Recap from last lecture

- Non-uniform sparsest cut

$G = (V, E)$ $c : E \to \mathbb{R}_+$ supply graph

Demand graph $H = (V, F)$

$\text{dem} : F \to \mathbb{R}_+$

$$\phi^* = \min_{S \subseteq V} \frac{c(\delta_G(S))}{\text{dem}(\delta_F(S))}$$

ρ^* = max concurrent flow for given demands in G.

$\rho^* \leq \phi^*$ flow $\leq$ cut

$$\rho^* = \min_{\substack{d : V \times V \to \mathbb{R}_+ \\ d \text{ metric}}} \frac{\sum_{uv \in E} c(uv)\, d(u,v)}{\sum_{st \in F} \text{dem}(st)\, d(s,t)}$$

above by LP duality.

We saw a rounding algorithm via ℓ_1-embeddings that

$$\phi^* \leq O(\log n)\, \rho^*$$

In fact we can prove

$$\phi^* \leq O(\log k)\, \rho^*$$

where $k = |F|$ is number of demand pairs.

Where did $\log n$ come from?

We used Bourgain's theorem to show that any n point metric can be embedded into l_1 with $O(\log n)$ <u>distortion</u>.

<u>Questions</u>:

1. Is Bourgain's theorem tight?
2. Is LP integrality gap, ie flow-cut gap $\Omega(\log n)$?
3. Are there special graph classes where one can obtain improved flow-cut gap or l_1-embeddings?
4. Uniform sparsest cut vs non-uniform sparsest cut?

Uniform Sparsest Cut

dem(s t) = 1 $\forall$ all s t

$\Rightarrow$ demand graph is a clique on V.

What is ϕ^* ?

$$\phi^* = \min_{S \subseteq V} \frac{c(\delta(S))}{|S|\,|V-S|}$$

Recall edge-expansion of G

$$\alpha(G) = \min_{\substack{S \subseteq V \\ |S| \leq \frac{|V|}{2}}} \frac{c(\delta(S))}{|S|}$$

$\Rightarrow$ $n\phi^* \approx \alpha(G)$ within factor of 2.

Can approx ϕ^* to $O(\log n)$

$\Rightarrow$ $\alpha(G)$ also to $O(\log n)$.

<u>Defn</u>: For a graph $G=(V,E)$ let $\alpha(G)$ be the worst ℓ_1-distortion of any metric induced on V by edge lengths on G.

We know $\alpha(G) = O(\log n)$ where $|V| = n$.

<u>Claim</u>: $\alpha(T) = 1$ for any tree T.

<u>Claim</u>: $\alpha(C) = 1$ for any cycle C.

<u>Theorem</u>. $\alpha(G) = 2$ for any series parallel graph.

<u>Theorem</u>: $\alpha(G) = O(\sqrt{\log n})$ for any

planar graph G. More generally for any graph G that comes from a proper minor closed family of graphs.

<u>Conjecture</u> $\alpha(G) = O(1)$ for planar graph G.

$\Rightarrow$ $O(\sqrt{\log n})$ flow-cut gap in planar graphs. But believed to be $O(1)$.

Major open problem

Bourgain's Theorem

Proof sketch following Vazirani's book.

(V, d) metric on $|V| = n$ vertices.

Want to embed into l_1. Or more generally l_p.

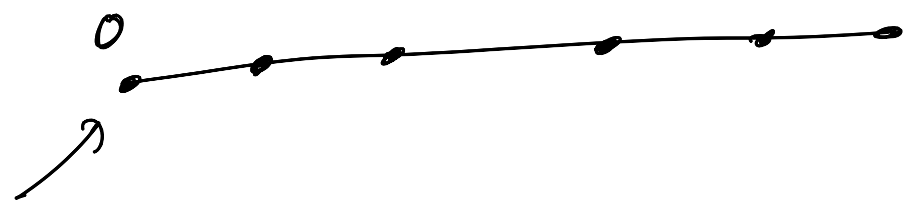
General technique: Use Frechet embeddings.

Fix a set $S \subseteq V$

Can define a 1-d embedding of V as follows.

For each $u \in V$ $f(u) = d(S, u)$

$$= \min_{s \in S} d(s, u)$$



$s \in S$ maps to 0

We find distance of each vertex u to S and put ~~u~~ on a line.

