Spring 2016, CS 583: Approximation Algorithms

Homework 5
Due: 4/27/2016

Instructions and Policy: Each student should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with; ideally you should discuss with no more than two other people.

Read through all the problems and think about them and how they relate to what we covered in the lectures. Solve as many problems as you can. Please submit your solutions to at least 4 problems.

Please write clearly and concisely - clarity and brevity will be rewarded. Refer to known facts as necessary. Your job is to convince me that you know the solution, as quickly as possible.

Problem 1 Problem 11.6 from Shmoys-Williamson book.

Problem 2 In the Node-weighted Multiway Cut problem we are given an undirected node-weighted graph $G = (V, E)$, k terminal nodes $S = \{s_1, s_2, \ldots, s_k\}$. Node v has a non-negative weight c_v. The goal is to remove a minimum weight subset of nodes $V' \subset V$ such that $G - V'$ has no path from s_i to s_j, $1 \leq i < j \leq k$. Assume for simplicity that terminals cannot be removed (they have infinite weight) and that they form an independent set (so that there is a feasible solution). Consider the following LP relaxation where there is a variable x_v for each $v \in V \setminus S$ indicating whether to remove v. Let $\mathcal{P}_{u,v}$ denote the set of all paths from u to v in G.

$$\min \sum_{v \in V} c_v x_v$$

$$\begin{array}{rcll} \displaystyle\sum_{v \in p} x_v & \geq & 1 & \quad p \in \mathcal{P}_{s_i,s_j}, i \neq j \\ x_v & \geq & 0 & \quad v \in V \end{array}$$

Let $\bar{x}$ be a feasible solution to the LP. Define $B_{\bar{x}}(s, r)$ where $s \in V$ and r is a real number to be the set of all nodes v such that there is a path P from s to v such that $(\sum_{u \in P} \bar{x}_u) - \bar{x}_v < r$. That is, we are counting only the length of the nodes in the path other than the endpoint v.

Consider the following rounding algorithm. First remove all nodes v such that $\bar{x}_v \geq 1/3$. Second, pick a θ uniformly at random from $(0, 1/3)$ and for each s_i remove all nodes that are adjacent to $B(s_i, \theta)$ but not in $B(s_i, \theta)$. Prove that for any θ the removed nodes form a multiway cut and that the expected weight of the nodes removed is at most $3 \sum_v c_v x_v$.

Problem 3 In the k-tree problem you are given an undirected edge-weighted graph $G = (V, E)$ with edge weights $c : E \to \mathbb{R}^+$ and an integer k. The goal is to find a tree $T = (V_T, E_T)$ in G of smallest edge weight ($\sum_{e \in E_T} c(e)$) such that $|V_T| \geq k$. Show that if there is an α-approximation for k-tree then there is an α-approximation for the Steiner tree problem. Recall that in the Steiner tree problem, the input is an edge-weighted graph $G = (V, E)$ and a set of terminals $S \subseteq V$; the goal is to find a tree T of minimum edge-weight that connects (contains) all the terminals S.

Problem 4 In this problem you will derive an $O(\log k \cdot \log n)$ approximation for the rooted k-Steiner-tree problem which is related to the previous problem. The input consists of an edge-weighted undirected graph $G = (V, E)$, a specified root vertex r and a set $S \subset V$ of terminals. The goal is to find a min-cost tree (V_T, E_T), a sub-graph of G, such that $r \in V_T$ and $|S \cap V_T| \geq k$. Obtain an approximation for this problem following the outline below.

- Consider the density variant of the problem where the goal is to find a tree $T = (V_T, E_T)$ rooted at r that minimizes the ratio $c(E_T)/|V_T \cap S|$. Write an LP relaxation for this problem using ideas similar to the one for Sparsest Cut (and Steiner tree). Using the scaling idea used to reduce Sparsest Cut to Multicut, reduce the density variant of k-Steiner-tree to the Steiner tree problem and obtain an $O(\log n)$ approximation. Recall that the Steiner tree LP has an integrality gap of 2.

