CS 583: Approximation Algorithms	**Lecture date: April 12, 2018**
Instructor: Chandra Chekuri	**Scribe: CC**

MultiwayCut is the following problem. The input is an undirected graph $G = (V, E)$ with non-negative edge weights $c : E \to \mathbb{R}_+$, and a set of terminals $\{s_1, s_2, \ldots, s_k\} \subseteq V$. The goal is to remove a minimum-cost subset of edges $E' \subseteq E$ such that $G - E'$ has no path from s_i to s_j for $i \neq j$. The $k = 2$ case is the same as the s-t mincut problem in undirected graphs, and hence can be solved optimally. MultiwayCut is NP-Hard, and APX-Hard to approximate, for $k \geq 3$. We saw in lecture a simple combinatorial algorithms called the Isolating-Cut heuristic which yields a $2(1 - 1/k)$-approximation. We also saw that a standard LP relaxation for cut problems, given below, has an integrality gap of $2(1 - 1/k)$.

$$\begin{aligned} \min \sum_e c_e x_e & & \\ \sum_{e \in P} x_e &\geq 1 & P \text{ is a path connecting } s_i, s_j,\ i \neq j \\ x_e &\geq 0 & e \in E \end{aligned}$$

It is useful to take another perspective on the LP and write it with distance variables. For this purpose we can assume without loss of generality that G is a complete graph (we can add dummy edges of zero cost to achieve this). For each *unordered* pair uv of distinct nodes we have a variable d_{uv}; the variable denotes the distance between u and v which is supposed to be 1 if u and v are separated and 0 otherwise. The relaxation requires d to be a metric subject to the constraint that $d_{s_i s_j} = 1$ for each pair of terminals $s_i s_j$. Triangle inequality constraints ensure that d is a metric.

$$\begin{aligned} \min \sum_{uv} c_{uv} d_{uv} & & \\ d(s_i s_j) &= 1 & i \neq j \\ d_{uv} + d_{vw} &\geq d_{uw} & u, v, w \\ d_{uv} &\geq 0 & uv \end{aligned}$$

It is not hard to convince oneself that the two relaxations are equivalent in terms of the optimum cost; we note that the two relaxations are very different in the polyhedral sense. Syntactically it is sometimes more convenient to use variable $d_{u,v}$ for the ordered pair (u, v) and impose symmetry explicitly in the constraints $d_{u,v} = d_{v,u}$ and then let d_{uv} stand for the common variable.

The following example shows that the worst case integrality gap of the preceding relaxations is $2(1 - 1/k)$. Consider a star with center v and k leaves which are the terminals. All edges have cost 1. It is easy to see than an optimum integral solution has cost $(k - 1)$, while the optimum fractional solution has cost $k/2$; setting each $x_e = 1/2$ achieves this cost.

Calinescu, Karloff and Rabani [2] were the first to obtain an approximation ratio better than 2 for MultiwayCut, and they achieved this via novel LP relaxation that has since been extensively studied. The current best upper bound on the integrality gap of this relaxation (henceforth called the CKR relaxation) is 1.2965 [5], and the best known lower bound on the relaxation is $6/(5 +$

$(1/(k-1))$ which approaches $6/5$ for large k [1]. Both are fairly recent results. Moreover, it is known, that under the unique games conjecture (UGC) [4], for every fixed k, the integrality gap of the CKR relaxation is the worst-case hardness of the problem! We do not know the precise worst-case integrality gap of the CKR relaxation for $k \geq 4$. For $k = 3$ it is known to be $12/11$.

In this notes we describe the CKR relaxation and a rounding scheme that achieves a 1.5 approximation which was the bound initially shown in [2]. This analysis is given in the approximation text books. The main purpose of this lecture notes is to highlight a different analysis which is based on the work of Chekuri and Ene [3] on the Submodular Multiway Partition (SubMP) problem which generalizes MultiwayCut. We believe that it gives a different perspective and also eliminates some of the intricacies in the standard analysis, at least as far as obtaining an approximation ratio of 1.5.

1 CKR **Relaxation**

The CKR relaxation is typically described as a geometric relaxation and this is well-justified. However, there are two other "natural" ways to derive it and we will discuss them later. The basic observation that leads to the relaxation is the following. It is useful to view MultiwayCut in undirected graphs as a *partition* problem rather than as an edge-removal problem. We can assume without loss of generality that G is connected. We observe that if $E' \subseteq E$ is a minimal feasible solution to disconnect the terminals, then $G - E'$ has exactly k connected components with exactly one terminal in each. Moreover, each edge in E' connects two of these components. Thus, we can reformulate MultiwayCut in the following equivalent way. Find a partition $V_1, V_2, \ldots, V_k$ of V such that $s_i \in V_i$ for $1 \leq i \leq k$, where the objective is to min $\frac{1}{2}\sum_{i=1}^{k} c(\delta(V_i))$. Once we have this partition based formulation, it is natural to use variables $x(u,i)$ for each vertex u and index $i \in [k]$; the variable indicates whether u belongs to V_i. We then write the following constraints that model the requirement that each u belongs exactly to one piece of the partition and that $s_i \in V_i$.

