



Lecture 16

NE Computation, PPAD and other TFNP classes

CS580

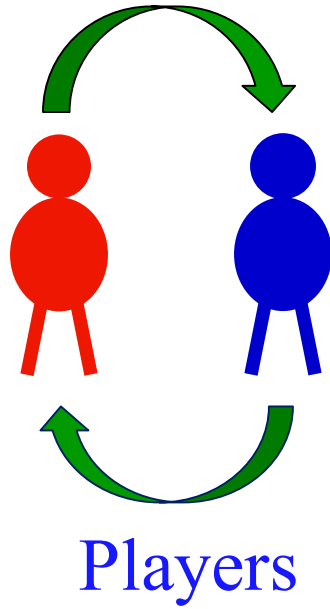
Ruta Mehta

Most slides are borrowed from Prof. C. Daskalakis's presentation.

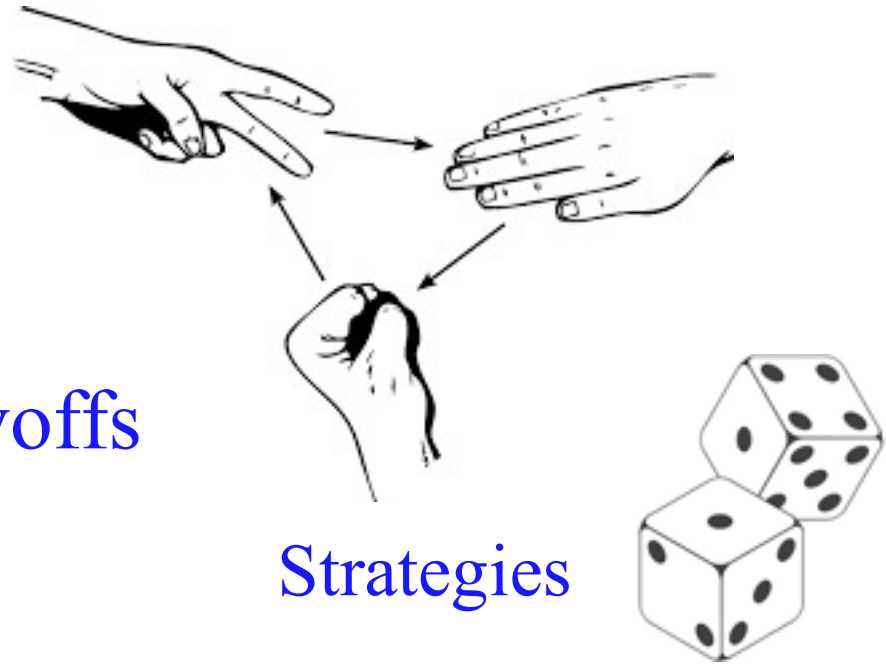
Menu

└→ Existence Theorems: **Nash**, Brouwer, Sperner

Games

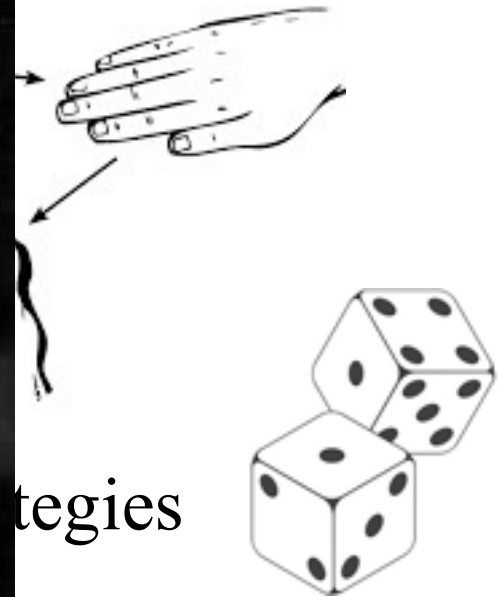
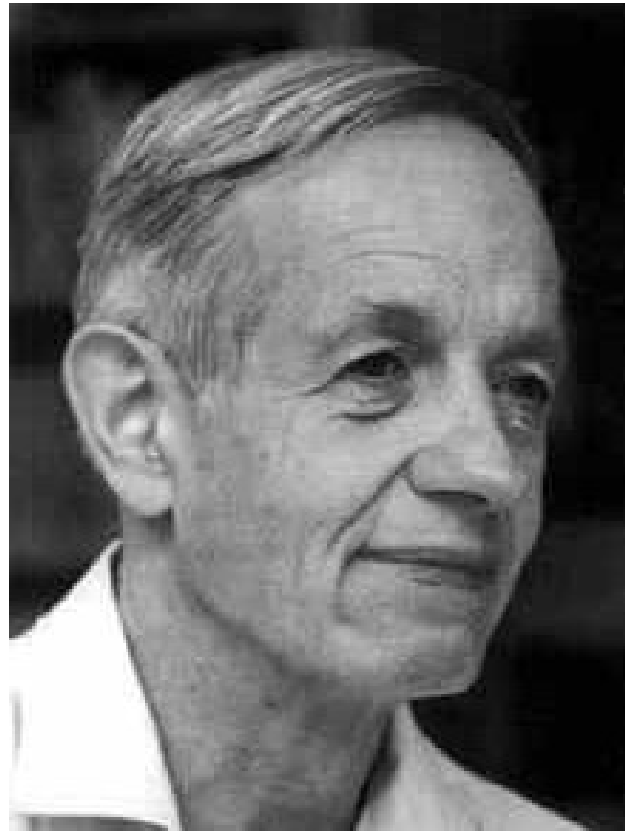
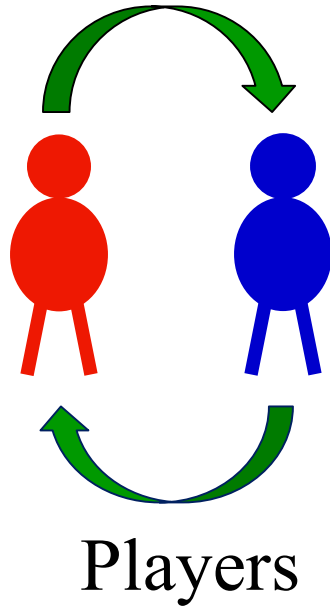


Payoffs



Randomize!

Equilibria



Randomize!

Nash (1950):

There exists a (stable) state where no player gains by unilateral deviation.

Nash equilibrium (NE)

Games and Equilibria

		2/5	3/5
	Kick Dive	Left	Right
1/2	Left	2 , -1	-1 , 1
1/2	Right	-1 , 1	1 , -1

Equilibrium:

A pair of randomized strategies so that no player has incentive to deviate if the other stays put.

[Nash '50]: *An equilibrium exists in every game.*

no poly-time algorithm known, despite intense effort

Menu

└→ Existence Theorems: Nash, **Brouwer**, Sperner

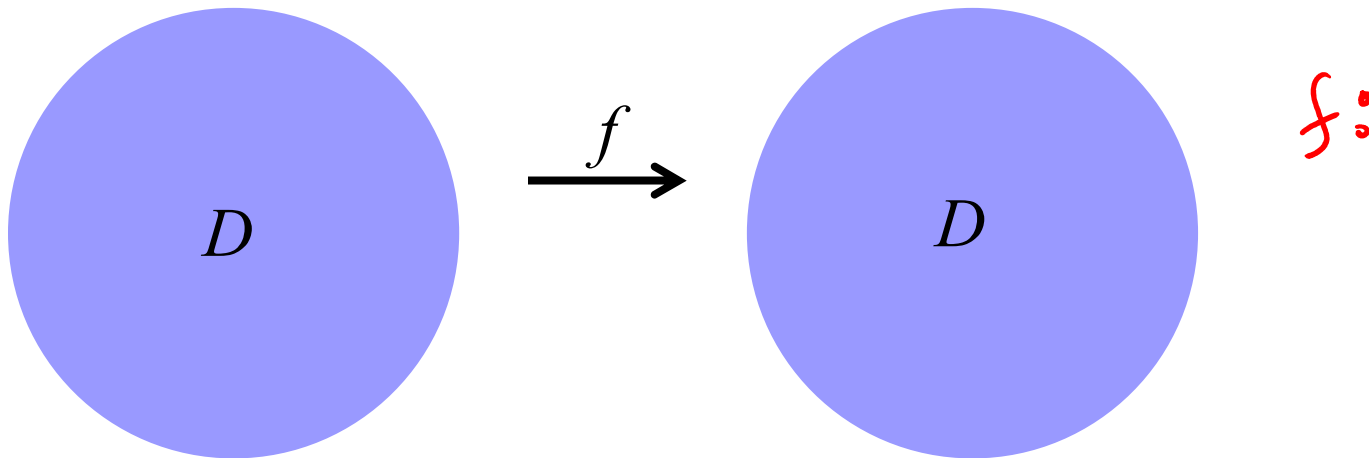
Brouwer's Fixed Point Theorem

[Brouwer 1910]: Let $f: D \rightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

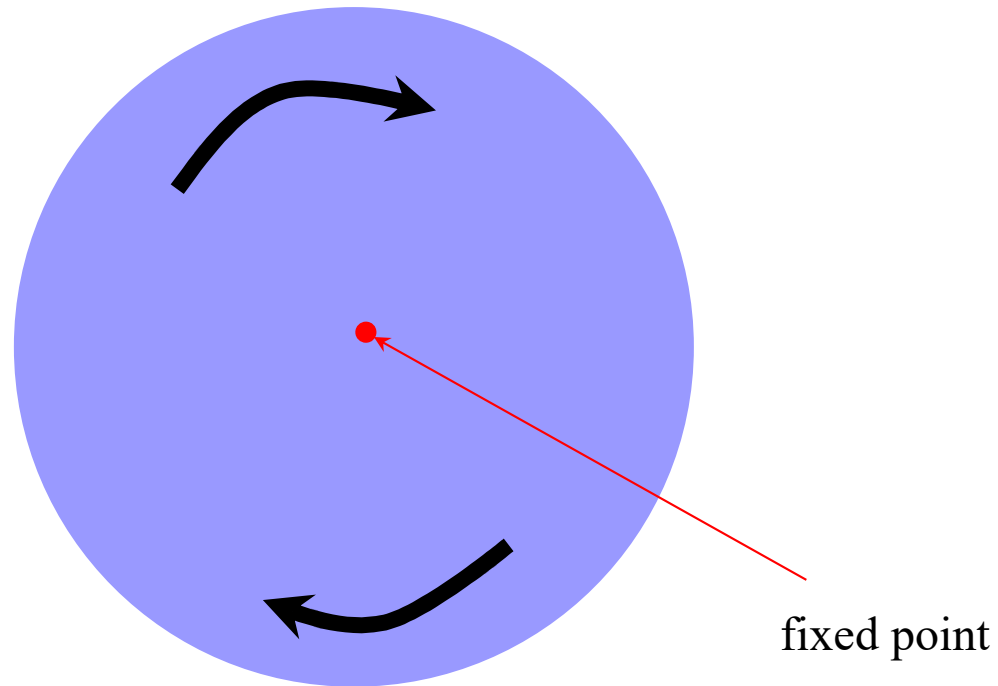
Then there exists an $x \in D$ s.t. $x = f(x)$.

closed and bounded

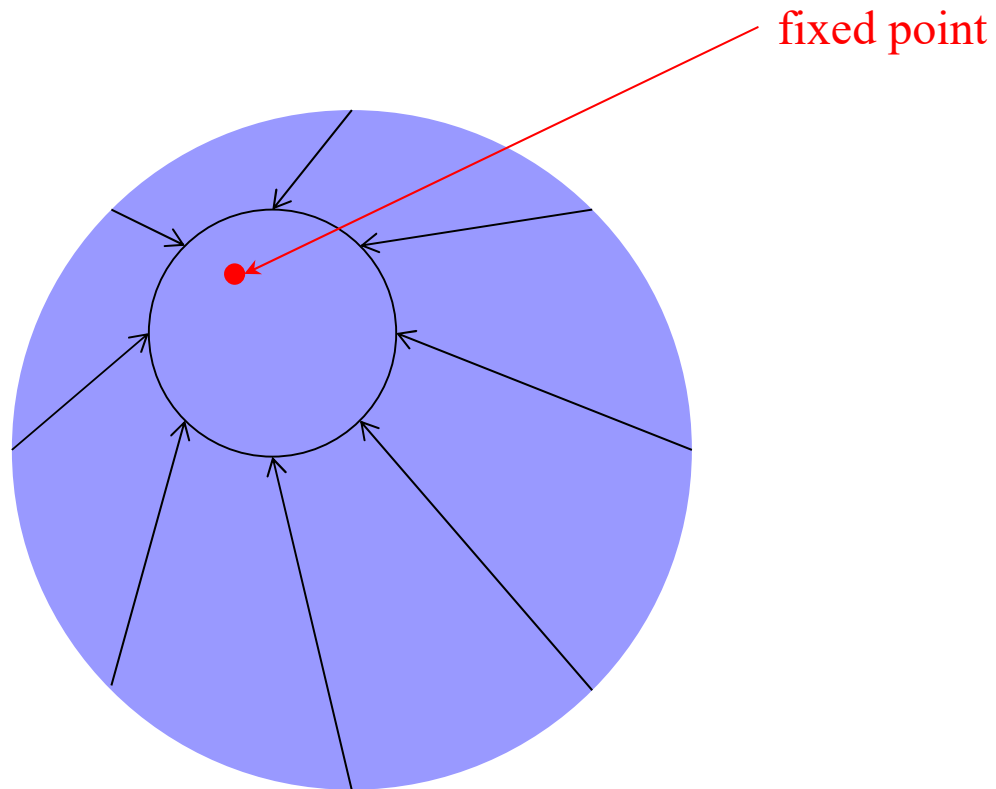
A few examples, when D is the 2-dimensional disk.



