CS 598CSC: Topics in Graph Algorithms	Lecture date: 08/29/2024
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Tree Packing

Given an undirected graph $G = (V, E)$, we are interested in finding edge-disjoint spanning trees. We let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees.

There is a beautiful theorem that provides a min-max formula for this. We introduce some notation: Let P be a partition of the vertex set V. We use $|P|$ to denote the number of parts in P. Let E_P denote the set of edges crossing the partition, i.e., $e \in E_P$ if and only if its endpoints are in different parts of P.

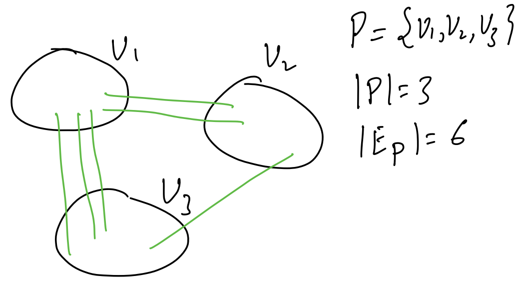

Figure 1: Illustration of P, E_P.

It is easy to see that any spanning tree must contain at least $|P| - 1$ edges from E_P. Thus, if G has k edge-disjoint spanning trees, then:

$$k \leq \frac{|E_P|}{|P| - 1}$$

A classical theorem in graph theory is the following.

Theorem 1 (Tutte-Nash-Williams)). *The maximum number of edge-disjoint spanning trees in a graph G is given by:*

$$\tau(G) = \lfloor \min_P \frac{|E_P|}{|P| - 1} \rfloor$$

The preceding theorem is a special case of a theorem on matroid base packing where it is perhaps more natural to see (see [Sch03]). A weaker version of the theorem is regarding fractional packing. In fractional packing, we allow one to use a fractional amount of a tree. The total amount to which an edge can be used is at most 1 (or $c(e)$ in the capacitated case). Clearly, an integer packing is also a fractional packing. The advantage of fractional packings is that one can write a linear program (LP) for it, and they often have some nice properties. Let $\tau_{\text{frac}}(G)$ be the fractional tree packing number. Clear $\tau_{\text{frac}}(G) \geq \tau(G)$.

One corollary of the Tutte-Nash-Williams theorem is the following:

Corollary 2.

$$\tau_{frac}(G) = \min_P \frac{|E_P|}{|P|-1}.$$

Do you see why the integer packing version implies the preceding corollary?

It is illustrative to see a tight example. Consider the cycle on n vertices. It is easy to see that $\tau(G) = 1$. We can see that $\tau_{\text{frac}}(G) \leq n/(n-1)$ since each tree has $n-1$ edges and there are n edges in the graph. This bound is tight by considering the n trees in the graph (corresponding to deleting each of the n edge) and assigning a fractional value of $1/(n-1)$ for each of them. Thus $\tau_{\text{frac}}(G) = n/(n-1)$ and the corresponding tight partition consists of the n singleton vertices.

A second important corollary that is frequently used is the following.

Corollary 3. *Let G be a capacitated graph and let $\lambda(G)$ be the global minimum cut, then:*

$$\tau_{frac}(G) \geq \frac{\lambda(G)}{2} \frac{|V|}{|V|-1}.$$

Proof. Consider the partition P that defines $\tau_{\text{frac}}(G)$. Let the parts of P be $V_1, V_2, \ldots, V_h$. Since $\lambda(G)$ is the mincut, $|\delta(V_i)| \geq \lambda(G)$ for each i. We have $|E_P| = \sum_{i=1}^h |\delta(V_i)|/2$ since each edge of E_P connects two parts. Thus

$$\tau_{\text{frac}}(G) = \frac{|E_P|}{h-1} \geq \frac{h\lambda(G)}{2(h-1)} \geq \frac{\lambda(G)}{2} \frac{h}{h-1} \geq \frac{\lambda(G)}{2} \frac{|V|}{|V|-1}.$$

■

The theorem and corollaries naturally extend to the capacitated case. For integer packing, we can assume c_e is an integer for each edge e, and the formula is changed to

$$\tau(G) = \lfloor \min_P \frac{c(E_P)}{|P|-1} \rfloor.$$

Typically, one uses the connection between tree packing and mincut to argue about the existence of many disjoint trees, since the global minimum cut is easier to understand than $\tau(G)$. However, we will see that one can use tree packing to compute $\lambda(G)$ exactly which may seem surprising at first due to the approximate relationship.

First, we prove the fractional version of the Tutte-Nash-Williams theorem via LP duality. This proof is taken from [CQX20]. We can write $\tau_{\text{frac}}(G)$ as the following LP problem: Let $\mathcal{T} = \{T : T \text{ is a spanning tree of } G\}$ be the set of all spanning trees of G.

