Polyhedral Aspects

Given SM instance. (Complete graph with equal sizes on both sides). Want to solve max weight (or min cost) stable matching.

Other measures.

Rothblum, Vande Vate gave simple descriptions of the SM polytope = convexhull$(\{\chi_M \mid M$ is a stable matching$\})$.

Simpler proof of SM polytope due to Sethuraman & Teo 1998.

i man m_i

j woman w_j

Variables: $x_{i,j} \in \{0,1\}$ if $(i,j) \in M$ or not.

~~no~~ perfect matching constraints

$$\sum_j x_{i,j} = 1 \quad \forall i$$

$$\sum_i x_{i,j} = 1 \quad \forall j$$

$$x_{i,j} \geq 0$$

SM:

$$x_{i,j} + \sum_{k: w_k <_{m_i} w_j} x_{i,k} + \sum_{k: m_k <_{w_j} m_i} x_{k,j} \leq 1 \quad \forall i,j$$

Why is SM inequality valid in $\{0,1\}$ setting? Either $X_{i,j}=1$ in which case the other sums are 0.

or if $X_{i,j}=0$ and if ineq is violated both $\sum_{k:} X_{i,k}=1$ & $\sum_{k:} X_{k,j}=1$

$\Rightarrow (m_i, w_j)$ will be a blocking pair.

Theorem: Polytope is integral.

$\Rightarrow$ Polytope is the SM polytope.

Lemma: Let x be a feasible solution.

Then $x_{i,j}>0 \Rightarrow x_{i,j}+\sum_{k} + \sum_{k} = 1$

Consider primal with objective

$$\max \left\{ \sum x_{i,j} : x \in Q \right\}$$

and its dual.

$$\min \quad \sum_i \alpha_i + \sum_j \beta_j + \sum_{i,j} \gamma_{i,j}$$

s.t.

$$\alpha_i + \beta_j + \gamma_{i,j} + \sum_{k : w_k >_{m_i} w_j} \gamma_{i,k} + \sum_{k : m_k >_{w_j} m_i} \gamma_{k,j} \geq 1 \quad \forall i,j$$

$$\gamma_{i,j} \geq 0 \qquad \alpha_i, \beta_j \text{ unconstrained}$$

Given feasible primal solution x consider dual solution α, β, γ

where $\gamma_{i,j} = x_{i,j}$

$\alpha_i = 0 \quad \beta_j = 0 \quad \forall i, \forall j.$

Can check that this is a feasible solution with objective $\sum x_{i,j}$

x is optimal for primal and $\alpha, \beta, \gamma = x$ is optimal for dual!

$\Rightarrow \gamma_{i,j} > 0 \Rightarrow$ primal constraint is tight

$\Rightarrow x_{i,j} > 0 \Rightarrow x_{i,j} + \sum_k + \sum_k = 1.$

Checking that $\alpha_i = 0$ $\beta_j = 0$ $\gamma_{i,j} = x_{i,j}$ is feasible for dual.

Need to check that

$$0+0+x_{i,j}+\sum_{k:w_k >_{m_i} w_j} x_{i,k}+\sum_{k:m_k >_{w_j} m_i} x_{k,j} \geq 1 \quad \forall i,j$$

Since $\sum_b x_{a,b}=1$ we have

$$\sum_{k:w_k >_{m_i} w_j} x_{i,k} = 1 - x_{i,j} - \sum_{k:w_k <_{m_i} w_j} x_{i,k}.$$

Similarly

$$\sum_{k:m_k >_{w_j} m_i} x_{k,j} = 1 - x_{i,j} - \sum_{k:m_k <_{w_j} m_i} x_{k,j}$$

So we need to check that

$$x_{i,j}+1-x_{i,j}-\sum_{k:w_k <_{m_i} w_j} x_{i,k}+1-x_{i,j}-\sum_{k:m_k <_{w_j} m_i} x_{k,j} \geq 1$$

but this is true because of primal. $\square$

We will prove that x can be written as a convex combination of perfect matchings.

Recall $\sum_j X_{i,j} = 1 \quad \forall i$ and

$$\sum_i X_{i,j} = 1 \quad \forall j.$$

We can view $X_{i,j}$ values $j = 1$ to n as partitioning interval $[0,1]$.

Create $2n$ row table.