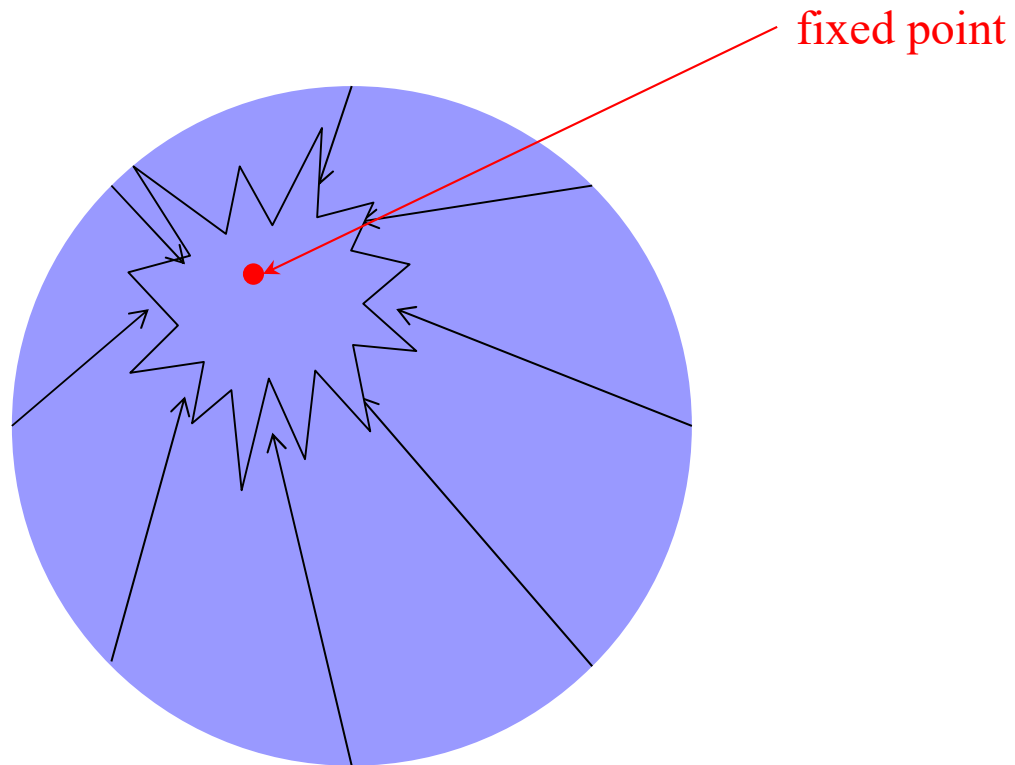
Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem





Brouwer $\Rightarrow$ Nash
(Nash'51)

Nash's Proof:

Randomized strategies of player 1

$$f: \Delta_m \times \Delta_n \rightarrow \Delta_m \times \Delta_n, \quad (x', y') = f(x, y)$$

$$\forall i, \delta_i = \max\{(Ay)_i - x^T Ay, 0\},$$

$$\forall i, \quad x'_i = \frac{x_i + \delta_i}{\sum_k (x_k + \delta_k)}$$

Lemma. If $x' = x$ then x is best for Alice against y
 $\equiv$ If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

$$\forall i, \quad \delta_i = \max\{(Ay)_i - x^T Ay, 0\},$$

$$\forall i, \quad x'_i = \frac{x_i + \delta_i}{\sum_k x_k + \delta_k}$$

argmax $z^T Ay$
 $z \in \Delta_m$

Lemma. If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

Pf:

$$k^* = \arg \max_{i=1}^m ((Ay)_i - x^T Ay)$$

$$\delta_{k^*} > 0$$

$$k^- = \arg \min_{i: x_i > 0} (Ay)_i$$

$$\delta_{k^-} = \max_{\substack{(Ay)_{k^-} - x^T Ay \leq 0 \\ \leq 0}} \{(Ay)_{k^-} - x^T Ay, 0\} = 0$$

$$\text{claim: } x'_{k^-} \neq x_{k^-}$$

$$x'_{k^-} = \frac{x_{k^-} + 0}{\left(\sum_{i=1}^m x_i + \sum_{i>0} \delta_i \right)} < x_{k^-}$$

Nash's Proof

$$f: \Delta_m \times \Delta_n \rightarrow \Delta_m \times \Delta_n, \quad (x', y') = f(x, y)$$

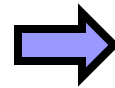
$$\forall j, \quad \tau_j = \max \left\{ (x^T B)_j - x^T B y, 0 \right\},$$

$$\forall j, \quad y'_j = \frac{y_j + \tau_j}{\sum_k y_k + \tau_k}$$

Lemma. If $y' = y$ then y is best for Bob against x
 $\equiv$ If $x^T B y < x^T B z$ for some $z \in \Delta_n$ then $y' \neq y$.

Visualizing Nash's Proof

Kick Dive	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1 , -1



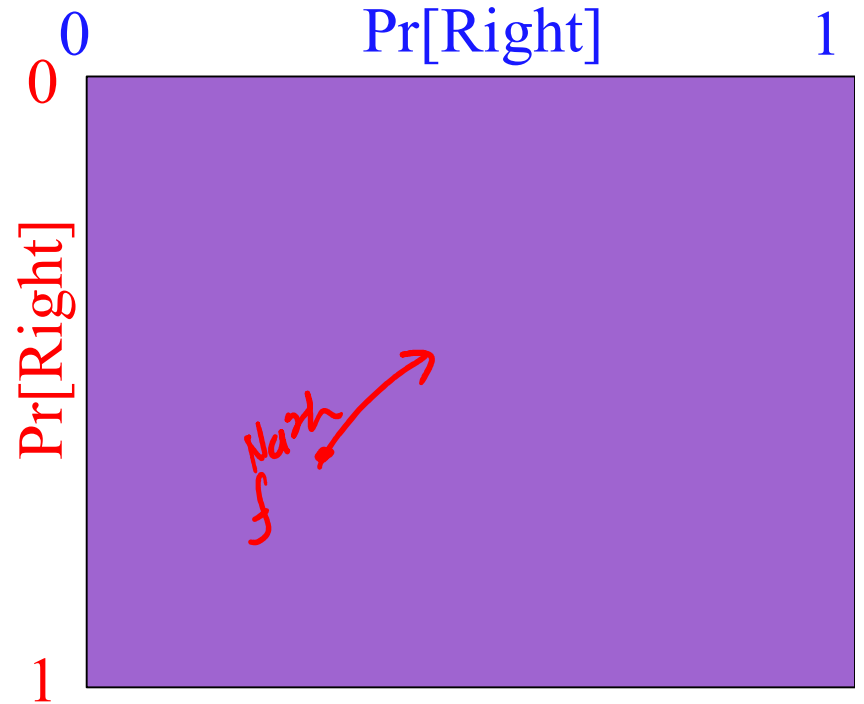
$f: [0,1]^2 \rightarrow [0,1]^2$, continuous
such that
fixed points $\equiv$ Nash eq.

Penalty Shot Game

Visualizing Nash's Proof

Kick Dive	Left	Right
	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1 , -1

Penalty Shot Game



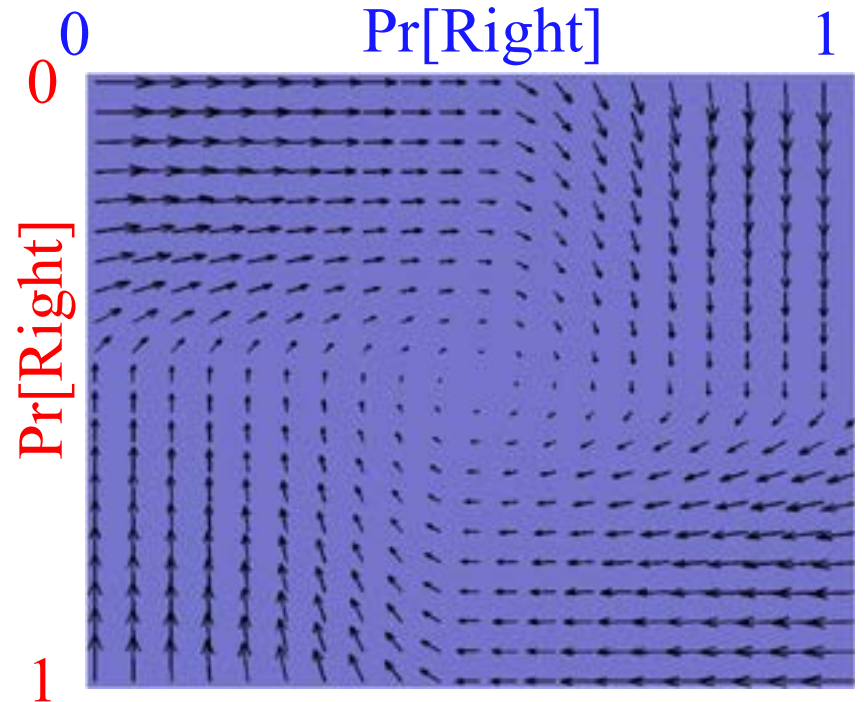
Visualizing Nash's Proof

Kick Dive	$(1-y)$ Left	y Right
$(1-x)$ Left	1 , -1	-1 , 1
x Right	-1 , 1	1 , -1

Penalty Shot Game

$$f: \Delta_2 \times \Delta_2 \rightarrow \Delta_2 \times \Delta_2$$

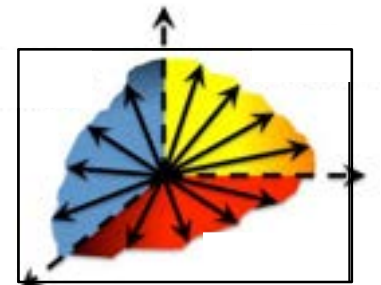
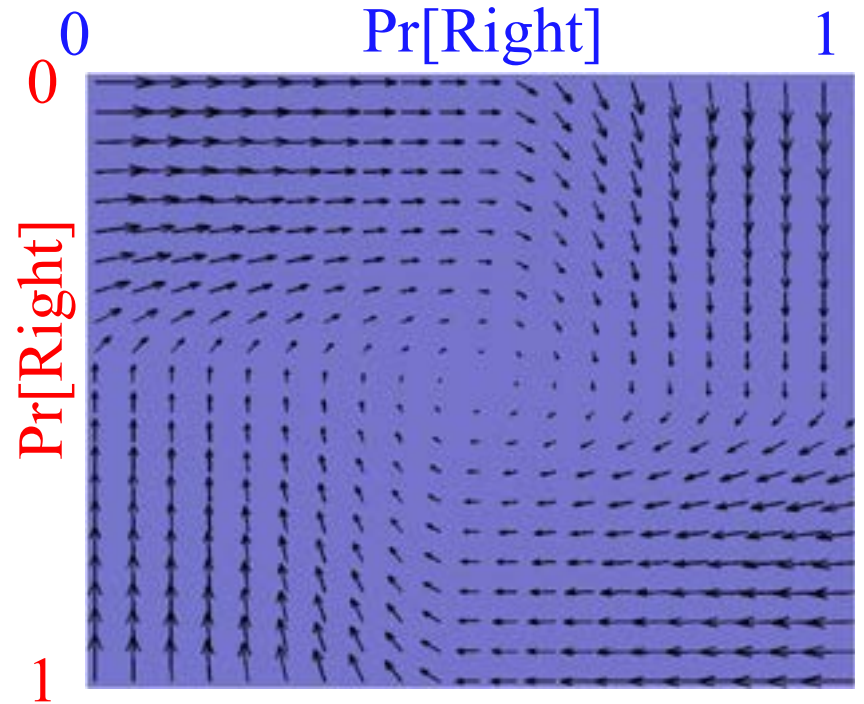
$$f: [0,1]^2 \rightarrow [0,1]^2$$



