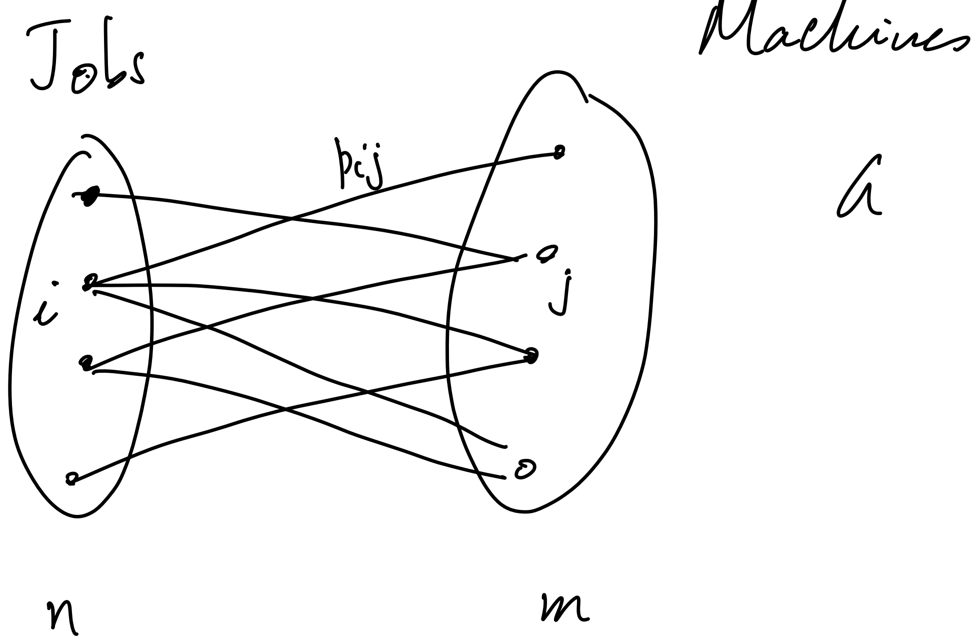
Generalized Assignment

Iterated rounding

Unrelated machine scheduling.



p_{ij} is load/processing time of job i on machine j

c_{ij} cost of scheduling i on j

b_j capacity of machine j.

<u>Theorem</u> [Shmoys-Tardos]
Suppose there is a feasible assignment of jobs to machines with cost C^*. There is an efficient alg that outputs an assignment of cost $\leq C^*$ and load on each machine j $\leq b_j$ + one job.
$\leq 2b_j$.

x_{ij} is a variable that i is assigned to j.

$ij \in E$.

$$\min \sum_{ij \in E} c_{ij} x_{ij}$$

$$\sum_{ij \in \delta(i)} x_{ij} = 1 \quad \forall\, i \in J$$

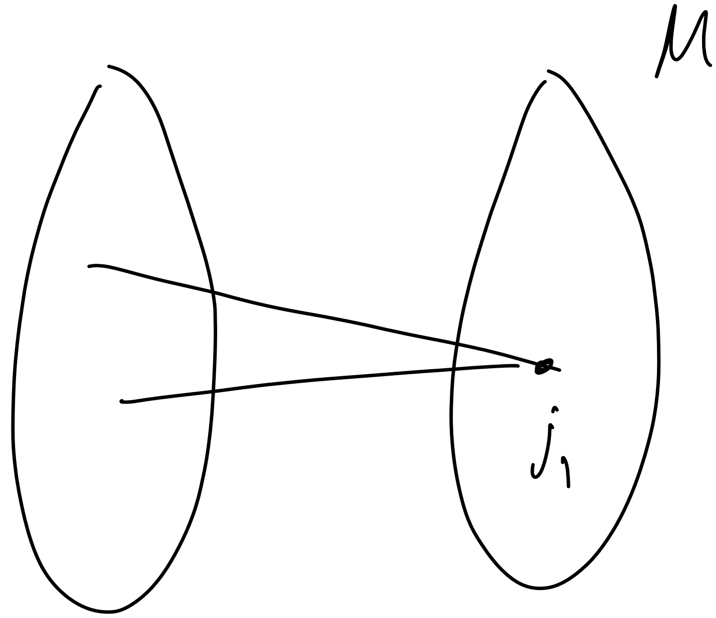
$$\boxed{\sum_{ij \in \delta(j)} p_{ij} x_{ij} \leq b_j \quad \forall\, j \in M.}$$

$$x_{ij} \geq 0.$$

Suppose $\exists$ a feasible solution y to above LP of cost C^* then we will find an assignment of cost $\leq C^*$

Scenario 1 $\quad b_j = \infty$

Scenario 2: $\quad$ in-degree of each machine ≤ 3.



Suppose y is a basic feasible solution to the LP.

