic Press, Toronto, ON, 1984.

[3] Lecture notes from Michael Goemans's class on Combinatorial Optimization. http://math.mit.edu/ goemans/18438/lec6.pdf. 2009.

[4] A. Schrijver. *Theory of Linear and Integer Programming.* Wiley, 1998.