<u>Claim:</u> $|d(S,u) - d(S,v)| \leq d(u,v)$

Hence it is a contraction

Suppose we pick k sets $S_1, S_2, \ldots, S_k$ and define

$$f(u) = (d(S_1,u), d(S_2,u), \ldots, d(S_k,u))$$

Then $\|f(u)-f(v)\|_1$

$$= \sum_{i=1}^{h} |d(S_i,u)-d(S_i,v)|$$

$$\leq h\, d(u,v)$$

and hence if we set

$$f(u) = \left(\frac{1}{h} d(S_1,u), \frac{1}{h} d(S_2,u), \dots, \frac{1}{h} d(S_h,u)\right)$$

we have $\|f(u)-f(v)\|_1 \leq d(u,v)$

and hence we will have a

<u>contraction</u>.

<u>Idea</u>: want to choose $S_1, \dots, S_h$

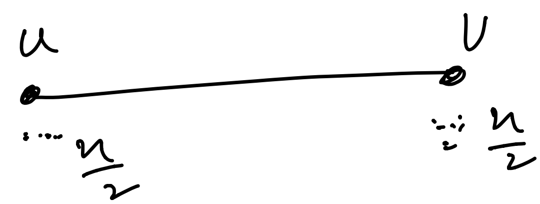
cleverly to ensure that

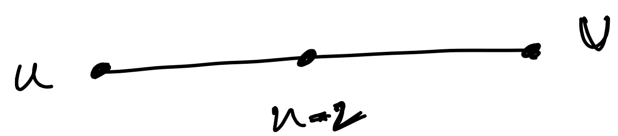
$$\|f(u)-f(v)\|_1 \geq c(n)\, d(u,v)$$

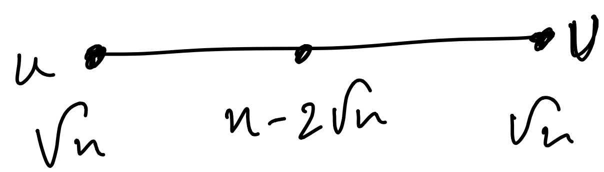
for some $c(n)$

<u>Examples:</u> Say $d(u,v)=1$ wlog.

Consider different scenarios.







Difficult to know the precise sets so pick sets of different sizes randomly.

① For $i = 1$ to $h = \log_2 n$ (assume n is power of 2)

S_i = random set where each v chosen into S_i with prob $\frac{1}{2^{i+1}}$ independently.

② $f(u)_i = \frac{1}{h} d(S_i, u)$

$E[|S_i|] = \frac{n}{2^{i+1}}$.

hence $|S_1|$ is roughly $\frac{n}{2}$

$|S_h| = \Theta(1)$ . . .

$$\|f(u)-f(v)\|_1 \leq d(u,v)$$

Want: to argue

$$\mathbb{E}\left[\|f(u)-f(v)\|_1\right] \geq \Omega\left(\frac{1}{h}\right) d(u,v).$$