$y_{ij} = 0$ or $y_{ij'} = 1$ or y_{ij} is $(0,1)$

→ strictly fraction

# of strictly fractional variables

$\leq m+n.$

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graph $G = (J \cup M, E)$

M' is a subset of machines

that are active in LP.

GAP-LP(G, M')

$$\min \sum_{ij \in E} c_{ij} x_{ij}$$

$$\sum_{ij \in \delta(i)} x_{ij} = 1 \quad \forall i \in J$$

$$\sum_{ij \in \delta(j)} p_{ij} x_{ij} \leq b_j \quad \forall j \in M'$$

$$x_{ij} \geq 0 \quad ij \in E.$$

<u>Lemma</u>: Let y be a basic feasible solution to GAP-LP(G, M').
Then one of the following properties holds.

1. $y_{ij} = 0$ or $y_{ij} = 1$ for some $ij \in E$.
2. $\deg(j) \leq 1$ for some machine $j \in M'$
3. $\deg(j) = 2$ and $\sum_{ij \in \delta(j)} y_{ij} \geq 1$ for some machine $j \in M'$.

GAP-Iterated-Rounding (G)

1. $F \leftarrow \emptyset$, $M' \leftarrow M$.

2. While ($|F| < n$) do

    2.1. Obtain bfs y for GAP-LP(G, M').

    2.2 If $\exists\, ij \in E$ s.t. $y_{ij} = 0$

    $G \leftarrow G - ij$

    Else If $\exists\, ij \in E$ s.t. $y_{ij} = 1$

    $F \leftarrow F \cup \{ij\}$

    $G \leftarrow G - i$

    $b_j \leftarrow b_j - p_{ij}$

    Else if $\exists\, j \in M'$ s.t. $d(j) = 1$

    $M' \leftarrow M' - j$

    Else if $\exists\, j \in M'$ s.t. $d(j) = 2$

    and $\sum_i y_{ij} \geq 1$

    $M' \leftarrow M' - j$

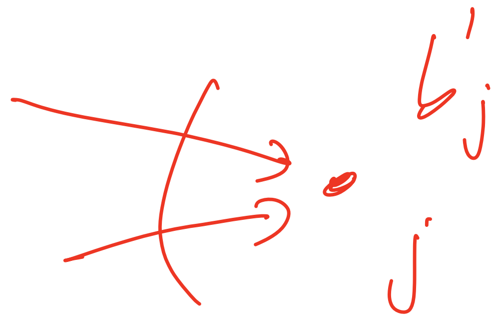
end while

3. Output assignment F.

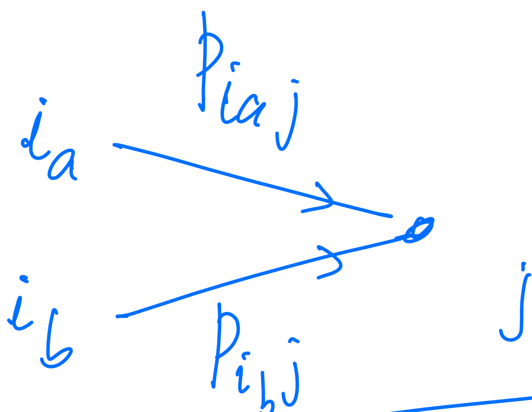
<u>Claim:</u> Algorithm terminates in $O(m+n)$ iterations and outputs a feasible assignment of jobs to machines.

<u>Claim:</u> $\text{cost}(F) \leq C^*$

<u>Claim:</u> $\text{load}(j) \leq b_j$ + one job.



$b_j - b'_j$



$p_{i_a} \le p_{i_b}$

$$\boxed{y_{i_a j} + y_{i_b j} \geq 1}$$

$$\boxed{b_j - \underline{b'_j} + p_{i_a} + p_{i_b} \le b_j + p_{i_b}}$$

$$p_{i_a j}\, y_{i_a y} + p_{i_b y}\, y_{i_b j} \leq b'_j -$$

$$\Rightarrow \quad \underline{\underline{b'_j \geq p_{i_a}}}.$$

Lemma: Let y be a basic feasible solution to GAP-LP(G, M').
Then one of the following properties holds.

1. $y_{ij} = 0$ or $y_{ij} = 1$ for some $ij \in E$.
2. $\deg(j) \le 1$ for some machine $j \in M'$
3. $\deg(j) = 2$ and $\sum_{ij \in \delta(j)} y_{ij} \geq 1$ for some machine $j \in M'$.

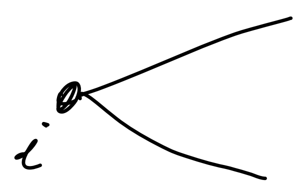
GAPLP(G, M') has $n + m'$ non-trivial constraints

Let y be a basic feasible solution

Assume that 1 and 2 don't hold.