Visualizing Nash's Proof

Dive \ Kick	Left	Right
	Left	Right
Left	1 , -1	-1 , 1
Right	-1 , 1	1 , -1

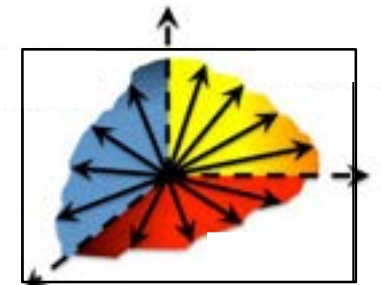
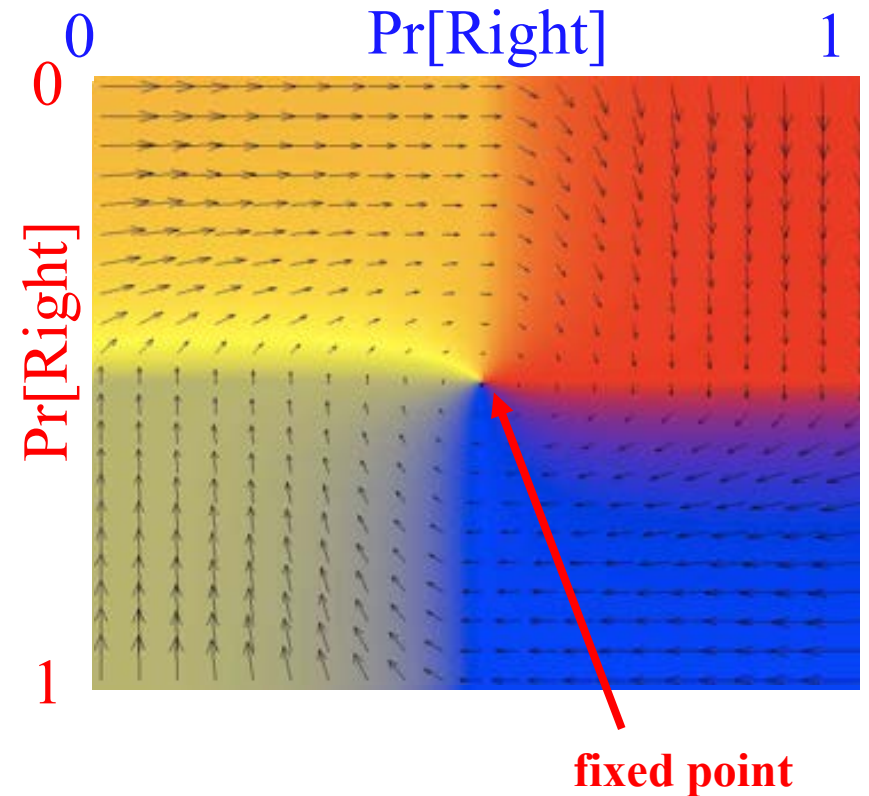
Penalty Shot Game



Visualizing Nash's Proof

		$\frac{1}{2}$	$\frac{1}{2}$
		Left	Right
$\frac{1}{2}$	Kick Dive		
$\frac{1}{2}$	Left	1 , -1	-1 , 1
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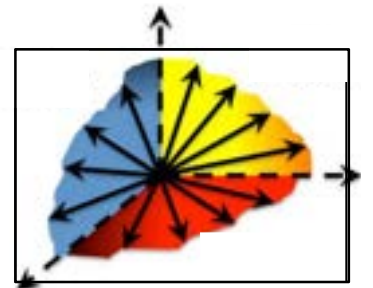
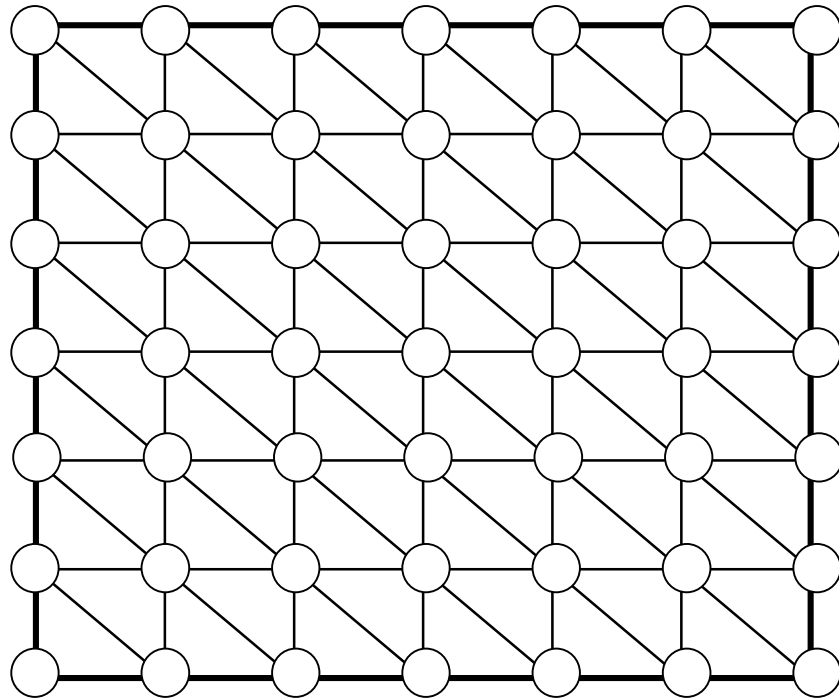
Penalty Shot Game



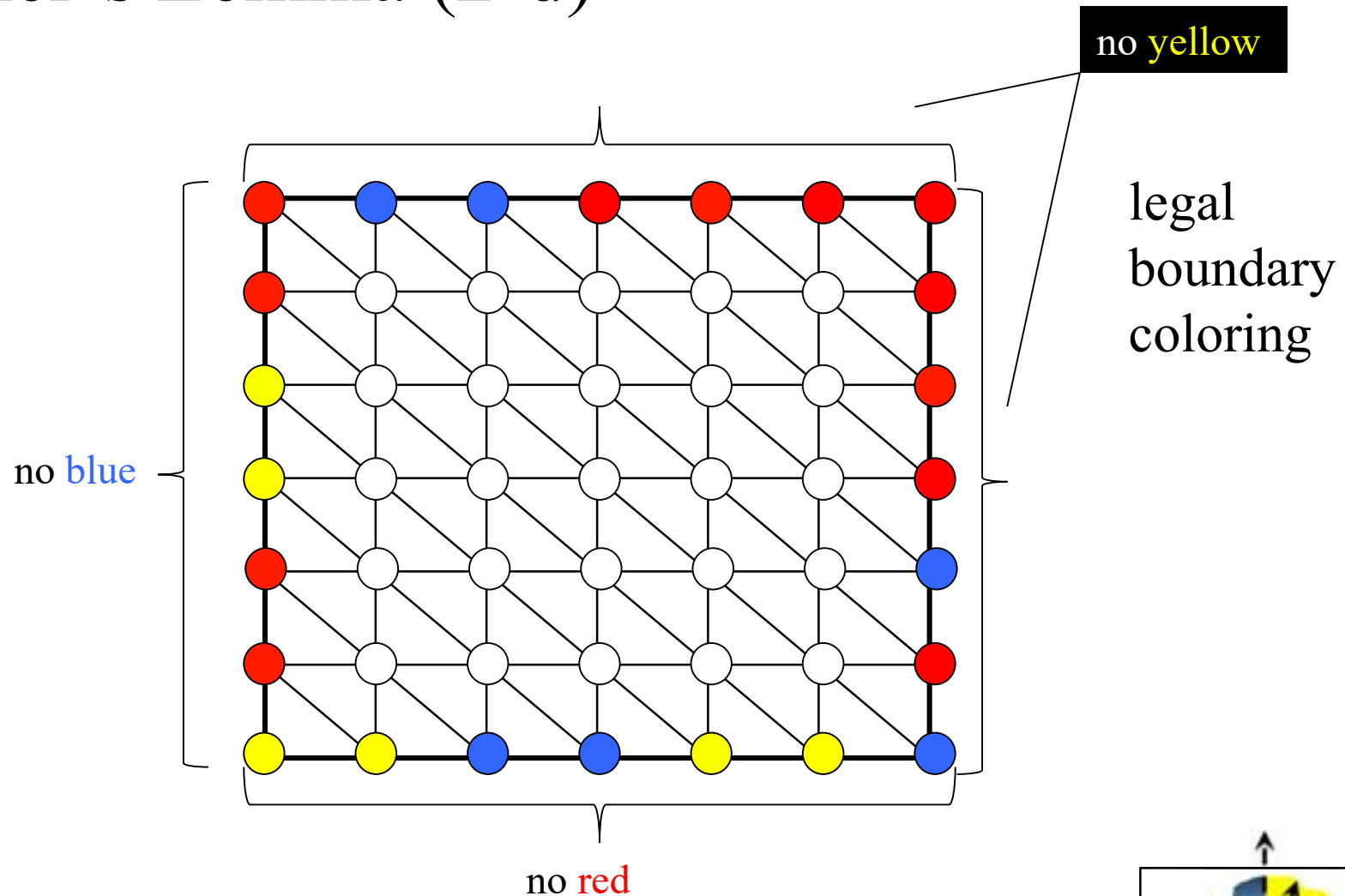
Menu

└→ Existence Theorems: Nash, Brouwer, **Sperner**

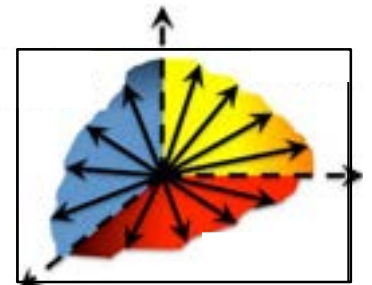
Sperner's Lemma (2-d)



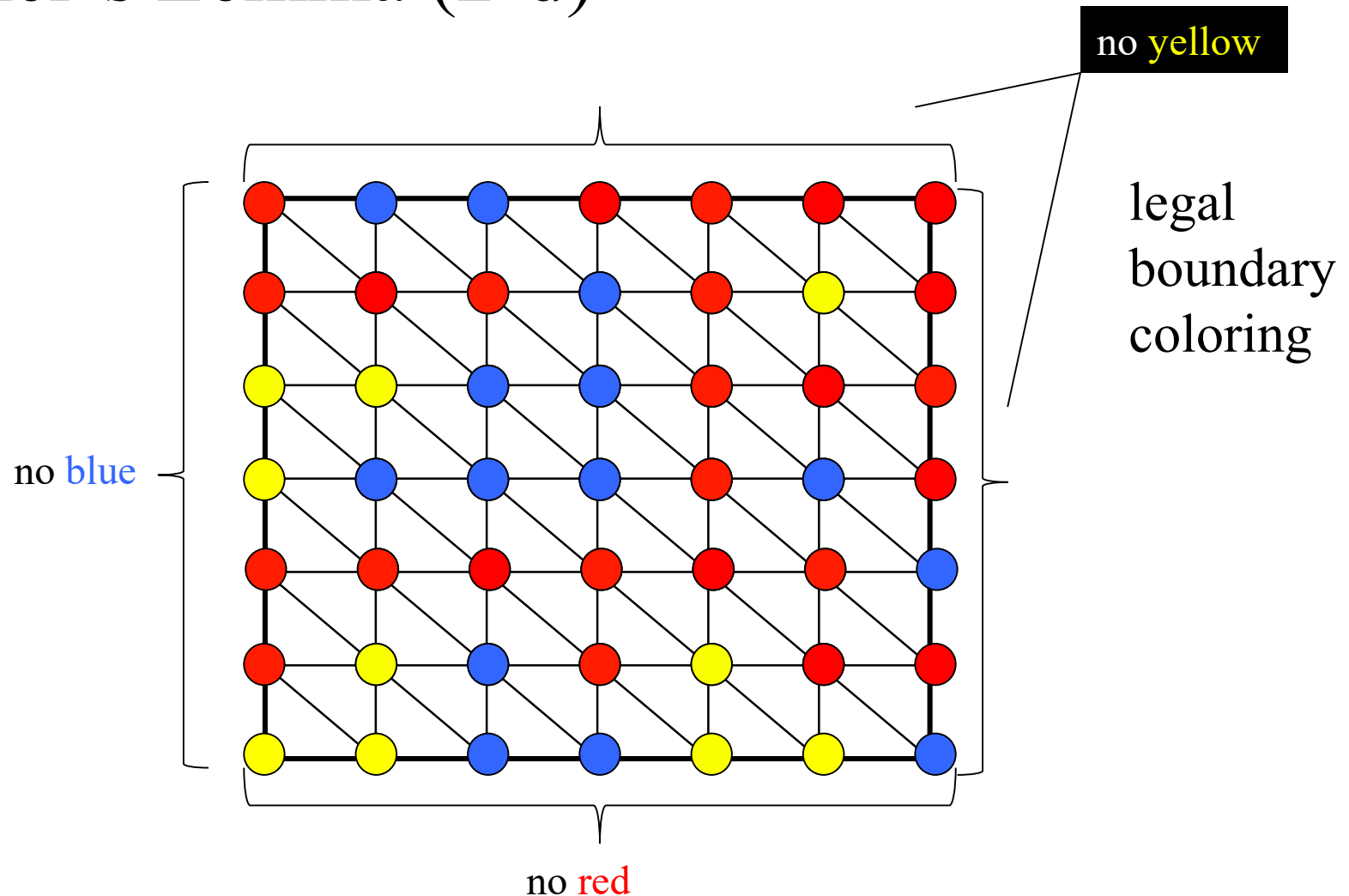
Sperner's Lemma (2-d)



[Sperner 1928]: Color the boundary using three colors in a legal way.