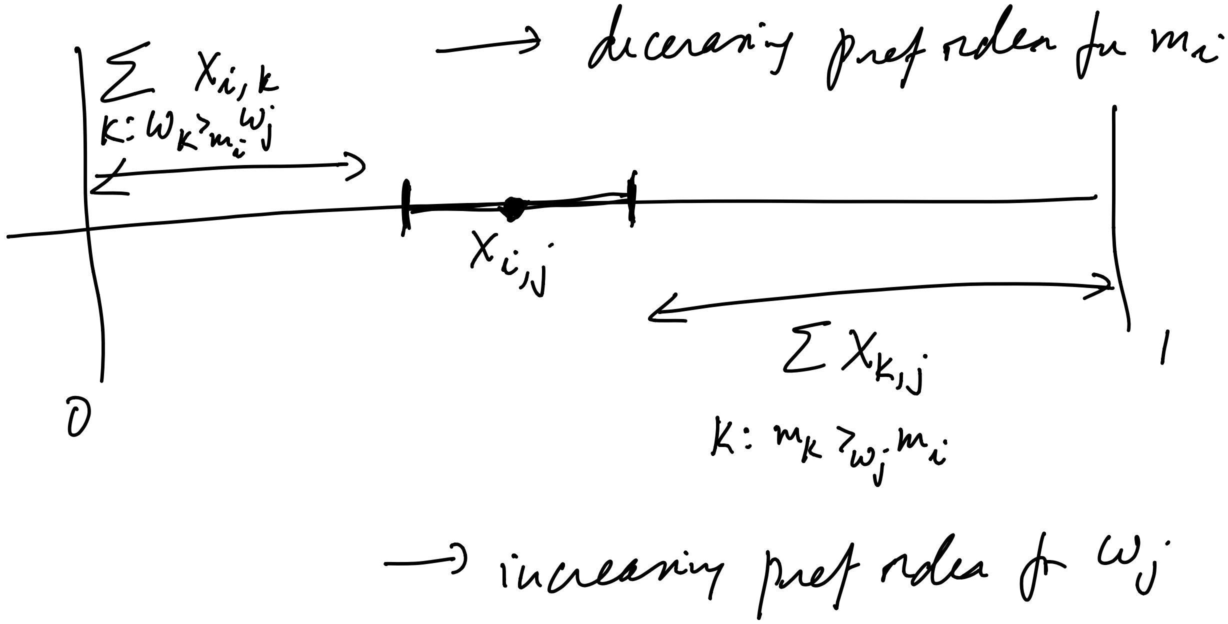
n rows for men and n for women.

In each man's row ~~as~~ split $[0,1]$ according to $X_{i,j}$ values $j = 1$ to n in <u>decreasing</u> preference order of women.

For each woman order in increasing order of preference order of men.

Note $X_{i,j} > 0$

$$\Rightarrow X_{i,j} + \sum_{k:\, w_k <_{m_i} w_j} X_{i,k} + \sum_{k:\, m_k <_{w_j} m_i} X_{k,j} = 1$$



$X_{i,j} > 0 \Rightarrow X_{i,j}$ is mapped in same location in interval for both i and j.

Pick $\theta \in (0,1)$ uniformly at random.

Match i to j if $x_{i,j} > 0$ and θ lands in interval corresponding to (i,j).

Exactly one for i. By above property exactly one for j as well.

So we get perfect matching.

Is it stable?

Suppose i prefers k over j

Inter $I^{(i)}_{i,k}$ to left $I^{(i)}_{i,j}$ in row i. By aligning property $I^{(k)}_{i,k}$ is in same position row k.

$\Rightarrow$ θ is to right of $I^{(i)}_{i,k}$ and preferences are increasing in row k

$\Rightarrow$ k matched to i' who she strictly prefers to i. So (i,k) cannot be a blocking pair.

$\Pr[(i,j) \in M] = x_{i,j}$

$\Rightarrow$ $\bar{X}$ can be decomposed into ...

$\square$

An Application of Stable Matchings to Disjoint Paths

Let $G = (V, E)$ be an undirected graph. Let s, x, t be 3 distinct vertices in G.

Suppose we are given k edge-disjoint paths from s to x say $\mathcal{P}$

and another set of k-edge disjoint paths from x to t say $\mathcal{Q}$

$\Rightarrow \lambda_G(s, x) \geq k$ and $\lambda_G(x, t) \geq k$

where $\lambda_G(a, b)$ is the edge-connectivity between a and b in G.

Claim: $\forall\, a, b, c \in V$,

$$\lambda(a,b) \geq \min\big(\lambda(a,c), \lambda(c,b)\big).$$

Proof: Exercise.

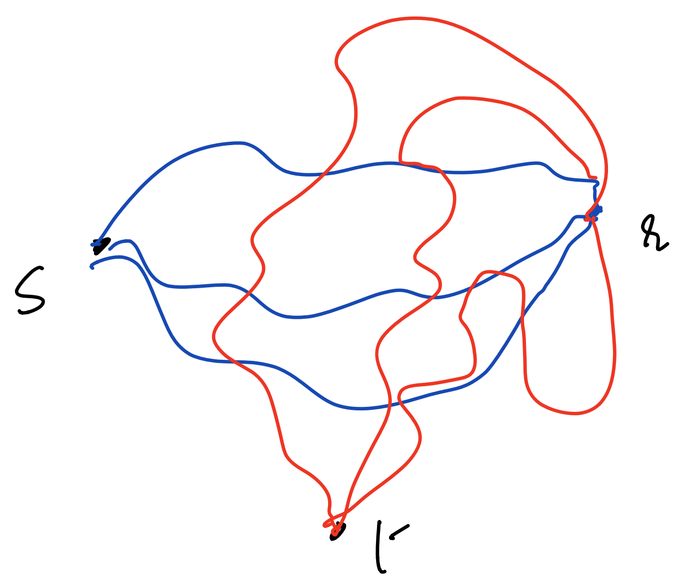
Therefore if $\exists$ k edge-disjoint paths from s to r and k-edge disjoint paths from r to t then $\exists$ k edge disjoint paths from s to t.

