Spring 2026, CS 583: Approximation Algorithms

Homework 3

Due: 04/07/2026 in Gradescope

- Each homework can be done in a group of size at most two. Only one homework needs to be submitted per group. However, we recommend that each of you think about the problems on your own first.
- Homework needs to be submitted in pdf format on Gradescope. See `https://courses.grainger.illinois.edu/cs374al1/fa2025/hw-pol.html` for more detailed instructions on Gradescope submissions.
- Follow academic integrity policies as laid out in student code. You can consult sources but cite all of them including discussions with other classmates and LLMs. Write in your own words. See the site mentioned in the preceding item for more detailed policies including specific instructions on using LLMs.
- Read through all the problems and think about them and how they relate to what we covered in the lectures. Solve as many problems as you can but I expect that you will submit 4 — if you submit only 4 then that set should solve at least one of Problems 5 and 6.
- Please write clearly and concisely - clarity and brevity will be rewarded. Refer to known facts as necessary.

Problem 1. The k-Center problem can be alternatively thought of as the following. Given a finite metric space (V, d) and a radius parameter $R > 0$, what is the smallest $\alpha \geq 1$ such that the points in V can be covered by k balls of radius αR where each ball is centered at a point in V? In *Priority* k-Center we are given (V, d), k, and for each $v \in V$ a radius $r(v) > 0$. The goal is to find smallest α such that there are k centers $C = \{c_1, c_2, \ldots, c_k\}$ with the property that $d(v, C) \leq \alpha r(v)$ for each $v \in V$.

- Describe a greedy style 2-approximation for this problem.
- Write a feasibility LP for the smallest α and show that one can obtain a 2-approximation with respect to the lower bound given by the LP relaxation.
- **Extra Credit:** Consider the *supplier* version of the problem. Here $V = F \uplus C$ where F is a set of facilities and C is a set of clients. The centers can be chosen only F and only clients need to be covered. Each client $v \in C$ has radius $r(v)$. Obtain a 3-approximation for this problem.

Problem 2. We discussed the Max-k-Cover problem in lecture and showed that it yields a $(1-1/e)$-approximation. A local-search algorithm is also natural. Start with an arbitrary collection of k-sets and swap one set at a time if it improves the coverage. Show that this algorithm yields a 1/2-approximation and also give an example to demonstrate that a local optimum is no better than 1/2-approximation.

Problem 3. Problem 5.6 from Shmoys-Williamson book on maximum directed cut via LP.

Problem 4. We saw local search examples in lecture. In *non-oblivious* local search we perform local search with an auxiliary function rather than the original objective. While local-search for Set Cover does not yield a good approximation algorithm, a clever non-oblivious one does. Read about it in this short and elegant recent paper (Section 2 suffices) and summarize the proof in your own words.

Problem 5. In the Node-weighted Multiway Cut problem we are given an undirected node-weighted graph $G = (V, E)$, k terminal nodes $S = \{s_1, s_2, \ldots, s_k\}$. Node v has a non-negative weight c_v. The goal is to remove a minimum weight subset of nodes $V' \subset V$ such that $G - V'$ has no path from s_i to s_j, $1 \leq i < j \leq k$. Assume for simplicity that terminals cannot be removed (they have infinite weight) and that they form an independent set (so that there is a feasible solution). Consider the following LP relaxation where there is a variable x_v for each $v \in V \setminus S$ indicating whether to remove v. Let $\mathcal{P}_{u,v}$ denote the set of all paths from u to v in G.

$$
\begin{aligned}
\min \sum_{v \in V} c_v x_v & & & \\
\sum_{v \in p} x_v & \geq 1 & & p \in \mathcal{P}_{s_i, s_j}, i \neq j \\
x_v & \geq 0 & & v \in V
\end{aligned}
$$

Let $\bar{x}$ be a feasible solution to the LP.

- Obtain a 2-approximation by generalizing the ball cutting algorithm that we saw for the edge-weighted case.
- Prove that an α-approximation for this problem implies an α-approximation for the Vertex Cover problem.

Problem 6. We saw the CKR relaxation for the Multiway Cut problem. It introduces a labeling perspective. The uniform metric labeling problem is a nice and powerful generalization which has led to an important rounding strategy. Solve Prob 5.10 in Williamson-Shmoys book.