$$\begin{aligned} \sum_{i=1}^{k} x(u,i) &= 1 \quad u \in V \\ x(s_i,i) &= 1 \quad i \in [k] \\ x(u,i) &\geq 0 \quad u \in V, i \in [k] \end{aligned}$$

How do we model the objective? Consider an edge uv and assume all the allocation variables are binary and satisfy the constraints. uv is cut iff there exists an i such that $x(u,i) \neq x(v,i)$. But if there is such an i then there will be an $i' \neq i$ such that $x(u,i') \neq x(v,i')$. Thus, one can see that uv is cut iff $\sum_{i=1}^{k} |x(u,i) - x(v,i)| = 2$. Hence the objective becomes

$$\min \sum_{uv \in E} c_{uv} \frac{1}{2} \sum_{i=1}^{k} |x(u,i) - x(v,i)|$$

The objective function involves the absolute value function and hence is not linear. Since we are minimizing, we can obtain an LP, with the help of additional variables $z_{uv,i}$ as follows:

$$\min \sum_{uv \in E} c_{uv} \frac{1}{2} \sum_{i=1}^{k} z_{uv,i}$$

$$\begin{aligned}
\sum_{i=1}^{k} x(u,i) &= 1 \quad u \in V \\
x(s_i, i) &= 1 \quad i \in [k] \\
z_{uv,i} &\geq x(u,i) - x(v,i) \quad uv \in E, i \in [k] \\
z_{uv,i} &\geq x(v,i) - x(u,i) \quad uv \in E, i \in [k] \\
x(u,i) &\geq 0 \quad u \in V, i \in [k] \\
z_{uv,i} &\geq 0 \quad uv \in E, i \in [k]
\end{aligned}$$

Geometric interpretation: Perhaps the most useful interpretation, as far as MultiwayCut is considered, is the geometric interpretation. Consider the k unit basis vectors $\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_k}$ in R^k (here $\mathbf{e_i}$ is the vector with 1 in the i'th coordinate and 0 in all other coordinates). The convex hull of these k vectors creates the k-simplex and it is easy to see that for each $u \in V$, the vector $x_u = (x(u,1), x(u,2), \ldots, x(u,k))$ lies in this convex hull. Note that the simplex lies on a hyperplane of R^k, and hence is not full-dimensional. The relaxation is embedding the terminals at the corners of the k-simplex, and mapping each node u to a point within the simplex. The distance between$u$ and v is half the ℓ_1 distances between the vectors x_u and x_v. The advanced rounding algorithms for MultiwayCut via the CKR relaxation make very strong use of the geometric view point, as well as the special structure enjoyed by the objective function for graphs, which decomposes in a nice fashion.

Interpretation as strengthening the distance relaxation: Another nice interpretation of the CKR relaxation (also from [2]) is as follows. We already discussed the basic distance LP and showed that it has a worst case integrality gap of $2(1-1/k)$. It is natural to consider *strengthening* the LP by adding inequalities that are *valid* for the integer program. Consider a node u. If we take the partition view then we note that u must belong to one of the components, which implies that it must have a distance 0 to the terminal whose component it belongs to, and distance 1 to the other $k-1$ terminals. Thus the following inequalites are valid for the integer program:

$$\sum_{i=1}^{k} d(us_i) = k - 1 \quad u \in V$$

We can also add another set of valid inequalities as follows. Consider distinct nodes $u, v \in V$. Let $T \subset [k]$ correspond to a subset of terminals. Then we claim that

$$d(uv) \geq \sum_{i \in T} [d(us_i) - d(vs_i)]$$

is a valid constraint. We leave it as an exercise to check this inequality's validity when the distances are induced by partitions. Note that there are an exponential number of inequalities of the second type since we have to write it for each edge uv and each $T \subset [k]$. It is possible to separate over these inequalities in polynomial time (see [2] or think about it). Interestingly, if we add the preceding two sets of valid inequalities to the basic distance LP we get the CKR relaxation!