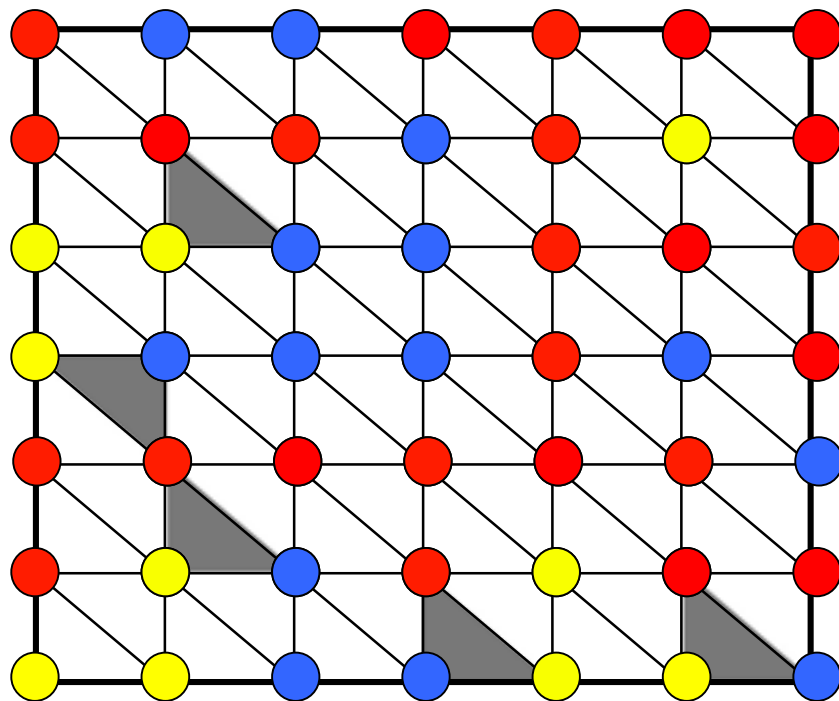


Sperner's Lemma (2-d)



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Sperner's Lemma (2-d)



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Sperner $\Rightarrow$ Brouwer

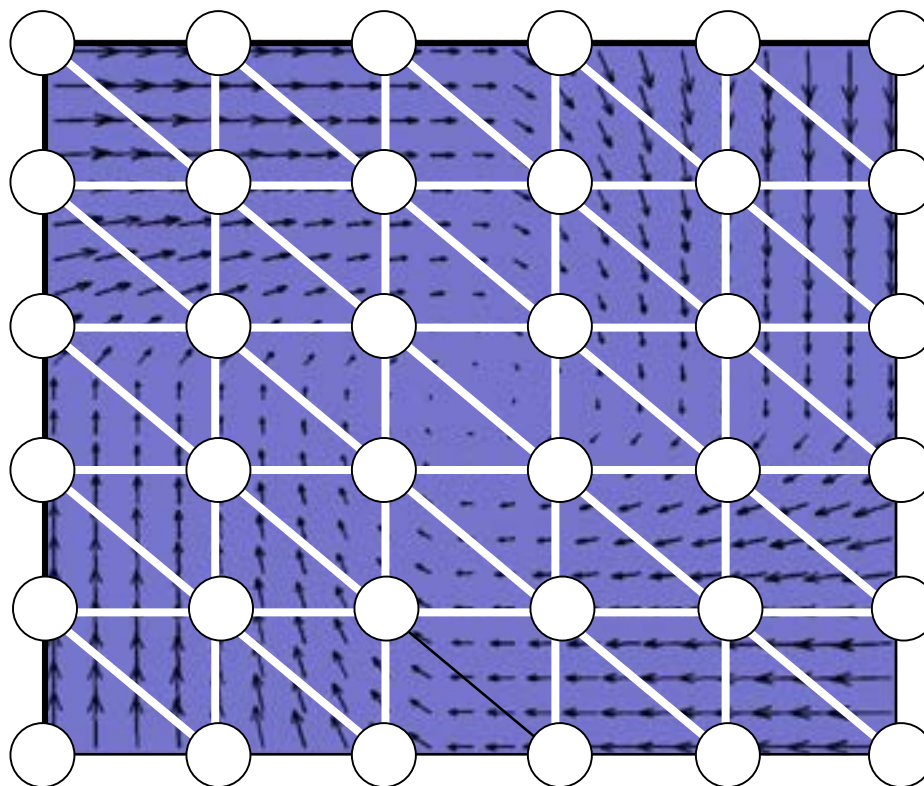
Sperner $\Rightarrow$ Brouwer (High-Level)

Given $f: [0,1]^2 \rightarrow [0,1]^2$

1) For all $\epsilon > 0$, existence of approximate fixed point $|f(x)-x| < \epsilon$,
can be shown via Sperner's lemma.

2) Then let $\epsilon \rightarrow 0$

For 1): Triangulate $[0,1]^2$;



Sperner $\Rightarrow$ Brouwer (High-Level)

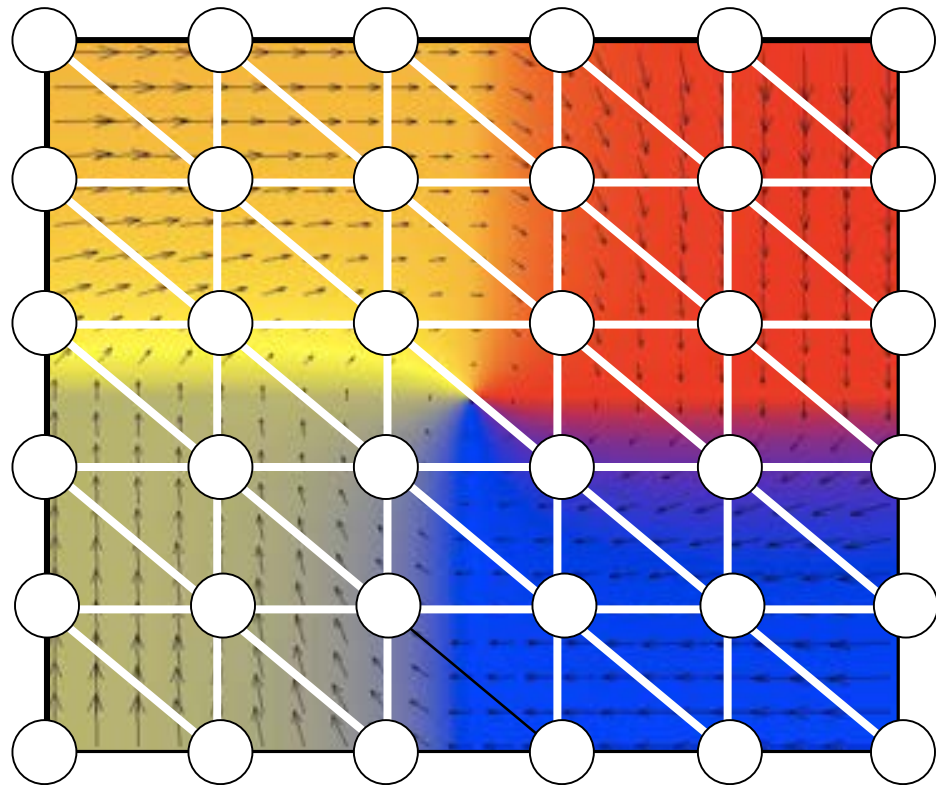
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For 1): Triangulate $[0,1]^2$;

Color points according
to the direction of $(f(x)-x)$;



Sperner $\Rightarrow$ Brouwer (High-Level)

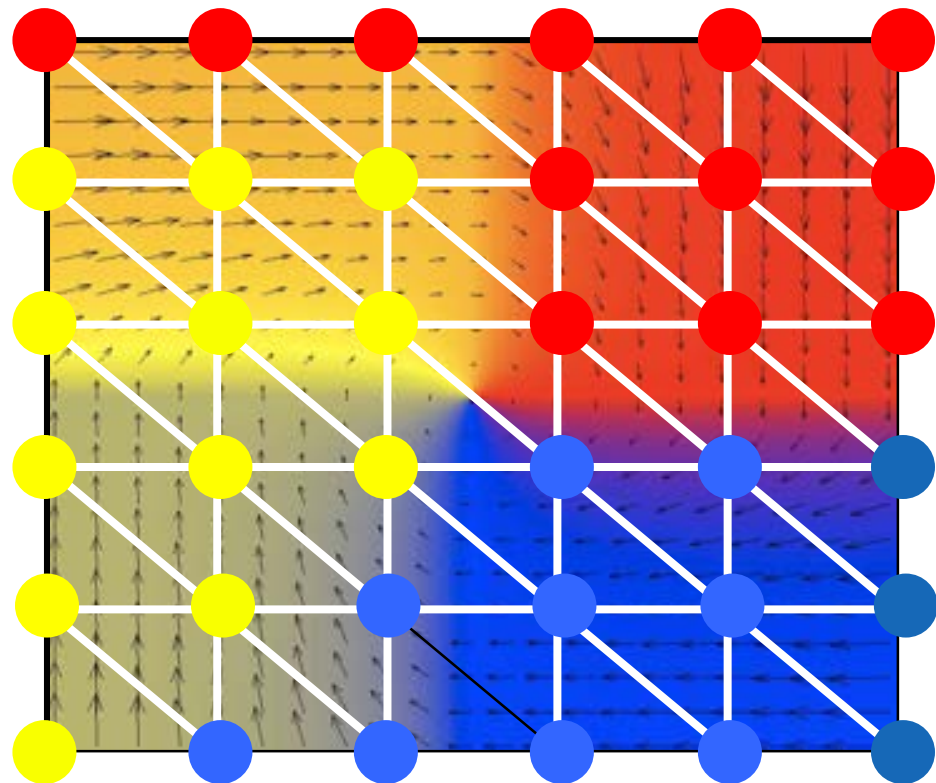
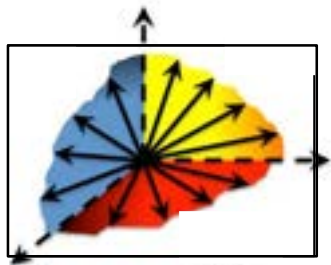
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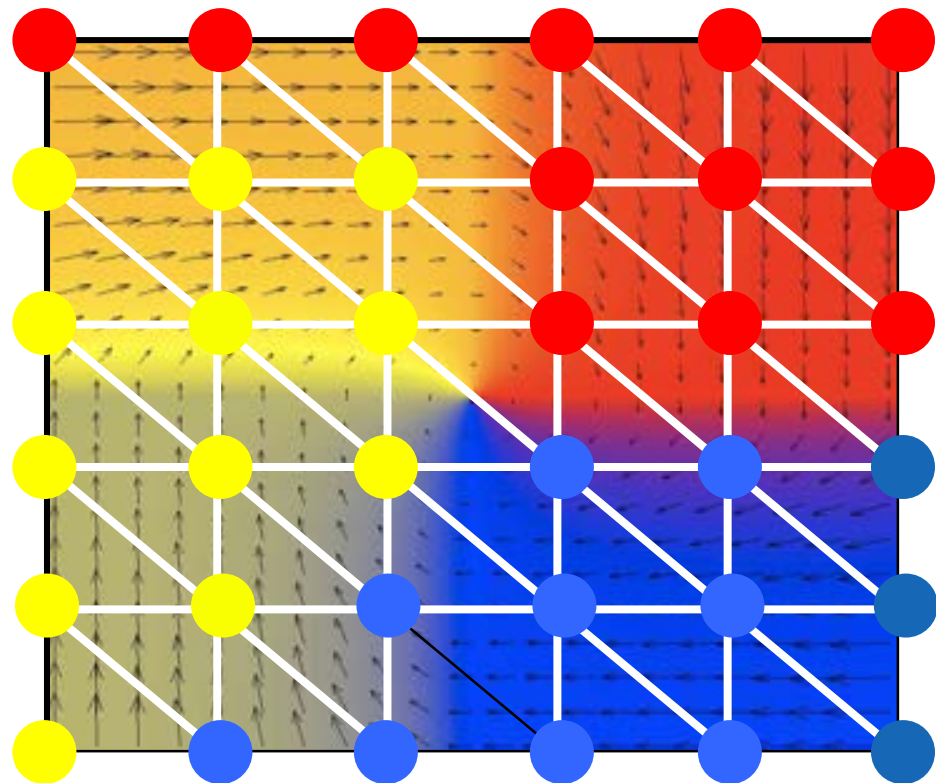
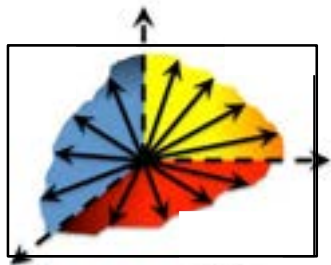
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Color points according
to the direction of $(f(x)-x)$;

Apply Sperner.