Q: How to find them?

We can take the union of the edges in P, Q and

Compute max-flow in the resulting graph between s and t and find paths from the flow.

Can we avoid above and use the fact that we have P and Q?



$P = \{p_1, \dots, p_k\}$ $\quad Q = \{q_1, \dots, q_k\}$

Construct bipartite multigraph $H = (A \cup B, E)$ as follows.

$|A| = |B| = k$ $A = \{a_1, \dots, a_k\}$

$B = \{b_1, \dots, b_k\}$

If edge g is common to paths p_i and q_j add edge between a_i and b_j (note multiple edges can be added between p_i and q_j).

For each a_i the edges incident to it in H correspond to edges in p_i. Create ~~prio~~ priority on them with priority decreasing along

p_i from p to q.

For each b_j the edges incident to it in H correspond to edges in q_j. Give priority in decreasing order from q to r. Equivalently increasing order along r to q.

H is bipartite multigraph. Make it complete by adding dummy edges) which receive lowest priority.

(p_i, q_j are adjacent $\forall\, i,j$).

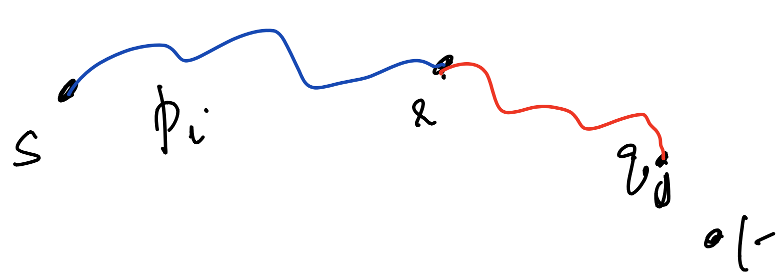
GS can be extended to show that even in this multigraph setting there is a stable matching.

Let M be any such stable matching. We will use M to create K edge disjoint paths from p to q.

One path for each edge $(p_i, q_j) \in M$.

<u>Case 1:</u> (p_i, q_j) is a <u>dummy</u> edge

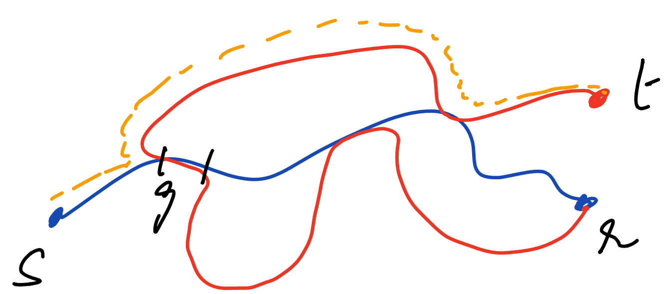
Concatenate p_i and q_j



Note that in this case p_i and q_j are edge disjoint - otherwise we will not have a dummy edge (p_i, q_j).

Case 2: (p_i, q_j) is a non-dummy edge.

$\Rightarrow$ p_i and q_j share an edge g and g is the first common edge between p_i and q_i along p_i



Create path s to g along p_i and then along q_j to t.

We get k paths from s to t?

Are they edge disjoint?

Yes. From stability of M.

Suppose not.

Say the two paths that share an edge were produced from two edges in M (p_{i_1}, q_{j_1}) and (p_{i_2}, q_{j_2}).

Call this edge g. Since p_{i_1}, p_{i_2} are disjoint g can belong only to one of them. Similarly for q_{j_1}, q_{j_2}.

So wlog g belongs to p_{i_1} and q_{j_2}

(the other case of belonging to p_{i_2} and q_{j_1} is similar).

Since (p_{i_1}, q_{j_1}) and (p_{i_2}, q_{j_2}) are from M $\Rightarrow$ p_{i_1} and q_{j_1} have common edge g_1 and p_{i_2} and q_{j_2} have common edge g_2.

g must occur before g_1 on p_{i_1} and g must occur after g_2 on q_{j_2}

but then g would be higher priority edge for a_{i_1} over (p_{i_1}, q_{j_1}) and also for b_{j_2} over (p_{i_2}, q_{j_2})

$\Rightarrow$ M is not stable.

contradiction.

Hence constructed paths are disjoint.

<u>Running time:</u> # of edges in H is $k^2 +$ # of edges in $P \cup Q$

GS algorithm runs in time linear in $E(H)$ so at most $k^2 + |P \cup Q|$. Can avoid k^2 time by not creating dummy edges explicitly and modifying GS.

$\Rightarrow$ linear in size $|P \cup Q|$.

17.