<u>Defn:</u> Ball (v, r) = ball of radius r around v

$$B(r,v) = \{u \mid d(v,u) \leq r\}.$$

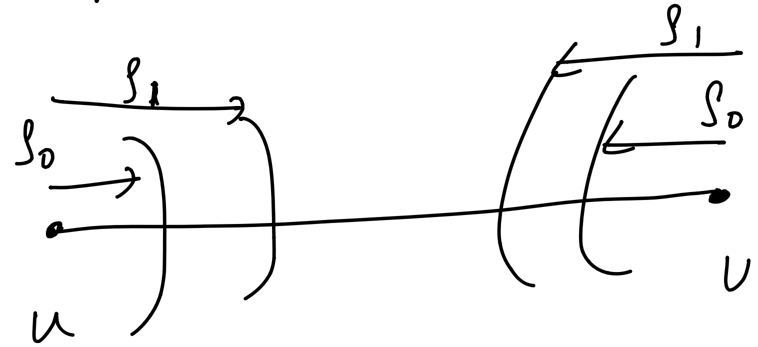
<u>Crucial defn:</u>

For $0 \leq l \leq h$

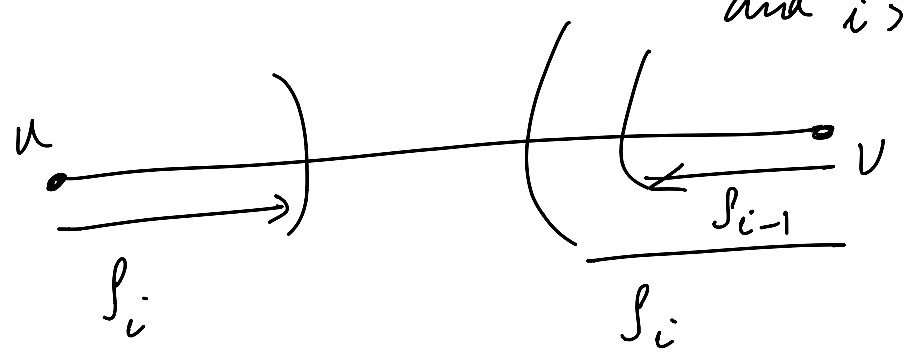
ρ_l = min radius ~~b~~ r such that $|B(u,r)| \geq 2^l$ <u>and</u> $|B(v,r)| \geq 2^l$

$\rho_0 = 0$. since $B(u,0) = \{u\}$
and $B(v,0) = \{v\}$.

$\rho_h \geq d(u,v)$ since
$|B(u,r)| = n \Rightarrow r \geq d(u,v)$.



Fix some i s.t. $\rho_i < \frac{d(u,v)}{2}$
and $i > 0$



Claim: $E[|d(S_i, u) - d(S_i, v)|]$

$$\geq (\rho_i - \rho_{i-1}) \underbrace{\tfrac{1}{2}(1 - e^{-\frac{1}{4}})}_{c}$$

$$\geq c(\rho_i - \rho_{i-1})$$

for some absolute constant c

Proof: $|B(u, \rho_i)| = 2^i$ or $|B(v, \rho_i)| = 2^i$

wlog say $|B(u, \rho_i)| = 2^i$.

Let $X = B(u, \rho_i)$ $\quad |X| = 2^i$

Let $Y = B(v, \rho_{i-1})$ $\quad |Y| \geq 2^{i-1}$.

$$|d(S_i, u) - d(S_i, v)| \geq \rho_i - \rho_{i-1}$$

if $S_i \cap Y \neq \phi$ and $S_i \cap X = \phi$.

What is

$$\Pr[S_i \cap Y \neq \phi \text{ and } S_i \cap X = \phi]?$$

X, Y disjoint so

$$\Pr[S_i \cap Y \neq \phi \text{ and } S_i \cap X = \phi]$$
$$= \Pr[S_i \cap Y \neq \phi] \cdot \Pr[S_i \cap X = \phi]$$

(independence).

$$\Pr[S_i \cap X = \phi] \geq \frac{1}{2}$$

Why?

$$E[|S_i \cap X|] = |X| \cdot \frac{1}{2^{i+1}} \leq \frac{1}{2}$$

$\Rightarrow \Pr[|S_i \cap X| \geq 1] \leq \frac{1}{2}$ by Markov.

$$
\begin{aligned}
\Pr\left[S_i \cap Y \neq \phi\right] &= 1 - \Pr\left[S_i \cap Y = \phi\right] \\
&= 1 - \prod_{v \in Y} \left(1 - \frac{1}{2^{i+1}}\right) \\
&= 1 - \left(1 - \frac{1}{2^{i+1}}\right)^{|Y|} \\
&\geq 1 - \left(1 - \frac{1}{2^{i+1}}\right)^{2^{i-1}} \\
&\geq 1 - \left(1 - \frac{1}{2^{i+1}}\right)^{\frac{2^{i+1}}{4}} \\
&\geq 1 - e^{-\frac{1}{4}}
\end{aligned}
$$

$\square$.

Let $i^* = \max i$ s.t. $\rho_i \leq \frac{d(u,v)}{2}$

For i^*+1

$$E\left[\left|d(S_{i^*+1}, u) - d(S_{i^*+1}, v)\right|\right] \geq c\left(\frac{d(u,v)}{2} - \rho_{i^*}\right)$$

Proof similar to previous Lemma.

Now

<u>Lemma</u>: $E\left[\|f(u) - f(v)\|_1\right] \geq c\, \frac{d(u,v)}{2} \cdot \frac{1}{h}$

and $\|f(u) - f(v)\|_1 \leq d(u,v)$.