2D-Brouwer on the Square

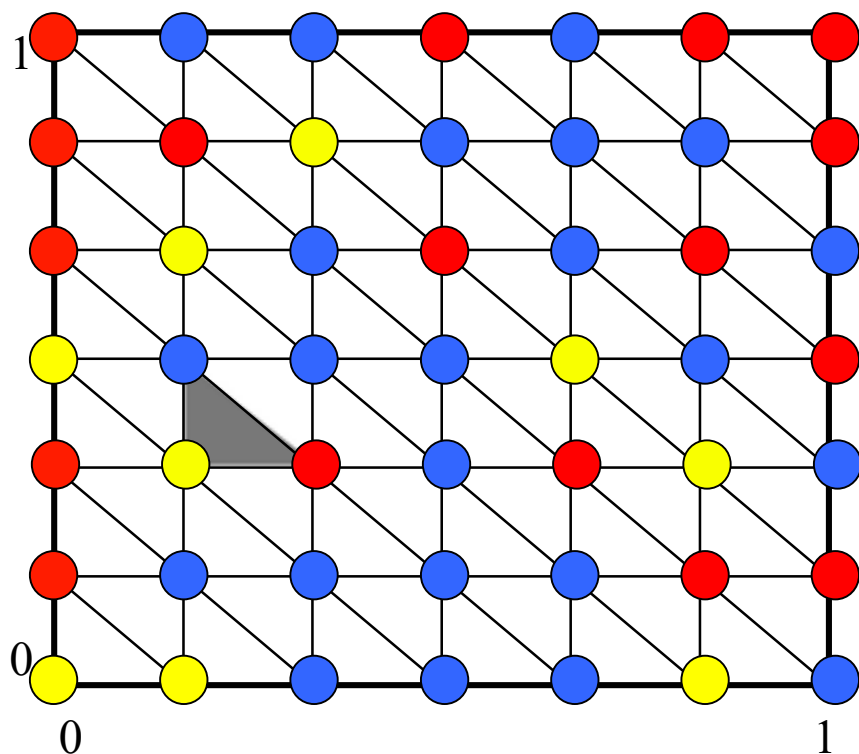
d be l_∞ norm

Suppose $f: [0,1]^2 \rightarrow [0,1]^2$, continuous

$\hookrightarrow \forall \epsilon, \exists \delta(\epsilon) > 0, s. t. \quad$ (by the [Heine-Cantor theorem](#))
 $d(x, y) < \delta(\epsilon) \Rightarrow d(f(x), f(y)) < \epsilon$



Choose small enough grid size so that..



Claim: If z a corner of a trichromatic triangle, then

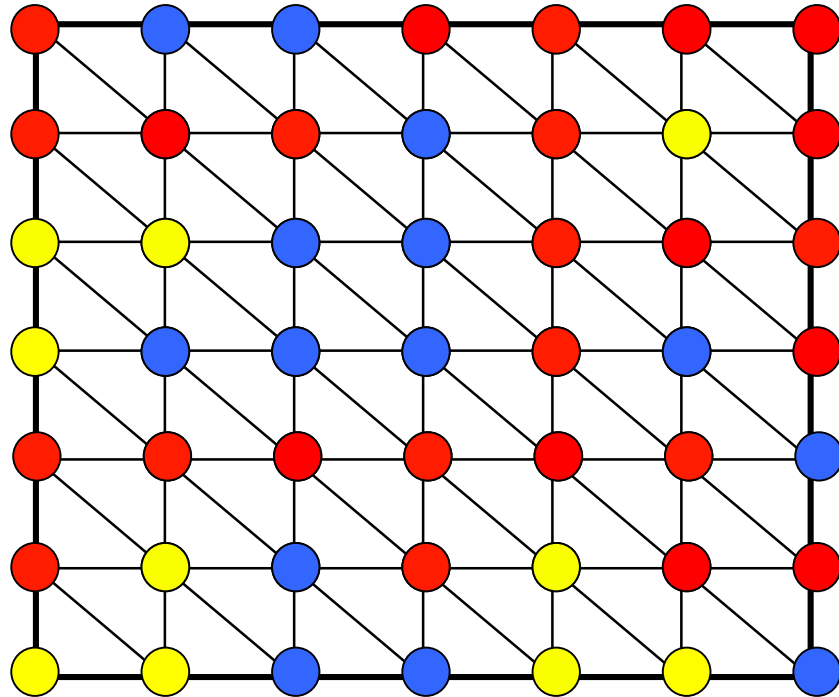
Choosing $\delta = \min\{\delta(\epsilon), \epsilon\}$

$$|f(z) - z|_\infty < c\delta, \quad c > 0$$

Menu

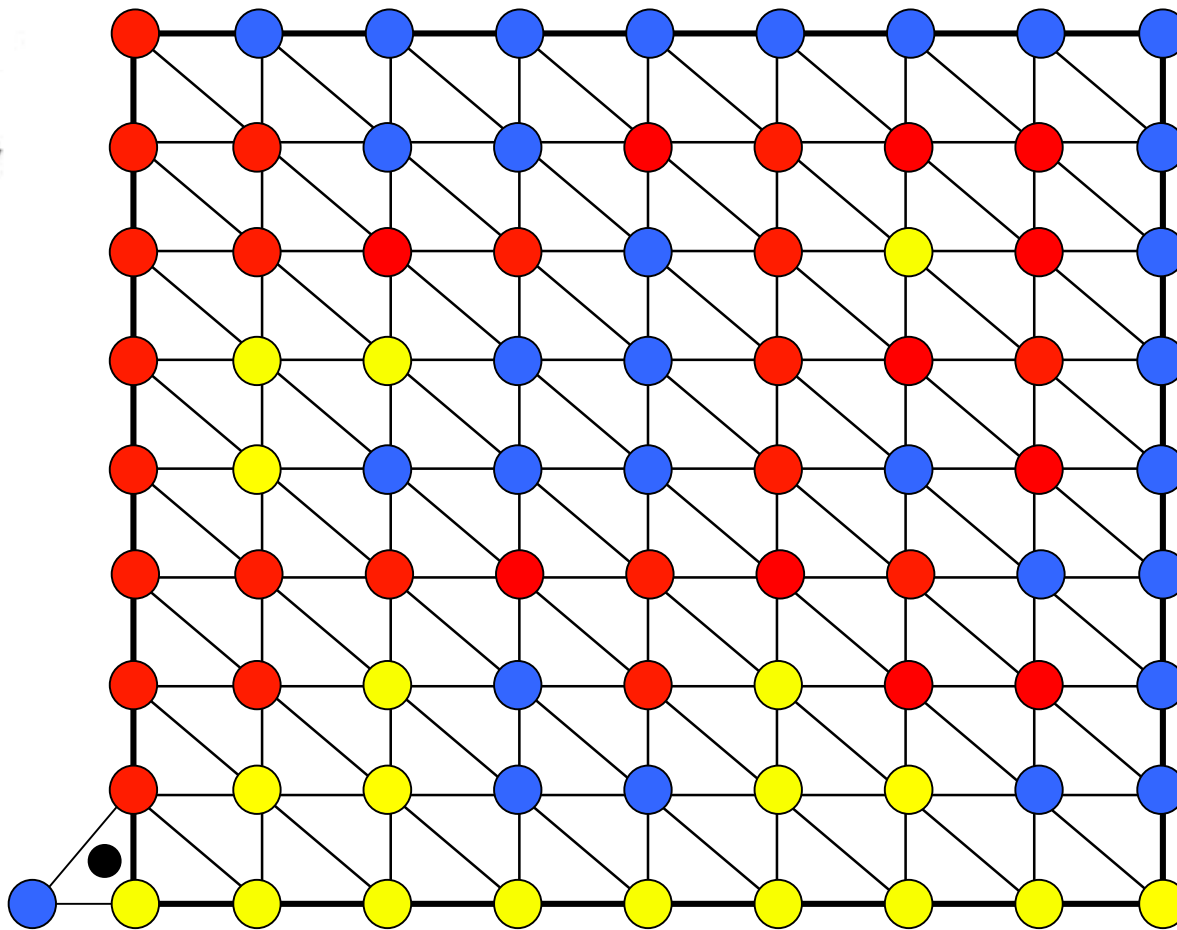
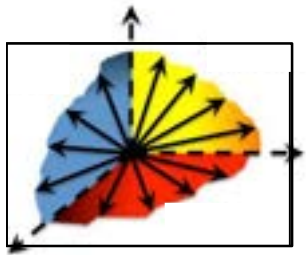
- Existence Theorems: Nash, Brouwer, Sperner
- (Constructive) proof of Sperner $\rightarrow$ PPAD.

Proof of Sperner's Lemma



[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof of Sperner's Lemma



For convenience introduce an outer boundary, that does not create new tri-chromatic triangles.

Also introduce an artificial tri-chromatic triangle.

Next define a directed walk starting from the artificial tri-chromatic triangle.

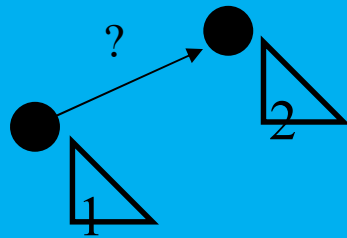
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Proof of Sperner's Lemma

Space of Triangles

Transition Rule:

*If $\exists$ **red** - **yellow** door cross it with **red** on your left hand.*



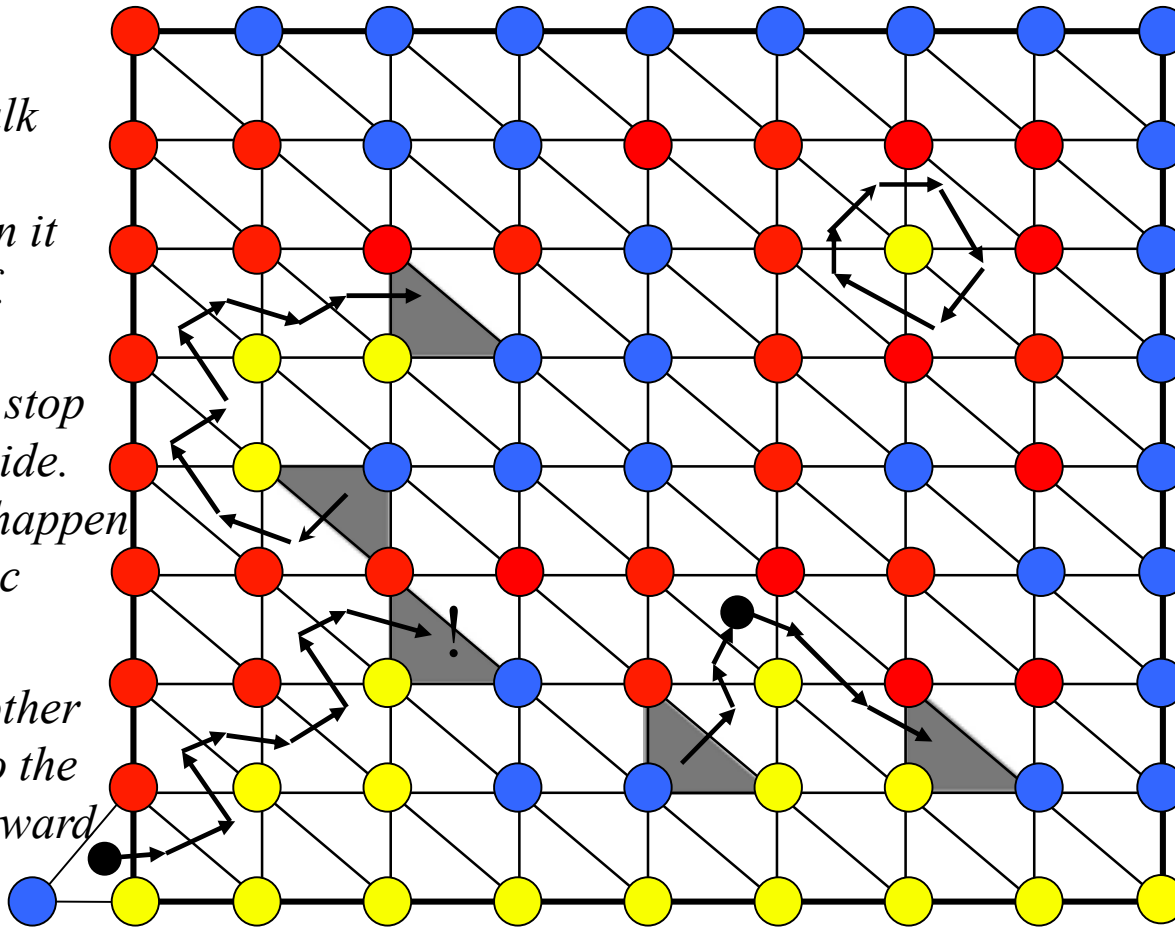
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Proof of Sperner's Lemma