Primal LP:

$$\text{maximize} \sum_{T \in \mathcal{T}} y_T \quad \text{subject to} \quad \sum_{T \ni e} y_T \leq c_e \quad \forall e \in E, \quad y_T \geq 0 \quad \forall T \in \mathcal{T}$$

Dual LP:

$$\text{minimize} \sum_{e \in E} c_e x_e \quad \text{subject to} \quad \sum_{e \in T} x_e \geq 1 \quad \forall T \in \mathcal{T}, \quad x_e \geq 0 \quad \forall e \in E$$

The dual can be interpreted as a relaxation for the minimum cut problem. In fact, if x_e is required to be in $\{0, 1\}$, it is an exact relaxation. Let y^* and x^* be optimum fractional solutions to the primal and dual LPs. Both exist because the problems have finite optima. By strong duality, we have:

$$\sum_{T \in \mathcal{T}} y_T^* = \sum_{e \in E} c_e x_e^*$$

Claim 4. *Suppose $x_e^* = 0$ for some e. Let $G' = (V, E')$ be the graph obtained by contracting e. Then:*

$$\tau_{frac}(G') = \tau_{frac}(G)$$

Proof. Contracting an edge cannot decrease the fractional tree packing value. Thus, $\tau_{\text{frac}}(G') \geq \tau_{\text{frac}}(G)$. We can see that if x^* is restricted to E', it is feasible for G' (why?) . Thus, by weak duality,

$$\sum_{e \in E'} c_e x_e^* \geq \tau_{\text{frac}}(G').$$

Putting together, we have $\tau_{\text{frac}}(G') = \tau_{\text{frac}}(G)$. ■

Thus, we can essentially contract all edges e such that $x_e = 0$. This does not affect the value of the packing. An intuitive reason is the following. If $x_e^* = 0$ we can effectively increase c_e to ∞ without affecting the value of the dual solution which means that e is not a bottleneck in the primal tree packing and hence safe to contract.

Claim 5. *Suppose $x_e^* > 0$ for all $e \in E$. Then consider the partition P consisting of the singletons. Then:*

$$\tau_{frac}(G) = \frac{\sum_{e \in E} c_e}{|V| - 1}.$$

Proof. By complementary slackness, since $x_e^* > 0$, we have that the corresponding primal constraint is tight. That is,

$$\sum_{T \in \mathcal{T} : e \in T} y_T^* = c_e.$$

Let $n = |V|$.

$$\begin{aligned} (n-1) \sum_{T \in \mathcal{T}} y_T^* &= \sum_{T} \sum_{e \in T} y_T^* \\ &= \sum_{e \in E} \sum_{T \ni e} y_T^* \\ &= \sum_{e \in E} c_e. \end{aligned}$$

Thus,

$$\sum_{T \in \mathcal{T}} y_T^* = \frac{1}{n-1} \sum_{e \in E} c_e = \frac{c(E_P)}{|P| - 1}$$

where P is the partition consisting of each vertex in a separate part. ■

We can use the preceding claims give us the desired conclusion $\tau_{\text{frac}}(G) = \min_P \frac{c(E_P)}{|P|-1}$ via induction. By the preceding claim it is true if $x_e^* > 0$ for all e. Otherwise, we contract edges with $x_e^* = 0$ and reduce to this case.

Finding an optimum tree packing: Can we solve the LP efficiently to obtain $\tau_{\text{frac}}(G)$? Note that this also yields an algorithm for the value of the integer packing $\tau(G)$ since it is the floor of $\tau_{\text{frac}}(G)$. We may also be interested in finding a (compact) representation of the packing itself. First, we note that the Ellipsoid method applied to the dual of the tree packing LP. The dual has variables x_e for each $e \in E$. We need a separation oracle. Given $x \in \mathbb{R}^E$, is it the case that $\sum_{e \in T} x_e \geq 1$ for all $T \in \mathcal{T}$? If not, find a tree T such that $\sum_{e \in T} x_e < 1$. This corresponds to solving Minimum Spanning Tree (MST) problem. Hence, the dual admits an efficient solution via the Ellipsoid method. One can convert an exact algorithm for the dual to finding an exact algorithm for the primal. There are combinatorial algorithms for solving the tree packing problem (both integer version and fractional versions) in strongly polynomial-time. See [Sch03].

Approximate tree packing : Can we find a faster algorithm for tree packing if one allows approximation? [CQ17] showed that a $(1-\varepsilon)$-approximate fractional tree packing can be found in $O(\frac{1}{\epsilon^2} m \log^3 n)$ time using an adaptation of the Multiplicative Weights Update (MWU) method and data structures for MST maintenance. We may see it later in the course.

Exercise 1. *Suppose x is a $(1-\varepsilon)$-approximate solution to the dual. How do we use it to find a $(1+O(\epsilon))$-approximat approximation for the optimum Tutte-Nash-Williams partition?*

References

[CQ17] Chandra Chekuri and Kent Quanrud. Near-linear time approximation schemes for some implicit fractional packing problems. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 801–820. SIAM, 2017.

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[Sch03] Alexander Schrijver. *Combinatorial optimization: polyhedra and efficiency*, volume 24. Springer Science & Business Media, 2003.