Spring 2018, CS 583: Approximation Algorithms

Homework 5

Due: 04/20/2018

Instructions and Policy: Each student should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with; ideally you should discuss with no more than two other people. You may be able to find solutions to the problems in various papers and books but it would defeat the purpose of learning if copy them. You should cite all sources that you use.

Read through all the problems and think about them and how they relate to what we covered in the lectures. Solve as many problems as you can. Please submit your solutions to at least 4 problems.

Please write clearly and concisely - clarity and brevity will be rewarded. Refer to known facts as necessary. Your job is to convince me that you know the solution, as quickly as possible.

Problem 1 Read about the prima-dual based augmentation framework for covering a proper function in the survey of Goemans and Williamson. This problem is designed to make you read the relevant parts. Let $f : 2^V \to \mathbb{Z}_+$ be a proper function. Recall that f is proper if it is symmetric, $f(V) = 0$, and it is maximal, i.e., ($f(A \cup B) \leq \max\{f(A), f(B)\}$ for disjoint A, B).

- Prove that if A, B, C is a partition of V then the maximum of f over these three sets cannot be attained by only one of them.
- Let p be an integer and define $g_p : 2^V \to \mathbb{Z}_+$ as $g_p(S) = \min\{p, f(S)\}$. Show that g_p is proper.
- Supose $G = (V, E)$ is a graph and $F \subseteq E$ such that for all $S \subset V$ $|\delta_F(S)| \geq \min\{p - 1, f(S)\}$. Define $h : 2^V \to \{0, 1\}$ as $h(S) = 1$ if $g_p(S) = p$ and $|\delta_F(S)| = p - 1$, otherwise $h(S) = 0$. Prove that h is uncrossable. That is, if $h(A) = h(B) = 1$ then $h(A \cap B), h(A \cup B) = 1$ or $h(A - B), h(B - A) = 1$.

Assuming that the problem of covering a uncrossable function admits a 2-approximation, derive a $2k$-approximation for the problem of covering a proper function where $k = \max_S f(S)$ is the maximum requirement.

Problem 2 Let $G = (V, E)$ be an undirected graph and let f be an *uncrossable* function on the vertex set, i.e., a $\{0, 1\}$ valued function that satisfies

- $f(V) = 0$; and
- if $f(A) = f(B) = 1$ for any $A, B \subseteq V$, then either $f(A \cup B) = f(A \cap B) = 1$ or $f(A - B) = f(B - A) = 1$.

Recall that the primal-dual algorithms require the ability to answer the following questions.

- Given $F \subseteq E$, is F a feasible solution for f?
- Given $F \subseteq E$ what are the minimal violated sets with respect to F?

1. Given a set of edges $F \subseteq E$ the minimal violated sets of f with respect to F are those sets S such that $f(S) = 1$ and $\delta_F(S) = \emptyset$. Prove that if f is uncrossable then for any F, minimal violated sets of f with respect to F are disjoint.

2. Suppose f is an uncrossable function. In general there cannot be a polynomial time algorithm to test if F is a feasible solution by simply using the value oracle for f. For instance the following function h is uncrossable: there is a single set A such that $h(A) = 1$ and for all other sets B, $h(B) = 0$. Using just a value oracle it is infeasible to find A without trying all possible sets. However, suppose you have an oracle that given $F \subseteq E$ returns whether F is feasible or not. Show how you can use such an oracle to compute in polynomial time the minimal violated sets with respect to a collection of edges A. First prove that if $S \subset V$ is a *maximal* set such that $A \cup \{(i, j) : i, j \in S\}$ is not feasible then $V \setminus S$ is a minimal violated set for A. Then deduce that the set of minimial violated sets can be obtained by less than $|V|^2$ calls to the feasibility oracle.

3. (Extra credit:) Suppose f is a proper function, i.e., $f : 2^V \rightarrow \mathbb{N}$ is an (non-negative) integer valued function satisfying,

    - $f(V) = 0$,
    - f satisfies maximality (see Problem 3),
    - f is symmetric.

    Suppose f is accessible by a value oracle which when given a set $S \subset V$ returns the value $f(S)$; Show that there is a polynomial time algorithm to determine if F is a feasible solution.
    Hint: Consider the cuts in the Gomory-Hu tree T for the graph $G[F]$.

Problem 3 We discussed the Point-to-Point connection problem and mentioned that it can be cast as a special case of covering a proper function. See survey of Goemans and Williamson on network design. Here we consider the *unbalanced* point-to-point connection problem. The input consists of an edge-weighted undirected graph $G = (V, E)$ and two disjoint sets of vertices S and T where $|S| \leq |T|$. The goal is to find a min-cost subgraph

H of G such that in each connected component C of H there are at least as many nodes from T as there are from S; that is $|V(C) \cap S| \leq |V(C) \cap T|$. When $|S| = |T|$ we have the Point-to-Point connection problem. Show that if $|S| < |T|$ the resulting problem is not necessarily a special case of covering a proper or the more general problem of covering an uncrossable function.

Problem 4 Problem 23.23 in Vazirani's book.

Problem 5 Problem 11.3 from Williamson-Shmoys book.

Problem 6 Problem 11.5 from Williamson-Shmoys book.

Problem 7 Problem 11.6 from Williamson-Shmoys book.