Claim: *The walk cannot exit the square, nor can it loop into itself.*

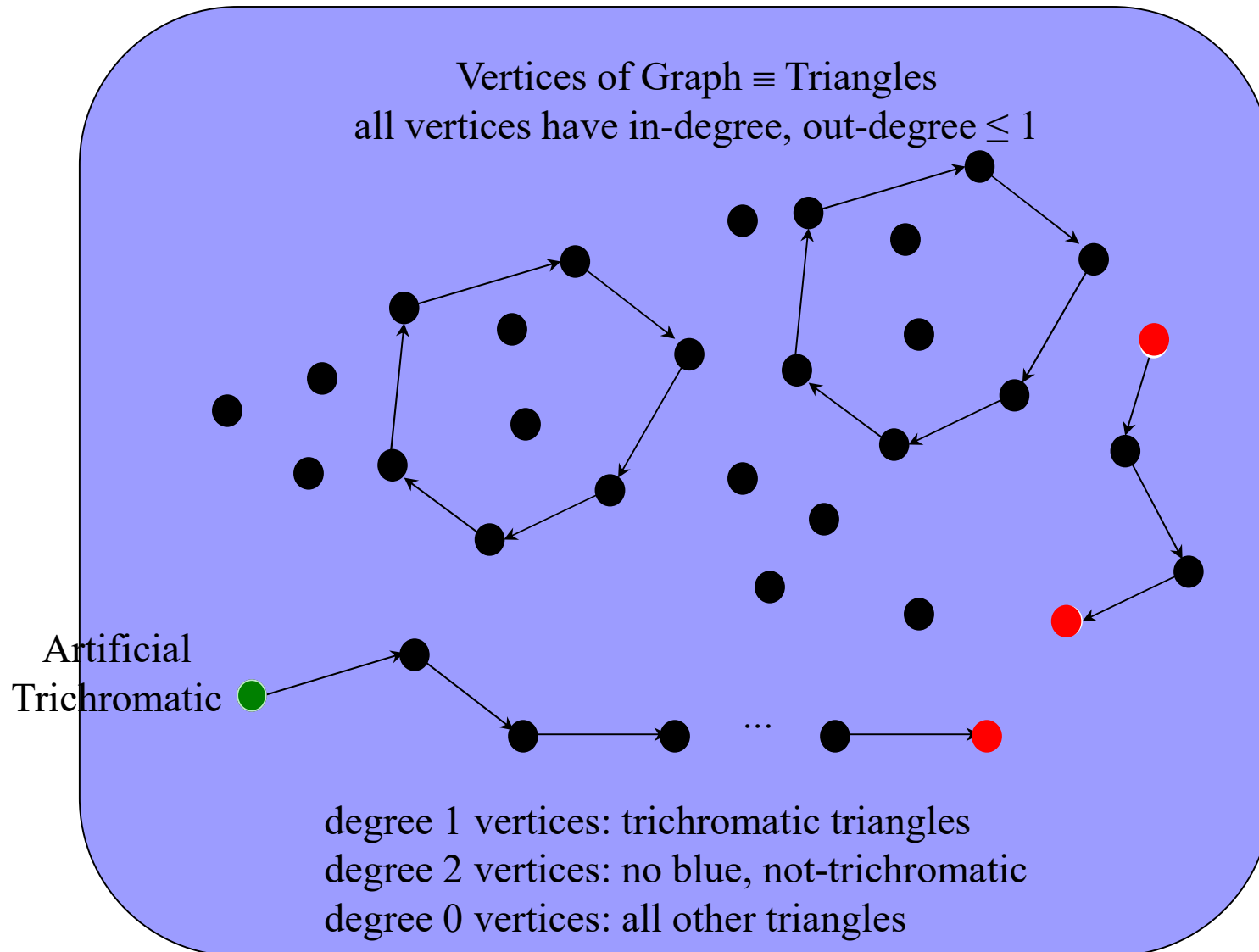
Hence, it must stop somewhere inside. This can only happen at tri-chromatic triangle...

Starting from other triangles we do the same going forward or backward.



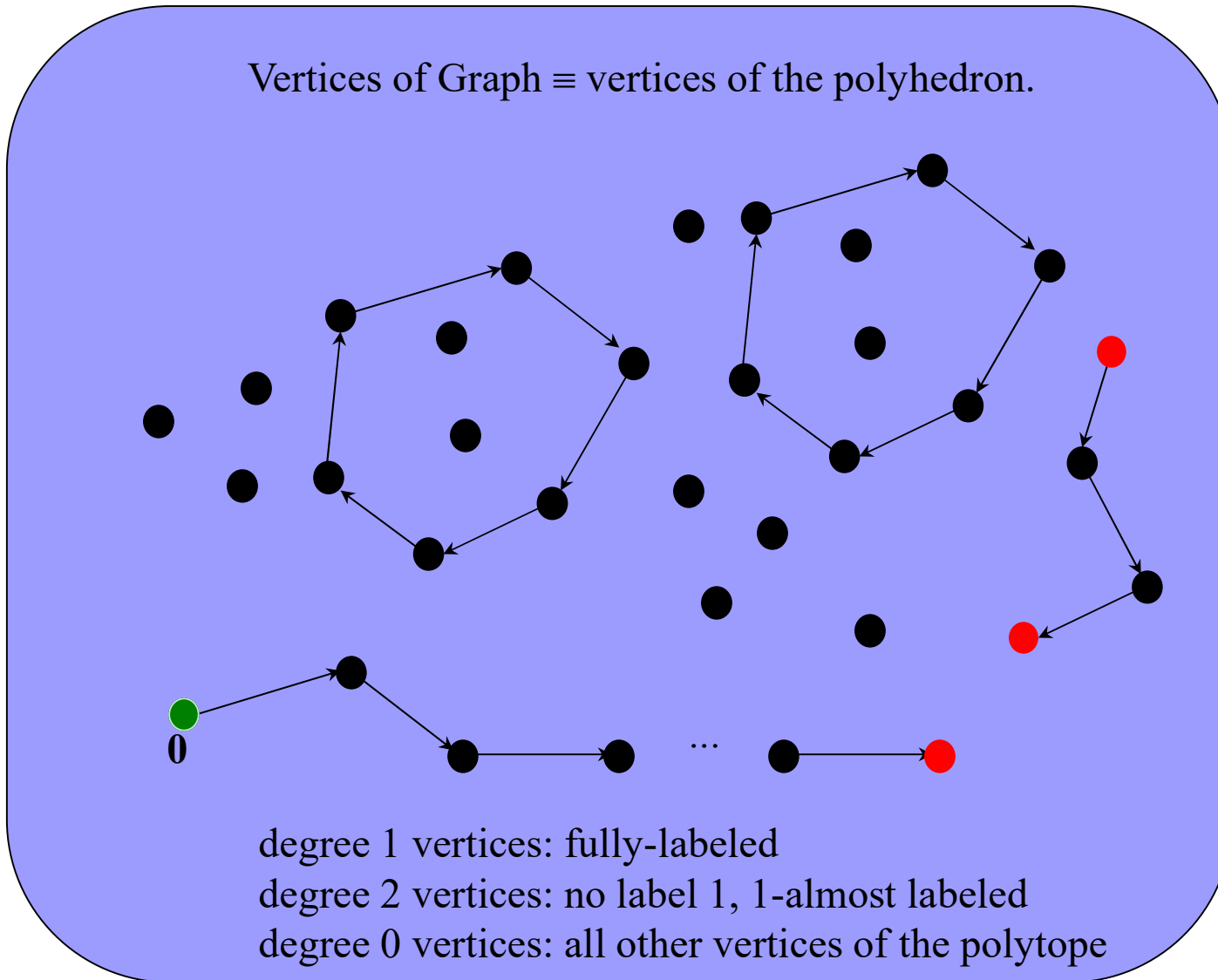
[Sperner 1928]: Color the boundary using three colors in a legal way. No matter how the internal nodes are colored, there exists a tri-chromatic triangle. In fact an odd number of those.

Proof Structure: A directed parity argument



Proof: $\exists$ at least one trichromatic (artificial one) $\rightarrow \exists$ another trichromatic

Recall: Lemke-Howson Structure for 2-Nash

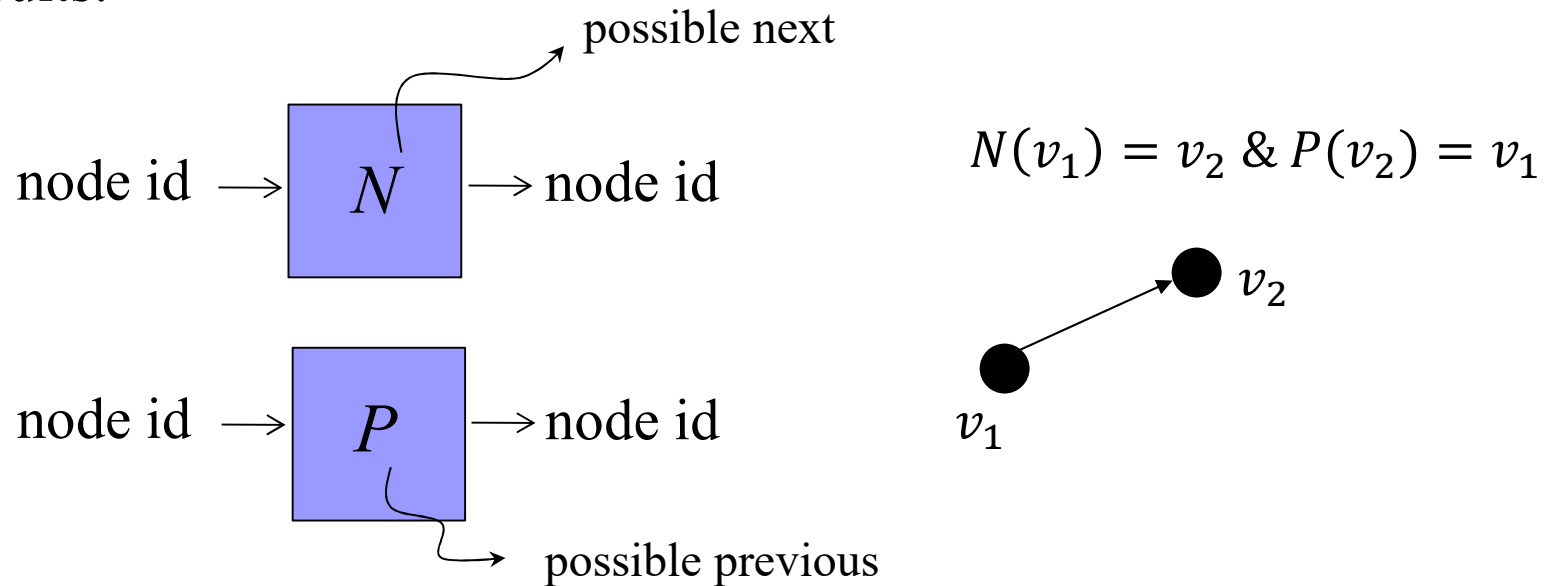


Proof: 0 fully-labeled $\Rightarrow \exists$ another fully-labeled

The PPAD Class [Papadimitriou '94]

(Polynomial Parity Argument for Directed Graph)

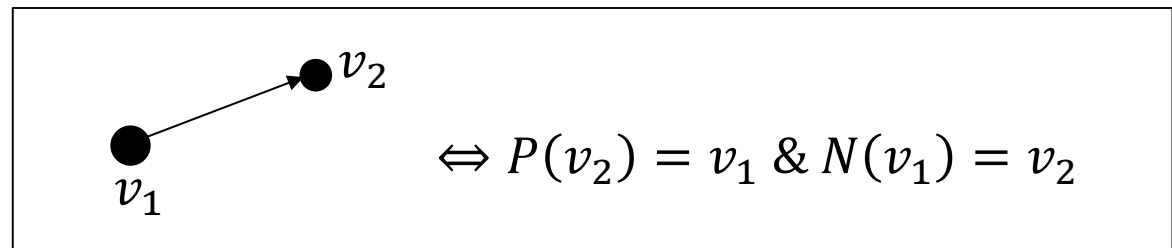
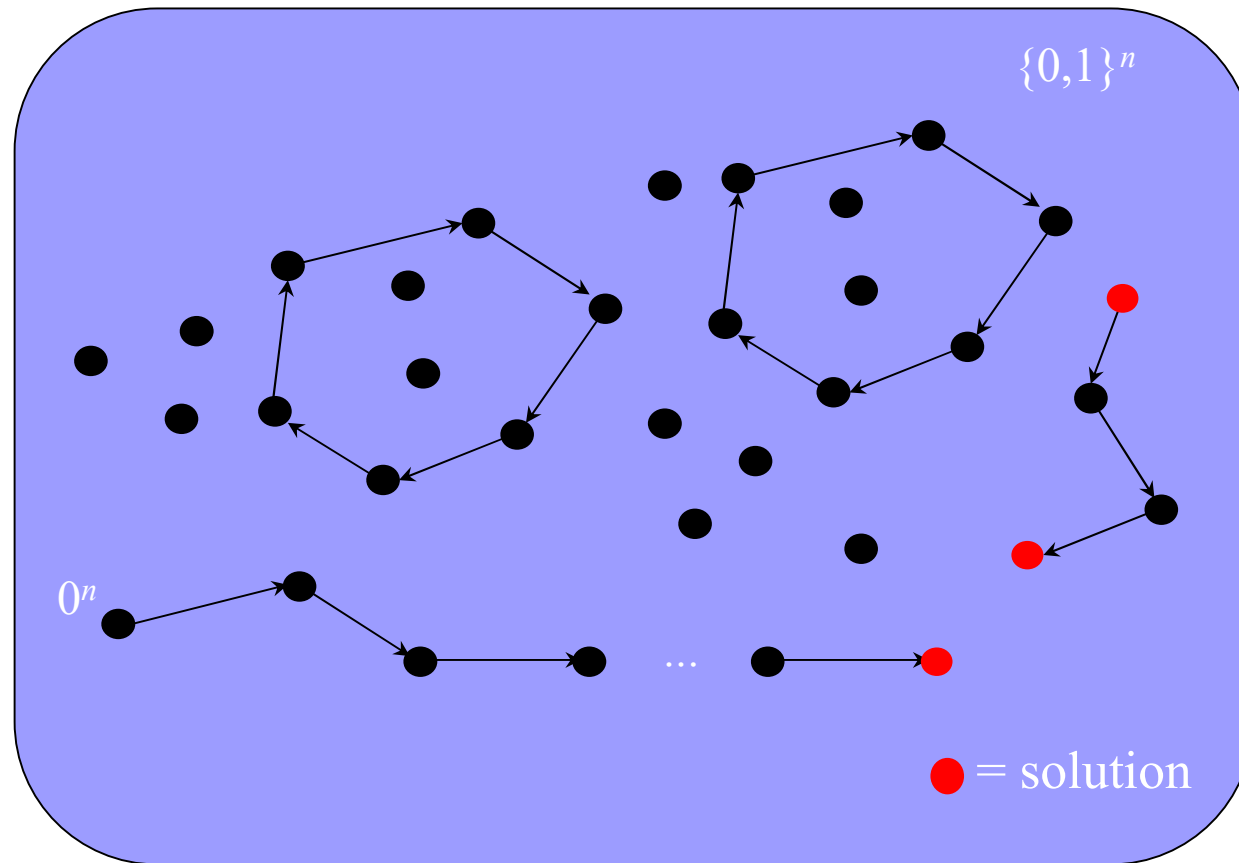
Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by two circuits:



