The Gale-Shapley algorithm for the Stable marriage problem has been highly influential and even lead to a Nobel Prize in Economics. It addresses an important problem that initially arose in matching residents to hospitals.

We will discuss, first, the most basic setting of the Stable Marriage (SM) problem that the Gale-Shapley algorithm (1962).

We are given n men and n women. Each man m_i has a total preference order over all the women and similarly each woman has a total preference order over the men.

We will write $p <_a q$ if according to person a's preferences p is less desirable than q.

We will assume for now that there are no ties, i.e. preference orders are strict.

In the SM setting we think of a complete bipartite graph between the men and women.

Given a matching M a pair (m, w) is a <u>blocking pair</u> for M if $(m, w) \notin M$ and m prefers w to its current partner in M (or m is unmatched in M) <u>and</u> w prefers m over its current partner in M (or is unmatched in M).

A matching M is <u>stable</u> if there is no blocking pair for M.

It is not clear that even in the SM setting where the graph is bipartite and complete and preference lists are strict and complete that there always exists a stable matching.

It is easy to see that given a matching M one can efficiently check whether M is stable or not.

Gale-Shapley showed via an efficient algorithm that every SM instance has a stable matching.

Ex: $n=4$

1	2	4	1	3
2	3	1	4	2
3	2	3	1	4
4	4	1	3	2

Men's preferences

1	2	1	4	3
2	4	3	1	2
3	1	4	3	2
4	2	1	4	3

Women pref.

$\{(1,4), (2,3), (3,2), (4,1)\}$ is stable

$\{(1,4), (2,1), (3,2), (4,4)\}$ is stable

$\{(1,1), (2,3), (3,2), (4,4)\}$ is not

Why? $(1,4)$ is a blocking pair.

GS algorithm

Men propose, women dispose.

Alg:

- all people are free.
- While $\exists$ man free do

    w = first woman on m's list that he has not proposed to yet.

    if w is free pair up (m, w).

    else if w is engaged to m'

        if w prefers m to m'

            pair (m, w), free m'

        else

            w rejects m. and m is free.

end;

Output matching of engaged pairs.

Note that the algorithm as stated is non-deterministic in that one can choose free men in any order. In fact they can all act in parallel in each round.

It is not obvious that the algorithm will terminate and whether it will output a stable matching.

However it is clear that we can make it terminate because a man does not propose to a woman more than once (he moves down his list).

Exercise: Rewrite algorithm such that it terminates and argue that it can be implemented in $O(n^2)$ time.

The main claims are the following.

1. At end of algorithm all people are paired up.
2. Matching is stable.

Observation: Once a woman gets matched she will remain matched and her partner's ranking in her order can only monotonically improve.

Observation: Every man is paired.

Suppose all women on m's list reject him. If a woman rejects m $\Rightarrow$ w paired with some one else and by previous observation they will remain paired. But n men and n women $\Rightarrow$ not all women can reject m.

$\Rightarrow$ all people paired.

Lemma: Matching M output by algorithm is stable.

Suppose (m, w) is a blocking pair.

m proposed to w before he proposed to $p_M(m)$. But w rejected m $\Rightarrow$ w has a better partner than m at time of rejection $\Rightarrow$ w has a better partner than m at end of alg. $\therefore$ (m, w) cannot be a blocking pair.

$\square$.

<u>Man-optimal</u> <u>and</u> <u>woman-optimality</u>

Even though the GS alg seems to suggest that women dispose the matching process in fact prefers men!

Theorem: In the GS algorithm each man is matched to the best ranked woman in any stable matching and each woman is matched to the least ranked man in any stable matching.

Proof: For each m define $best(m)$ to be the highest ranked women that m can be matched to in some stable matching.

Claim is that in any run of GS $(m, best(m))$ will be m's partner. Unique in all runs.

Suppose not. Consider a run of GS and let $(m, w = \text{best}(m))$ be the first pair that is "rejected".

$\Rightarrow$ w rejects m for m' or w already matched to m'.

$\Rightarrow$ w prefers m' to m.

Let S be stable matching with $(m, w) \in S$. Who is m''s partner in S? Say w'.

Clearly m' prefers w' to w otherwise (m', w) is blocking pair in S.

But when m was rejected by w m' had already considered w' before ~~m~~ w and must have been rejected

$\Rightarrow$ m' was rejected by w' before w rejected m $\Rightarrow$ (m, w) not first rejection in execution of G.S.

$\Rightarrow$ $(m, \text{best}(m))$ ~~$\in$~~ is output of any run of G.S.

Now for second part.

Suppose M_0 is man-optimal stable matching. Let w be a woman and suppose $(m, w) \in M_0$.

Suppose $\exists$ another stable matching M s.t. $(m', w) \in M$ and w prefers m over m'.

But then (m, w) is a blocking pair in M.

$\square$.

Simple Extensions of SM:

① Unequal Sizes.
Set $|X| \neq |Y|$ but preference lists are complete and strict.
(m, w) blocking for M if (m, w) prefer each other to current partners. assuming being married is always preferable to not being married.

Can reduce to $|X| = |Y|$ by adding dummy people to the smaller set.

<u>Theorem:</u> Stable matchings exist. All stable matchings match every one from smaller set and larger set partitioned into two sets, one of which is matched in all stable matchings and the other unmatched in all stable matchings.

(2) <u>Incomplete lists</u> <u>of</u> <u>unacceptable partners</u>

Bipartite graph $(A \cup B, E)$

Need not be complete

$(u,v) \in E \Rightarrow$ u is ok with v and vice versa

$(u,v) \notin E \Rightarrow$ u is not ok with v or v is not ok with u.

Each node has a strict preference list over its neighbors.

Now some nodes need not be matched.

<u>Theorem:</u> Stable matching exists and nodes are partitioned into two sets, one set is matched

in all stable matchings and the other is unmatched in every stable matching.

(3) Ties and Indifference

What if preference lists have ties? Then picture gets complicated because different notions of stability.

Not going to cover now. See references

Hospital Residents problem

HR is the original application and can be considered as the capacitated generalization of SM.

R set of residents
H set of ~~hosp~~ hospitals.

each $h \in H$ has a capacity u_h

each resident r has a preference order over hospitals and each hospital has a preference order over residents.

Now we are interested in assignments of residents to hospitals that respect the

Capacity Constraints.
Easy to generalize notion of Stability

Can reduce to SM by duplicating
h into U_h nodes and ordering
them arbitrarily but consistently.

Rural Hospitals Theorem

In any stable assignment same
capacity used up in each
hospital.