Interpretation via Lovász-extension of submodular functions: Recall that the objective function that came out naturally from the allocation variables $x(u,i)$ was the following.

$$\min \sum_{uv \in E} c_{uv} \frac{1}{2} \sum_{i=1}^{k} |x(u,i) - x(v,i)|$$

We had to add additional variables to model this objective as a linear function. However, one can observe that the function $\sum_{uv \in E} c_{uv} \frac{1}{2} \sum_{i=1}^{k} |x(u,i) - x(v,i)|$ is convex in the variables (in fact it is a piece-wise linear convex function). Thus, we can model our problem as a convex optimization problem with linear constraints. Why is the objective convex? Is there a more general phenomenon going on here? Indeed, that is the case. We can cast MultiwayCut as a special case of the following more general problem. Let $f : 2^V \to \mathbb{R}_+$ be a submodular function and let $\{s_1, s_2, \ldots, s_k\} \subseteq V$. In SubMP the goal is to partition V into $V_1, V_2, \ldots, V_k$ such that $s_i \in V_i$ for $1 \le i \le k$, and to minimize $\sum_i f(V_i)$. Note that MultiwayCut is the special case of SubMP where f is the cut function of a graph $G = (V, E)$. In fact the cut function is not only submodular but also symmetric, and hence MultiwayCut is a special case of SymSubMP (Symmetric Submodular Multiway Partition) which is obtained by restricting f to be a symmetric submodular function. It is natural to write a relaxation for SubMP via the same allocation variables $x(u,i)$ as we did for MultiwayCut. The main difficulty is with the objective function f. How do we deal with a generic submodular function that is only available as a value oracle? It turns out that there are useful ways to *extend* a submodular set function f over a discrete ground set V to a continuous function g over the entire real cube $[0,1]^{|V|}$. Lovász described a particular extension $\hat{f}$, named after him, that turns out to be convex and useful in the sense that it can be evaluated at any point in the real cube efficiently given f as a value oracle. One can thus write a convex relaxation for SubMP (and hence also SymSubMP) via the Lovász-extension. When specialized to MultiwayCut this relaxation turns out to be the same as the CKR relaxation! We refer the reader to [3] for more details.

2 Rounding Algorithm and Analysis

In this section we show a rounding algorithm that gives a 1.5 approximation for MultiwayCut. As we already mentioned better ratios are known. The algorithm and analysis here is an adaptation of the algorithm and analysis from [3] for the SymSubMP problem. The original algorithm and analysis from [2] can be be found in the approximation text books.

The informal description is as follows. It picks a random $\theta \in [0,1)$ and for each terminal s_i we consider A_i to be ball of radius $(1-\theta)$ around s_i. We observe that $A_i = \{v \mid x(u,i) \ge \theta\}$ because

$$d(s_i v) = \frac{1}{2} \sum_{j=1}^{k} |x(s_i, j) - x(u,j)| = 1 - x(u,i).$$

Note that the sets $A_1, A_2, \ldots, A_k$ can overlap. We say that v is unallocated if $v \in V \setminus (\cup_i A_i)$. Since we are seeking a partition we need to fix the overalapping sets. We do this via an uncrossing operation. If $A_i \cap A_j \neq \emptyset$ we replace them with either $A_i - A_j, A_j$ or $A_i, A_j - A_i$. We will argue that this does not increase the cost. This way of understanding the algorithm is based on [3]. The algorithm from [2] avoids the overlap by picking a random permutation of the terminals, and A_i is defined to set of all vertices in the ball of radius $(1 - \theta)$ around s_i that have not already been allocated to a terminal that came earlier in the permutation. The random permutation is implicity using the fact that sets can be uncrossed without increasing the cost.

The analysis hingest on understanding the expected cost of $\delta(A_i), i \in [k]$ and the expected cost of $\delta(U)$. The algorithm is formally described in the box.

```
RoundCKR(G,S):
Let x̄ be an optimum solution to the CKR relaxation on G with terminals S
Pick θ uniformly at random from [0,1)
For i = 1 to k let A_i = {v | x(v,i) ≥ θ}
Let U = V \ (∪_{i=1}^k A_i) be the unallocated nodes
// Uncross the sets A_1, A_2, ..., A_k so that they are disjoint
While there exist i ≠ j such that A_i ∩ A_j ≠ ∅ do
  If c(δ(A_i)) + c(δ(A_j − A_i)) ≤ c(δ(A_i)) + c(δ(A_j))) then
      A_j = A_j − A_i
  Else
      A_i = A_i − A_j
endWhile
Output the partition A_1 ∪ U, A_2, ..., A_k
```

Let OPT be the cost of the CKR relaxation. For the sake of the analysis we let $A_1, A_2, \ldots, A_k$ denote the *random* sets obtained in the first randomized step of the algorithm, and let $A'_1, \ldots, A'_k$ denote the sets after the uncrossing step. We make some basic observations that are easy to verify. They show that the final partition is indeed a valid solution to the problem.

- For each i, $s_i \in A_i$, and $s_i \in A'_i$.
- $\cup_i A_i = \cup_i A'_i$.
- The uncrossing step terminates in $O(k^2)$ iterations. Why?

The cost analysis is based on the following facts that we will prove.

