Fall 2021, CS 583: Approximation Algorithms

Homework 6

Due: 12/09/2021 in Gradescope

Instructions and Policy: Each student should write up their own solutions independently. You need to indicate the names of the people you discussed a problem with; ideally you should discuss with no more than two other people. You may be able to find solutions to the problems in various papers and books but it would defeat the purpose of learning if copy them. You should cite all sources that you use and write in your own words.

Read through all the problems and think about them and how they relate to what we covered in the lectures. Please submit solutions to three or more problems. Some problems are closely related so it may benefit you to solve them together or view them as parts of an extended problem.

Please write clearly and concisely - clarity and brevity will be rewarded. Refer to known facts as necessary.

Problem 1 Consider the feedback edge set problem (FES). The input is an edge-weighted undirected graph $G = (V, E)$ and the goal is to remove a minimum-weight set $E' \subset E$ such that $G - E'$ has no cycles. Note that FES can be solved in polynomial-time by taking the complement of a maximum-weight spanning tree. Nevertheless we will consider an analysis based on the following natural LP. There is a variable x_e for each $e \in E$ that indicates whether to remove e.

$$\begin{aligned} \min \sum_{e \in E} c_e x_e & & \\ \sum_{e \in C} x_e &\geq 1 & \text{for each cycle } C \\ x_e &\geq 0 & e \in E \end{aligned}$$

- Assuming the existence of a constant degree graphs whose *girth* is $\Omega(\log n)$ (the degree is bigger than 2 otherwise the cycle is an easy example) prove that the integrality gap of this LP is $\Omega(\log n)$. The girth of a graph is the length of its shortest cycle.
- Section 7.2 in Williamson-Shmoys describes a primal-dual algorithm that proves that the integrality gap is $O(\log n)$ (for the more general Feedback Vertex Set problem). Here we illustrate a different proof via a reduction to the Multicut analysis. Given a feasible solution $\bar{x}$ for the LP for FES, remove all edges e with $\bar{x}_e > 1/3$ and then define a Multicut instance where each a pair of vertices uv is to be separated of $d_{\bar{x}}(uv) \geq 1/3$;

that is, the distance from u to v according to edge-lenghts given by $\bar{x}$ is at least $1/3$. Define a feasible fractional solution for this Multicut instance on G by appropriately scaling up $\bar{x}$ and use the Multicut analysis to give an $O(\log n)$ upper bound on the integrality gap of the LP.

- *Extra Credit:* Use the above ideas to obtain an $O(\log n)$-approximation for the Subset Feedback Edgeset problem. Here we are given edge-weighted graph $G = (V, E)$ and a subset of terminals $S = \{s_1, \ldots, s_k\}$ and the goal is to remove a minimum-weight set of edges E' such that $G - E'$ has no cycle containing a terminal.

Problem 2 Consider an instance of the non-uniform Sparsest Cut problem on an undirected *cycle/ring* $G = (V, E)$ with edge capacities $c : E \to \mathbb{R}_+$. Note that in addition to the graph we are given demand pairs $(s_1, t_1), \ldots, (s_k, t_k)$ with non-negative demand values $D_1, \ldots, D_k$.

- Describe a simple combinatorial algorithm to find a sparsest cut in a cycle.
- Prove that any ring metric isometrically embeds into ℓ_1. And use this argue that the LP for sparsest cut has integrality gap 1 on cycles.

Problem 3 Problem 4 in HW 4 (Problem 7.2 in Williamson-Shmoys book) considered Multicut in trees and showed that the integrality gap is 2. Use this and and the result on tree embeddings to prove that the integrality gap of the natural LP for Multicut in general graphs is $O(\log n)$. Note that we already saw this directly but this problem is to illustrate the power of tree embeddings.

Problem 4 In the k-MST problem you are given an undirected edge-weighted graph $G = (V, E)$ with edge weights $c : E \to \mathbb{R}_+$ and an integer k. The goal is to find a tree $T = (V_T, E_T)$ in G of smallest edge weight ($\sum_{e \in E_T} c(e)$) such that $|V_T| \geq k$. Recall that in the Steiner tree problem, the input is an edge-weighted graph $G = (V, E)$ and a set of terminals $S \subseteq V$; the goal is to find a tree T of minimum edge-weight that connects (contains) all the terminals S.

Now consider the rooted k-Steiner tree problem. The input consists of an edge-weighted undirected graph $G = (V, E)$, a specified root vertex r and a set $S \subset V$ of terminals. The goal is to find a min-cost tree (V_T, E_T), a sub-graph of G, such that $r \in V_T$ and $|S \cap V_T| \geq k$. It is not hard to see that k-Steiner tree is a further generalization of k-MST but they are equivalent (do you see why?).

- Show that if there is an α-approximation for k-MST then there is an α-approximation for the Steiner tree problem.
- Obtain a randomized $O(\log n)$ approximation for k-Steiner tree problem via probabilistic embedding into tree metrics.

There is a 2-approximation for k-MST problem but it is rather involved.

Problem 5 Consider MAX-CUT with the additional constraint that specified pairs of vertices be on the same/opposite sides of the cut. Formally, we are given two sets of pairs of vertices, S_1 and S_2. The pairs in S_1 need to be separated, and those in S_2 need to be on the same side of the cut sought. Under these constraints, the problem is to find a maximum-weight cut.

1. Give an efficient algorithm to check if there is a *feasible* solution.
2. Assuming there is a feasible solution, give a strict quadratic program and vector program relaxation for this problem. Show how the algorithm for MAX-CUT we saw in class can be adapted to this problem while maintaining the same approximation ratio.

Problem 6 Problem 6.6 from the Williamson-Shmoys book. SDP for directed cut.