END OF THE LINE: P and N are given. If 0^n is an unbalanced node, find another unbalanced node. Otherwise output 0^n .

PPAD = $\{ \text{Problems reducible to END OF A LINE} \}$

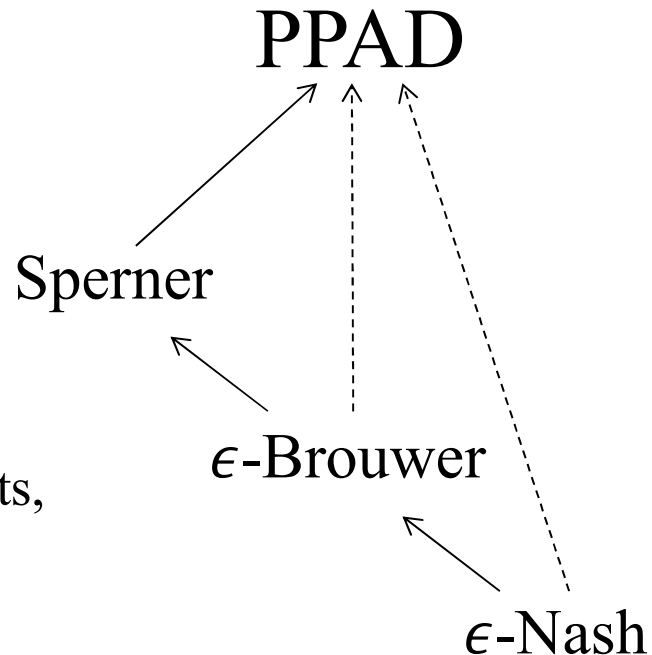
END OF A LINE



[Papadimitriou '94]

PPAD-complete:

Nash eq. (even 2-player games),
Market eq., Sperner, Brouwer,
win-lose games, sparse games,
competitive eq. with equal income,
clearing payments in financial markets,
Fractional hypergraph matching,
Fractional stable path problem, ...



ϵ -Brouwer: Given $f: D \rightarrow D$, find $x \in D$, s. t. $|f(x) - x| < \epsilon$

ϵ -Nash: Profile from which no player can deviate and gains by more than ϵ .

∴ Exact could be irrational

Menu

- Existence Theorems: Nash, Brouwer, Sperner
- (Constructive) proof of Sperner, and PPAD
- Why not use NP?
Total Search problems.

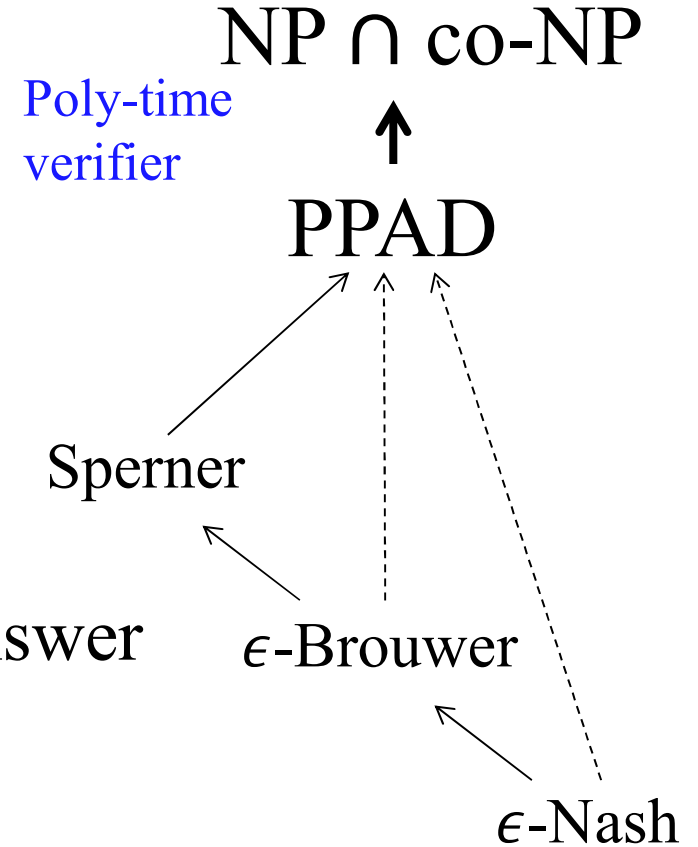
NP, co-NP vs PPAD

Can they be NP-hard? **NO!**

If there exists a solution?

- NP: poly-time verifier for YES answer
- co-NP: poly-time verifier for NO answer

- Here the answer is always YES!
 - The problem is to find a solution.



Function NP (FNP): Search problems

Either find a solution or say there is none!

Problem $L \in \text{FNP}$ has a poly-time verifier A_L s.t. $A_L(x, y) = 1$ if y is a soln. of $x \in L$

It is *poly-time (Karp) reducible* to another problem $L' \in \text{FNP}$, associated with $A_{L'}$, iff there exist poly-time functions f, g such that

(i) $f: \{0,1\}^* \rightarrow \{0,1\}^*$ maps inputs x to L into inputs $f(x)$ to L'

(ii) $\forall x, y: A_{L'}(f(x), y) = 1 \Rightarrow A_L(x, g(y)) = 1$
 $\forall x: A_{L'}(f(x), y) = 0, \forall y \Rightarrow A_L(x, y) = 0, \forall y$

can't reduce SAT to
SPERNER, NASH
or BROUWER

A search problem L' is *FNP-complete* iff

$L' \in \text{FNP}$

$\forall L \in \text{FNP}, L$ is poly-time reducible to L' .

e.g. SAT

SPERNER, NASH, BROUWER $\in \text{FNP}$.

Total Function NP

Function NP (FNP): Search problems

Either find a solution or say there is none!

Total FNP: A search problem is called *total* iff a solution is guaranteed

$$\text{PPAD} \subseteq \text{TFNP} \subseteq (\text{NP} \cap \text{co-NP})$$

Complexity Theory of TFNP:

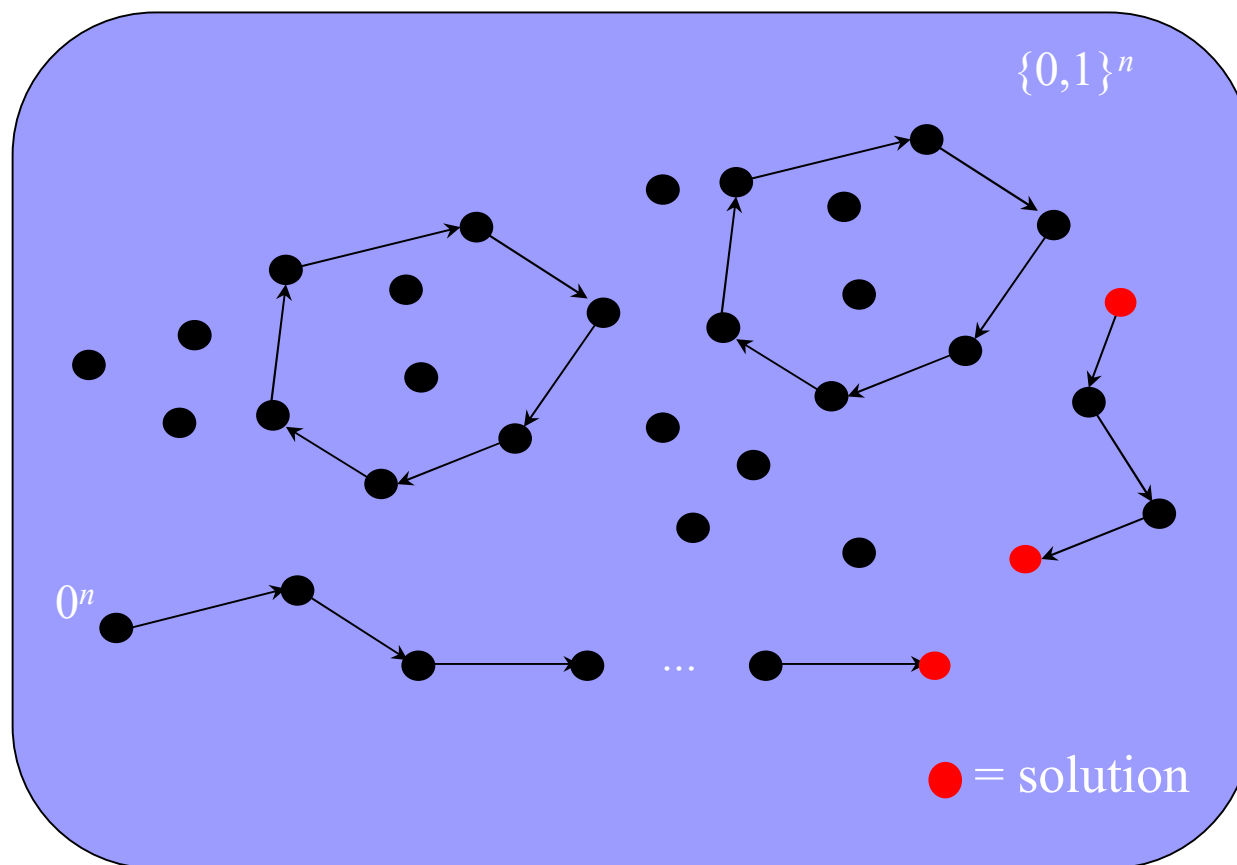
1. identify the combinatorial argument of existence, responsible for making these problems total;
2. define a complexity class inspired by the argument of existence;
3. make sure that the complexity of the problem was captured as tightly as possible (via completeness results).

PPAD:

In & out degree ≤ 1

0^n with in=0, out=1

$\exists$ another such node



Other arguments of existence, and resulting complexity classes

“If an undirected graph has a node of odd degree, then it must have another.”

PPA

“Every directed acyclic graph must have a sink.”

PLS

“If a function maps n elements to $n-1$ elements, then there is a collision.”

PPP

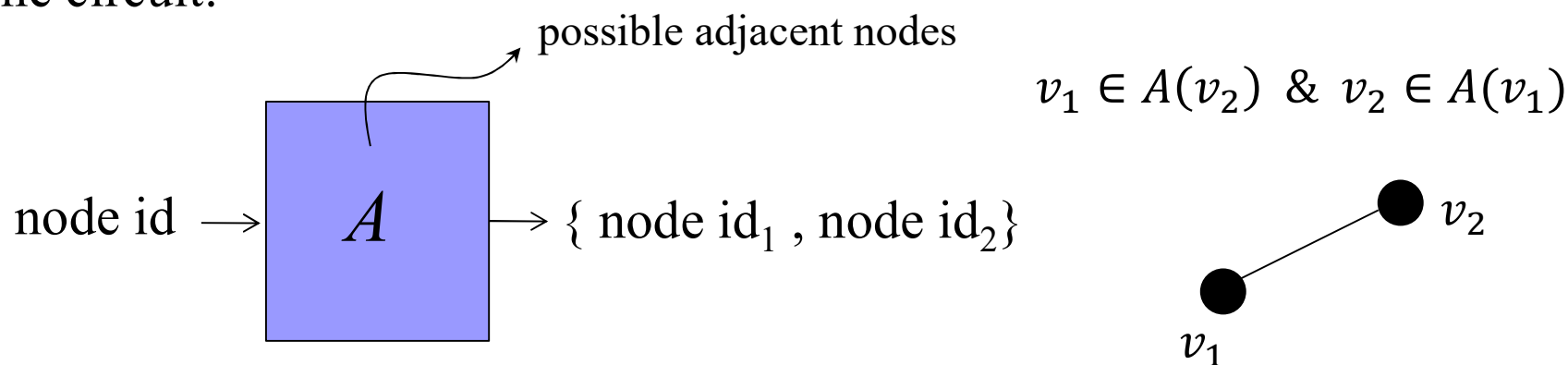
Formally?