<u>Proof</u>: We already saw that

$\|f(u) - f(v)\|_1 \leq d(u,v)$

$$\|f(u)-f(v)\|_1 = \sum_{i=1}^{h} |d(S_i,u)-d(S_i,v)|$$

$$E\left[\|f(u)-f(v)\|_1\right] = \sum_{i=1}^{h} E\left[|d(S_i,u)-d(S_i,v)|\right]$$

$$\geq \sum_{i=2}^{i^*+1} E\left[|d(S_i,u)-d(S_i,v)|\right]$$

$$\geq c \sum_{i=2}^{i^*} (\rho_i - \rho_{i-1})$$

$$+ c\left(\frac{d(u,v)}{2} - \rho_{i^*}\right)$$

$$\geq c\,\frac{d(u,v)}{2}.$$

$\square$

From expectation to high probability.

For a single pair (u,v)

$$E[\|f(u)-f(v)\|_1] \leq d(u,v)$$

$$\geq \frac{c}{2} d(u,v)$$

Need to preserve distances of all pairs!

Need for a single pair a high probability bound.

Obtained by repeating $\log n$ times and using Chernoff bounds.

① For $i = 1$ to $h = \log_2 n$ (assume n is power of 2)

S_i = random set where each v chosen into S_i with prob $\frac{1}{2^{i+1}}$ independently.

② $f(u)_i = \frac{1}{h} d(S_i, u)$

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① For $i = 1$ to $h = \ln_2 n$ do

For $j = 1$ to $l = c' \log n$

$S_{i,j}$ = random set where each $v \in S_{i,j}$ independently with prob $\frac{1}{2^{i+1}}$

② $f(u)$ is a hl dimensional vector with $f(u)_{i,j} = \frac{1}{hl} d(S_{i,j}, u)$.

Claim: $\|f(u)-f(v)\|_1 \leq d(u,v)$

and $\|f(u)-f(v)\|_1 \geq \frac{c}{4} d(u,v)$

with prob $\geq 1-\frac{1}{n^3}$

By union bound

$\|f(u)-f(v)\|_1 \leq d(u,v)$

and $\|f(u)-f(v)\|_1 \geq \frac{c}{4} d(u,v)$

for all uv with prob $\geq 1-\frac{1}{n}$.

$\square$.

For sparsest cut we only need to preserve distances of k pairs.

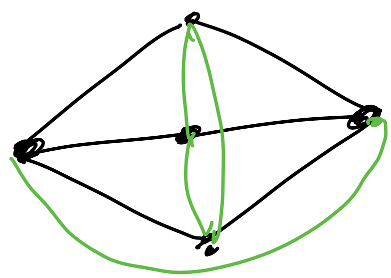
So $O(\log k)$ suffices.

Lower bounds

$$\rho^* \le \Phi^* \le \rho^* O(\log k).$$

How tight.

Example:



G is a series-parallel graph.

G satisfies cut condition for H.

$\Rightarrow \Phi^* \geq 1$.

What is ρ^*?

Total capacity is 6.

Each demand pair - shortest path length is at least 2.

4 demand edges

$\Rightarrow \; 4 \times \rho^* \times 2 \leq 6.$

$\Rightarrow \; \rho^* \leq \frac{6}{8} \leq \frac{3}{4}$

$\rho^* \leq \frac{3}{4} \qquad \phi^* \geq 1$

$\Rightarrow \; \frac{\phi^*}{\rho^*} \geq \frac{4}{3}.$

For series parallel graph

$$\frac{\phi^*}{\rho^*} \leq 2$$

and $\exists$ examples where $\phi^* \geq (2-\varepsilon)\rho^*$.

Non-trivial!

What about general graphs?
And especially for uniform sparsest cut.

Lower bound of $\Omega(\log n)$ via expander graphs.

<u>Defn</u>: Given multigraph $G=(V,E)$
edge-expansion $\alpha(G) = \min\limits_{\substack{S \subseteq V \\ |S| \leq \frac{|V|}{2}}} \frac{|\delta(S)|}{|S|}$.

<u>Theorem</u>: For all fixed integer $d \geq 3$ and all n even and sufficiently large $\exists$ d-regular n-vertex graphs with $\alpha(G) \geq c$ for some

absolute constant c.

Proof: Via probabilistic method.

Pick random d-regular graph and show it satisfies expansion property.

$\square$

Hard to do explicit constructions and deep work in this area.