  See Lecture 19 from spring 2009, and Prob 8.6 from the Shmoys-Williamson book to learn about the scaling idea mentioned.

- Use an approximation algorithm for the density variant above in an iterative greedy fashion to create a tree rooted at r with at least k terminals. What is the density of this tree when compared to the density of the optimal solution to the original k-Steiner-tree problem?

- If the tree you have in the previous step has many more than k terminals, *prune* it to have k' terminals where $k \leq k' \leq 2k$ such that the density of the resulting tree is not much worse than the tree you started with.

- How can you connect the pruned tree to the original root r without incurring too much cost? Assuming you know the optimal cost, can you preprocess the instance to ensure that this connection cost is not too much?

Problem 5 Consider the feedback edge set problem (FES). The input is an edge-weighted undirected graph $G = (V, E)$ and the goal is to remove a minimum-weight set $E' \subset E$ such that $G - E'$ has no cycles. Note that FES can be solved in polynomial-time by taking the complement of a maximum-weight spanning tree. Nevertheless we will consider an analysis based on the following natural LP. There is a variable x_e for each $e \in E$ that indicates whether to remove e.

$$
\begin{aligned}
\min \sum_{e \in E} c_e x_e & & \\
\sum_{e \in C} x_e \;\geq\; 1 & \qquad \text{for each cycle } C \\
x_e \;\geq\; 0 & \qquad e \in E
\end{aligned}
$$

- Assuming the existence of a constant degree graphs whose *girth* is $\Omega(\log n)$ (the degree is bigger than 2 otherwise the cycle is an easy example) prove that the integrality gap of this LP is $\Omega(\log n)$. The girth of a graph is the length of its shortest cycle.

- We saw an upper bound of $O(\log n)$ via the primal-dual technique. See Section 7.2 in Williamson-Shmoys book for the more general Feedback Vertex Set problem. Here we will see a proof via a reduction to the Multicut analysis. Given a feasible solution $\bar{x}$ for the LP for FES define a Multicut instance where each a pair of vertices uv is to be separated of $d_{\bar{x}}(uv) \geq 1/3$; that is, the distance from u to v according to edge-lenghts given by $\bar{x}$ is at least $1/3$. Define a feasible fractional solution for this Multicut instance on G by appropriately scaling up $\bar{x}$ and use the Multicut analysis to give an $O(\log n)$ upper bound on the integrality gap of the LP.

- *Extra Credit:* Use the above ideas to obtain an $O(\log n)$-approximation for the Subset Feedback Edgeset problem. Here we are given edge-weighted graph $G = (V, E)$ and a subset of terminals $S = \{s_1, \ldots, s_k\}$ and the goal is to remove a minimum-weight set of edges E' such that $G - E'$ has no cycle containing a terminal.

Problem 6 Prove that any ring metric isometrically embeds into ℓ_1.

Problem 7 Given a graph $G = (V, E)$ with edge-weights $c : E \to \mathbb{R}^+$, you wish to partition G into $G_1 = G[V_1], G_2 = G[V_2], G_3 = G[V_3]$ such that $\lfloor |V|/3 \rfloor \leq |V_i| \leq \lceil |V|/3 \rceil$ for $1 \leq i \leq 3$, and the cost of the edges between the partitions is minimized. Using an α-approximation for the sparsest cut problem, give a pseudo-approximation for this problem where you partition the graph into 3 pieces $G[V_1'], G[V_2'], G[V_3']$ such that $|V|/c_2 \leq |V_i'| \leq |V|/c_1$ for some constants $1 < c_1 < c_2$ and the cost of the edges between the partitions is $O(\alpha)\textsc{opt}$. What constants c_1, c_2 can you guarantee? Note that c_1 and c_2 should be *constants*, independent of the graph size. (Hint: this problem is similar to the one on partitioning into two pieces that is in Vazirani's book on applications of sparsest cut (Section 21.6.3).)