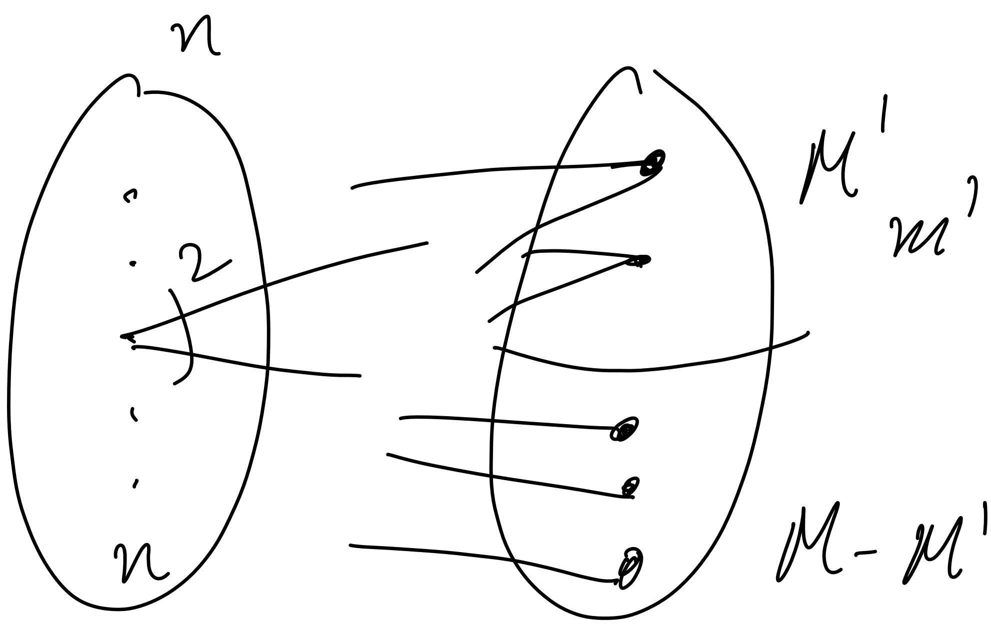
$\Rightarrow \quad y_{ij} \in (0,1) \quad \forall\, ij$ a

$\deg(i) \geq 2 \quad \forall$ jobs i



b

# of fractional variables $= n + m'$



\# $|E| = n + m'$

$deg(i) \geq 2 \quad \forall i \in [n]$

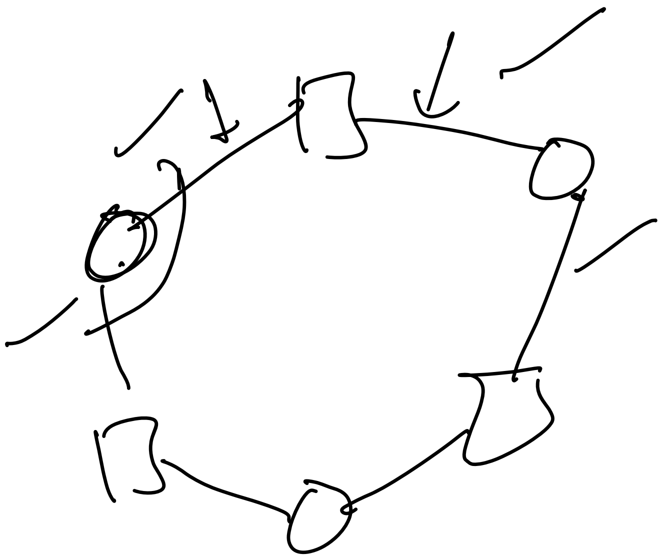
$deg(j) \geq 2 \quad \forall j \in M'$

$deg(j) \geq 0 \quad \forall j \in M - M'$

$\Rightarrow \quad deg(j) = 2 \quad \forall j \in M'$

$deg(j) = 0 \quad \forall j \in M - M'$

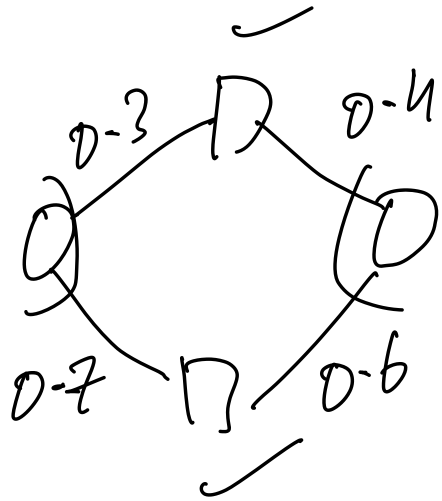
$d_G(i) = 2 \quad \forall i \in J$



$$\sum_{ij \in S} y_{ij} = \frac{|S|}{2}$$

$\sum \cancel{y}_{ij} \quad \exists j \quad s.t.$

$$\sum_{ij \in \delta(j)} y_{ij} \geq 1$$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