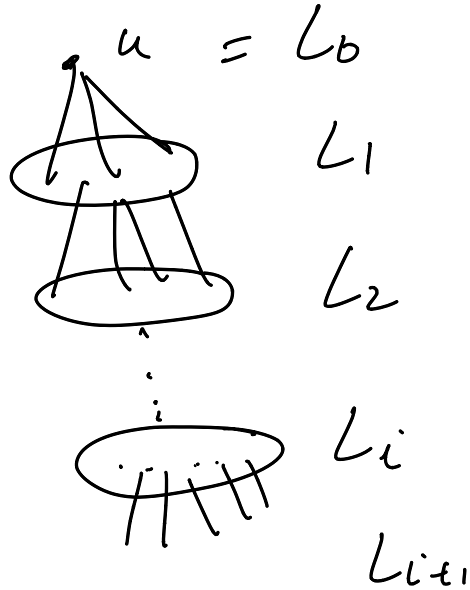
For now assume we have 3-regular α-expander for some fixed α.

Lemma: If G is a d-regular α-expander then $\mathrm{diam}(G) = \Omega(\log_d n)$ and $\mathrm{diam}(G) = O\left(\frac{d}{\alpha} \log n\right)$.

In particular for $d = 3$ and fixed α $\mathrm{diam}(G) = \Theta(\log n)$.

Proof sketch:

Fix vertex u and consider BFS layers.



$|L_{i+1}| \leq d\,|L_i|$ since degree $\leq d$

$\Rightarrow$ diam $= \Omega(\log_d n)$.

True for any constant degree graph.

Why is diam $\leq O(\frac{d}{\alpha} \log n)$ for expanders?

If $|L_1 \cup L_2 \cdots \cup L_i| \leq \frac{n}{2}$

$\Rightarrow |L_{i+1}| \geq |L_1 \cdots \cup L_i| \cdot \alpha \cdot \frac{1}{d}$

$\Rightarrow |L_1 \cup \cdots L_{i+1}| \geq (1 + \frac{\alpha}{d})^{i-1}$

grows exponentially at $(1+\frac{\alpha}{d})$ rate.

Fix u, v. Show that

# of BFS layers $\leq$ by considering

from u and v $\leq O(\ln_{1+\frac{\alpha}{d}} n)$

$$= O\left(\frac{d}{\alpha} \ln n\right).$$

$\square$.

Flow-cut example using expanders

deg 3 α expander.

Fix u.

$$\left| \mathrm{Ball}\left(u, \frac{\log n}{100}\right) \right| \leq \sqrt{n}$$

Since deg ≤ 3.

$\Rightarrow$ most vertices, $(n - \sqrt{n})$ are at distance $\geq \frac{\log n}{100}$ from u.

Consider uniform sparsest cut instance. $\mathrm{dem}(u\,v) = 1 \quad \forall\, u \neq v$.

$$\phi^* = \min_{S \subseteq V} \frac{|\delta(S)|}{|S|\,|V-S|}$$

$$|\delta(S)| \geq \alpha |S| \quad \forall S \subseteq V, |S| \leq \frac{|V|}{2}$$

$$\Rightarrow \frac{\delta(S)}{|S||V-S|} \geq \frac{\alpha}{\frac{n}{2}} \geq \frac{2\alpha}{n}.$$

$$\Rightarrow \phi^* \leq \frac{2\alpha}{n}.$$

What is ρ^*

$$\rho^* = \min_{\substack{d : V \times V \to R_+ \\ d \text{ metric}}} \frac{\sum_{uv \in E} d(uv)}{\sum_{uv} d(uv)}$$

Consider $x(uv) = \frac{100}{\log n}$.

Let $d(uv)$ be distance induced by edge lengths $x(u,v)$.

For each edge uv, $d(uv) = x(uv)$

So $$\sum_{uv \in E} d(uv) = \frac{100}{\log n} \cdot |E|$$

$$= \frac{300n}{\log n}.$$

What about

$$\sum_{uv} d(uv) = \sum_{u} \sum_{v} d(uv)$$

$$\geq \sum_{u} (n - \sqrt{n}) \cdot 1$$

$$\geq n(n-\sqrt{n}) \geq \frac{n^2}{2}$$

$$\Rightarrow \rho^* \leq \frac{300n}{\log n \cdot \frac{n^2}{2}} \leq \frac{600}{n \log n}.$$

Hence $\phi^* \geq \frac{2\alpha}{n}$

$$\rho^* \leq \frac{600}{n \log n}$$

and α is a fixed constant

$\Rightarrow \quad \phi^* = \Omega(\log n)\, \rho^*.$

Also implies that metric induced by expander requires $\Omega(\log n)$ distortion to embed into l_1.

Why?