PPA: Polynomial Parity Argument

[Papadimitriou '94]

“If a graph has a node of odd degree, then it must have another.”

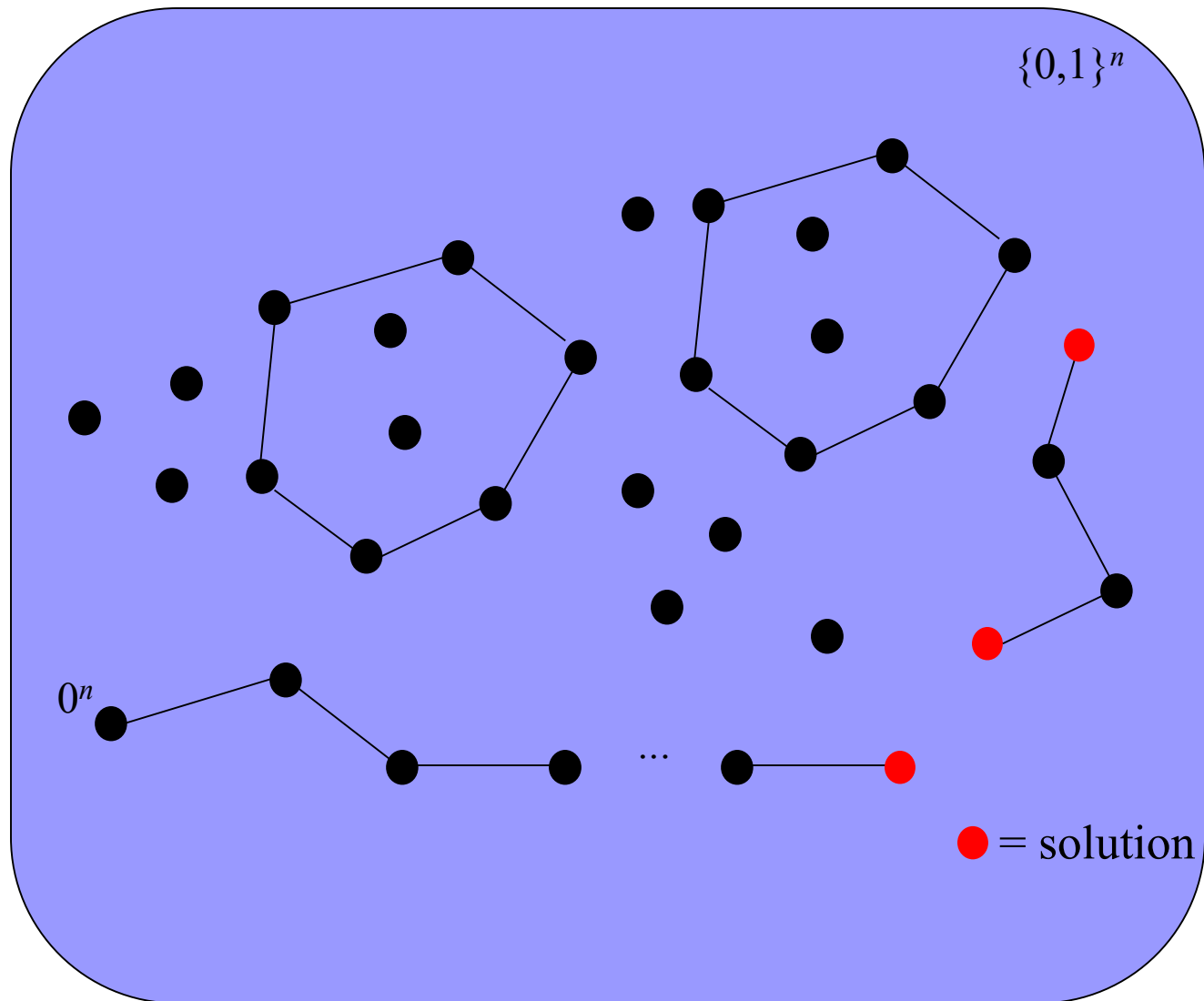
Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



ODD DEGREE NODE: Given C , if 0^n has odd degree, find another node with odd degree. Otherwise say “yes”.

PPA = $\{ \text{Search problems in FNP reducible to ODD DEGREE NODE} \}$

The Undirected Graph

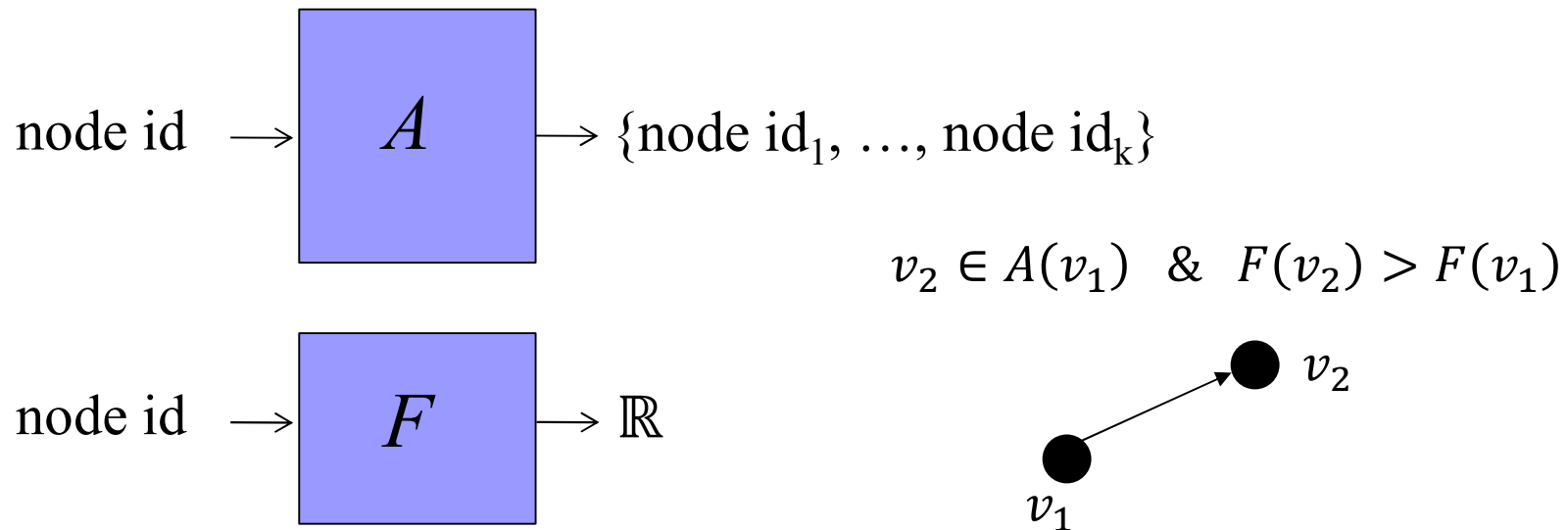


PLS: Polynomial Local Search

[Johnson, Papadimitriou, Yannakakis '89]

“Every DAG has a sink.”

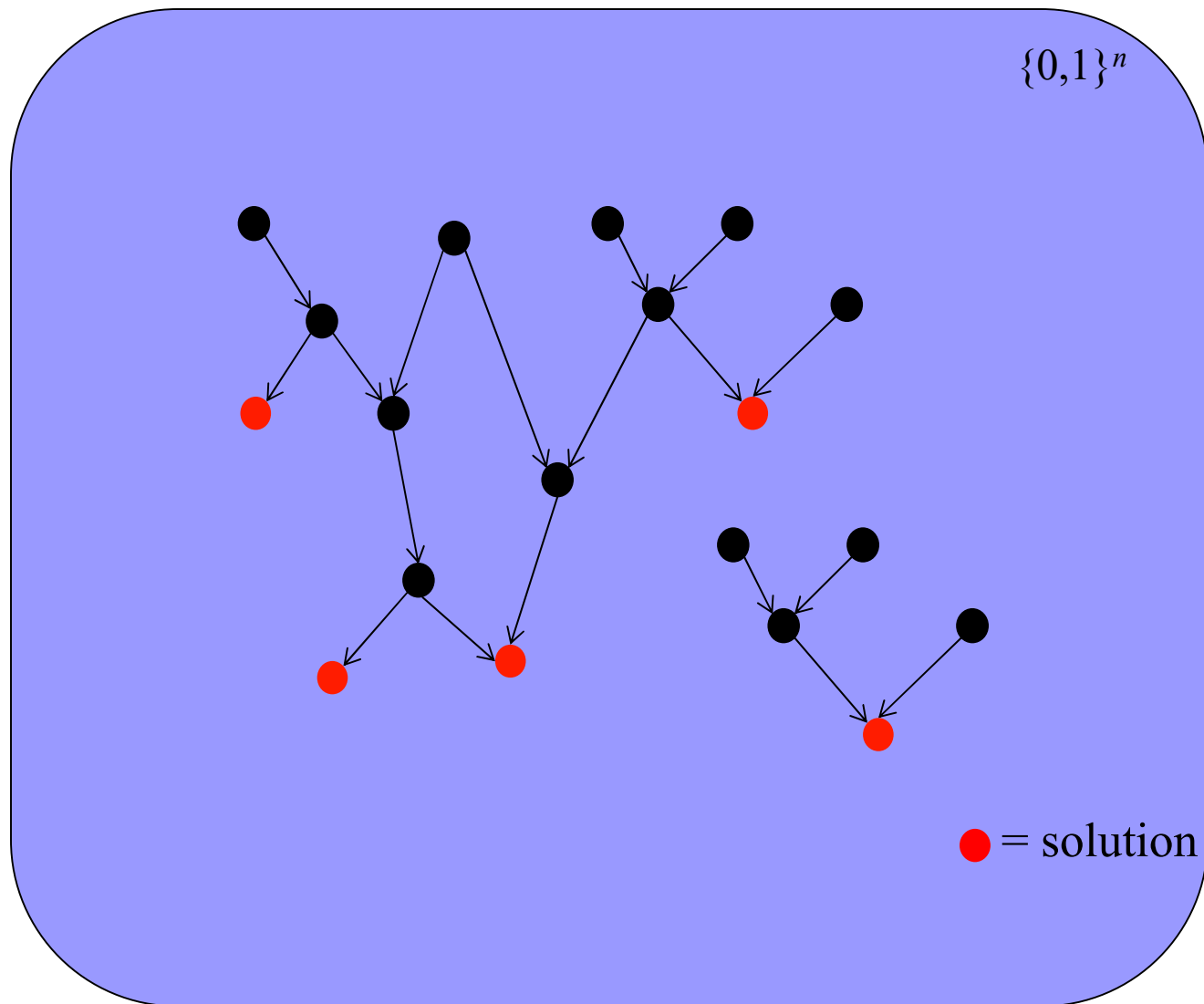
Suppose that a DAG with vertex set $\{0,1\}^n$ is defined by two circuits:



FIND SINK. Given C, F : Find x s.t. $F(x) \geq F(y)$, for all $y \in C(x)$.

PLS = $\{ \text{Search problems in FNP reducible to FIND SINK} \}$

The DAG

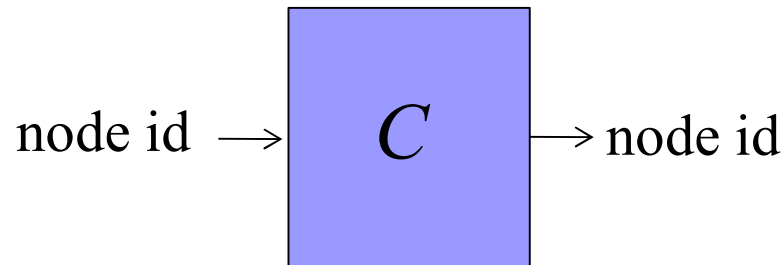


PPP: Polynomial Pigeonhole Principle

[Papadimitriou '94]

“If a function maps n elements to $n-1$ elements, then there is a collision.”

Suppose that an exponentially large graph with vertex set $\{0,1\}^n$ is defined by one circuit:



COLLISION. Given C : Find x s.t. $C(x) = 0^n$; or find $x \neq y$ s.t. $C(x) = C(y)$.

PPP = $\{ \text{Search problems in FNP reducible to COLLISION} \}$

