Homework 3

Topics in Graph Algorithms
CS598CCI, Spring 2020
Due: 3/27/2020 10am

Instructions and Policy: Each person should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with. Solutions to most of these problems can be found from one source or the other. Try to solve on your own first, and cite your sources if you do use them.

Please write clearly and concisely. Refer to known facts. You should try to convince me that you know the solution, as quickly as possible.

Do as many problems as you can. I expect you to do at least 3 and you should do Problem 4.

Problem 1. Given a graph $G = (V, E)$ let $\mathcal{I} = \{S \subseteq V \mid \text{there is a matching } M \text{ in } G \text{ that covers } S\}$. Prove that $(V, \mathcal{I})$ is a matroid.

Extra credit: Is this matroid a linear matroid?

Problem 2. Let $G = (V, \mathcal{E})$ be a hypergraph, that is each $e \in \mathcal{E}$ is a hyperedge, in other words $e \subseteq V$. If all edges have the property that $|e| = 2$ we have graphs but in general hypergraphs can have edges of different cardinalities. With each hypergraph G one can associate a bipartite graph $H_G = (V \cup \mathcal{E}, F)$ where V is on one side of the bipartite graph and $\mathcal{E}$ is on the other side and we connect v and e by an edge f if $v \in e$.

- There is an obvious attempt at generalizing the graphic matroid to hypergraphs with ground set E via the notion of acyclicity. To define this formally consider $X \subseteq \mathcal{E}$ and let $V(X) = \cup_{e \in X} e$ be the set of vertices incident to hyperedges in X. Consider the natural bipartite graph associated with the hypergraph $(V(X), X)$ and define X to be *acyclic* if this bipartite graph is acyclic. Now consider $\mathcal{I} = \{X \subseteq \mathcal{E} \mid X \text{ is acyclic}\}$. Prove via a counter example that $(\mathcal{E}, \mathcal{I})$ in general need not be a matroid.
- We say that $X \subseteq \mathcal{E}$ a *forest-representable* if one can choose for each $e \in X$ two nodes in e such that the chosen pairs when viewed as edges form a forest on V. Prove that $(\mathcal{E}, \mathcal{I})$ is a matroid where $\mathcal{I} = \{X \subseteq \mathcal{E} \mid X \text{ is forest-representable}\}$. This is called the hypergraphic matroid.
- **Extra credit:** Is the hypergraphc matroid a linear matroid?

Problem 3. Let $f : E \to \mathbb{Z}_+$ be an integer-valued monotone submodular function with $f(\emptyset) = 0$. Such a function is called a polymatroid. Recall that the rank function of a matroid $\mathcal{M} = (E, \mathcal{I})$ is a polymatroid with the additional property that $f(e) \leq 1$ for each $e \in E$. Can every polymatroid be understood via matroids? This problem shows that this is indeed the case. If f is a polymatroid then $f(e)$ can be an integer larger than 1. Given f over E construct a new set X where $X = \uplus_e X_e$ where X_e is a set of $f(e)$ elements (X_e and $X_{e'}$ for $e \neq e'$ are disjoint sets). We define a set function r over ground set X as follows. For $U \subseteq X$ let

$$r(U) = \min_{T \subseteq E}(f(T) + |U \setminus \bigcup_{e \in T} X_e|).$$

- Prove that r is the rank function of a matroid over X.
- Prove that for any $T \subseteq E$, $f(T) = r(\cup_{e \in T} X_e)$.

Problem 4. Letm $\mathcal{M} = (E, \mathcal{I})$ be a matroid on $|E| = n$ elements and let $c : E \to \mathbb{Z}_+$ be costs on the elements. Let $r_{\mathcal{M}}$ denote the rank function of $\mathcal{M}$ which we abbreviate by r. $A \in \{0,1\}^{m \times n}$ and $b \in \mathbb{Z}^m$ be a matrix and vector. We wish to solve the problem of finding a minimum cost basis in $\mathcal{M}$ that respects the packing constraints specified in the form $Ax \leq b$. More formally consider the following integer program.

$$\begin{aligned}
\min \sum_{e \in E} c_e x_e & & & \\
Ax & \leq & b & \\
\sum_{e \in S} x_e & \leq & r(S) & \quad \forall S \subseteq E \\
\sum_{e \in E} x_e & = & r(E) & \\
x_e & \in & \{0,1\} & \quad \forall e \in E
\end{aligned}$$

Let Δ denote an upper bound on number of nonzeroes in any column of A.

- Why is the bounded degree minimum spanning tree problem a special case of the preceding problem? What is Δ in this case?
- Suppose x^* is an extreme point solution of the LP relaxation and x^* is fully fractional. Show that x^* is the unique solution of n linearly independent tight constraints from the LP some of which come from the packing constraints $Ax \leq b$ and the rest from the tight constraints of the matroid polytope and these latter constraints can be chosen to form a chain.
- Use the characterization in the preceding part and a token counting argument to show that if x^* is fully fractional then there must be some row A_i in $Ax \leq b$ such that A_i has at most $b_i + \Delta - 1$ non-zero entries.
- Use the preceding part to develop an iterated rounding algorithm that outputs a base B such that $c(B) \leq \textsc{opt}$ and $A1_B \leq b + \Delta - 1$. In other words each packing is violated by at most an additive $\Delta - 1$ amount. If $\Delta - 1$ is hard to achieve prove a weaker bound of the form $\Delta + c$ for some fixed integer c. Since you are working with fully fractional solutions you may need to use matroid contraction in addition to deletion to handle the case when $x^*_e = 1$ for some e.