- The expected cost of $\frac{1}{2}\sum_{i=1}^k c(\delta(A_i))$ is at most OPT. That is

$$E[\frac{1}{2}\sum_{i=1}^{k} c(\delta(A_i))] \leq \text{OPT}.$$

- The expected cost of $\frac{1}{2}c(\delta(U))$ is at most OPT/2, that is,

$$E[c(\delta(U))] \leq \text{OPT}/2.$$

- The uncrossing steps do not increase the cost. That is, $\sum_i c(\delta(A'_i)) \leq \sum_i c(\delta(A_i))$. This is a deterministic statement for any sets $A_1, \ldots, A_k$.

Exercise: Argue, assuming the above properties, that the expected cost of the final partition is at most 1.5OPT.

We now prove the desired properties. The analysis will be on an edge by edge basis. Since the objective function is separable over the edges we can use linearity of expectation for the overall bounds. Recall that $z_{uv,i} = |x(u,i) - x(v,i)|$ and let $z_{uv} = \sum_i z_{uv,i}$ be the ℓ_1 distance between x_u and x_v. Note that the objective function has a

Lemma 1 $\Pr[uv \in \delta(A_i)] = z_{uv,i}$.

Proof: It is easy to see that $uv \in \delta(A_i)$ iff θ is in the interval $[\min\{x(u,i), x(v,i)\}, \max\{x(u,i), x(v,i)\}]$ which happens with probability $|x(u,i) - x(v,i)|$. □

The preceding lemma easily shows, via linearity of expectation, that $E[c(\delta(A_i))] = \sum_{uv} c_{uv} z_{uv,i}$ and establishes the first property.

We now show that uncrossing can be done without increasing the cost. Here we rely on submodularity and symmetry of the cut function.

Lemma 2 *Let $f : 2^V \to \mathbb{R}$ be a symmetric submodular function. For any two sets $A, B \subset V$ we have $f(A) + f(B) \geq f(A - B) + f(B)$ or $f(A) + f(B) \geq f(A) + f(B - A)$.*

Proof: Submodularity and symmetry implies posi-modularity for f. For any A, B

$$f(A) + f(B) \geq f(A - B) + f(B - A).$$

Hence

$$2f(A) + 2f(B) \geq [f(A) + f(B - A)] + [f(A - B) + f(B)].$$

This implies the desired claim. □

Exercise: Using the preceding lemma argue that $\sum_i c(\delta(A'_i)) \leq \sum_i c(\delta(A_i))$.

Now we consider the probability that $uv \in \delta(U)$. For this purpose, for each node v, we let α_v denote the quantity $\max_i x(v,i)$.

Lemma 3 $\Pr[uv \in \delta(U)] = |\alpha_u - \alpha_v|$.

Proof: Note that $v \in U$ iff $\theta > \alpha_v$; if $\theta \leq \alpha_v$ v will belong to some A_i. Thus $uv \in \delta(U)$ only if exactly one of u, v is in U. This happens only if θ is in the interval $[\min\{\alpha_u, \alpha_v\}, \max\{\alpha_u, \alpha_v\}]$. The lemma follows since θ is chosen uniformly from $[0, 1]$. □

Now, all that remains is the following technical lemma, which is not quite intuitive at first glance but whose proof is not difficult. We leave it as an exercise to the reader.

Lemma 4 $|\alpha_u - \alpha_v| \leq \frac{1}{2} \sum_i |x(u,i) - x(v,i)|$.

From Lemma 3, and linearity of expectation, we obtain

$$E[c(\delta(U))] \leq \frac{1}{2} \sum_{uv \in E} c_{uv} z_{uv}$$

and hence

$$\frac{1}{2} E[c(\delta(U))] \leq \frac{1}{2} \text{OPT}$$

as desired.

References

[1] H. Angelidakis, Y. Makarychev, P. Manurangsi. An Improved Integrality Gap for the Călinescu-Karloff-Rabani Relaxation for Multiway Cut. *Proc. of IPCO*, 2017.

[2] G. Calinescu, H. J. Karloff, and Y. Rabani, "An improved approximation algorithm for multiway cut," *Journal of Computer and System Sciences*, vol. 60, no. 3, pp. 564–574, 2000. Preliminary version in STOC 1998.

[3] C. Chekuri and A. Ene, "Approximation algorithms for submodular multiway partition,", *Proceedings of IEEE FOCS*, 2011.

[4] Rajsekar Manokaran, Joseph Seffi Naor, Prasad Raghavendra, and Roy Schwartz. Sdp gaps and ugc hardness for multiway cut, 0-extension, and metric labeling. In *Proceedings of the fortieth annual ACM symposium on Theory of computing*, pages 11–20. ACM, 2008.

[5] Ankit Sharma and Jan Vondrák. Multiway cut, pairwise realizable distributions, and descending thresholds. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing*, pages 724–733. ACM, 2014.