Lecture 8
Games and Nash Equilibrium

CS 580

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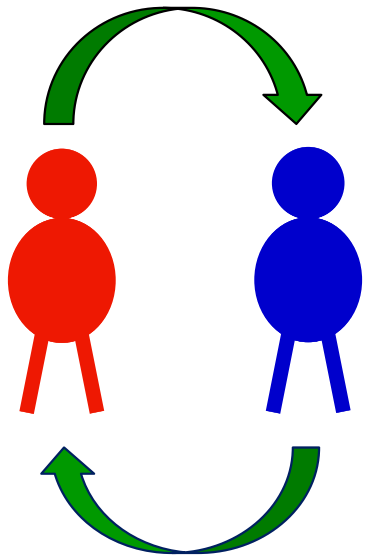
Agenda

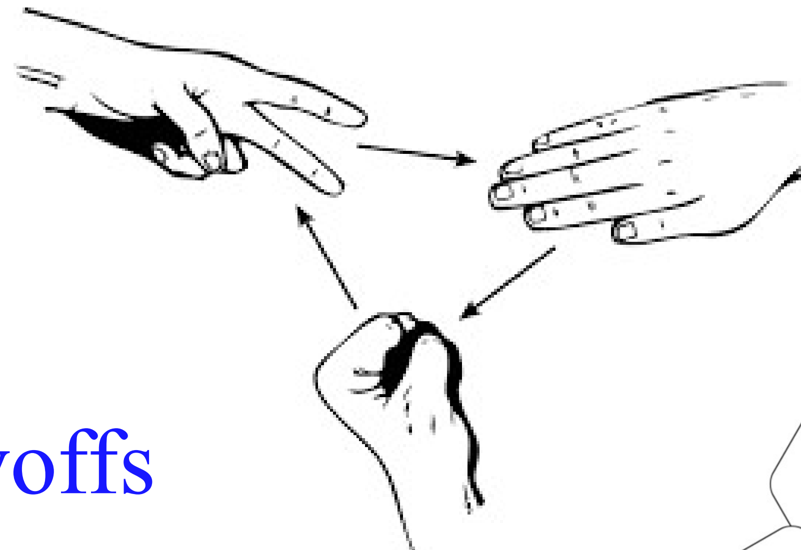
- Two-player Games
- Nash Equilibrium (NE)
  - NE existence
  - NE characterization
- Zero-sum games
  - Minmax Theorem
  - LP-duality

Games

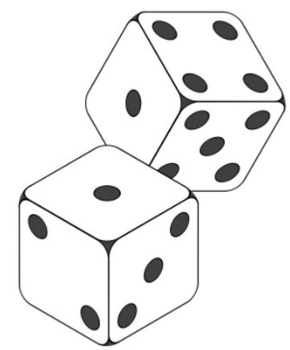


Players

Payoffs

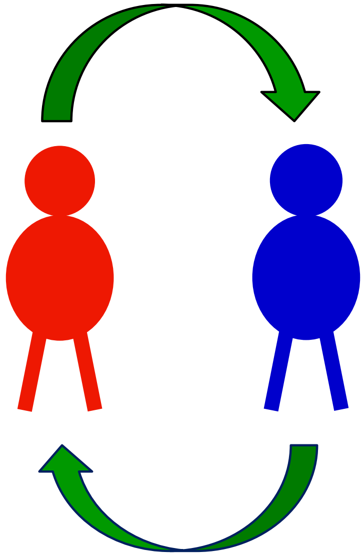


Strategies

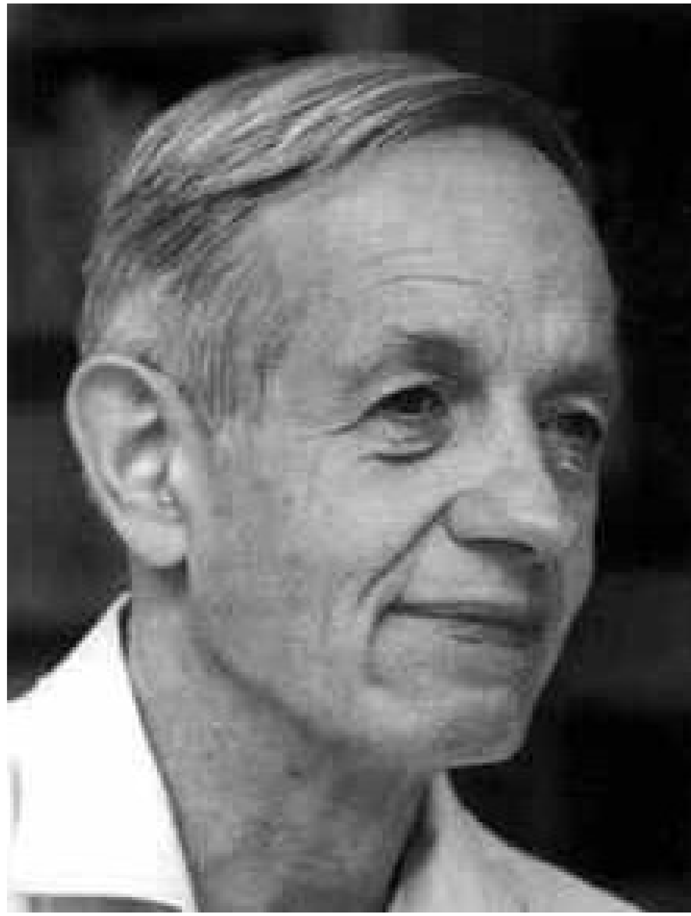


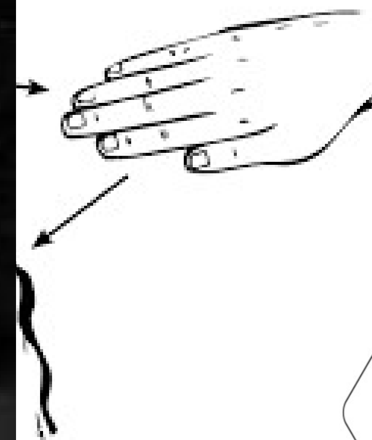
Randomize!

Games

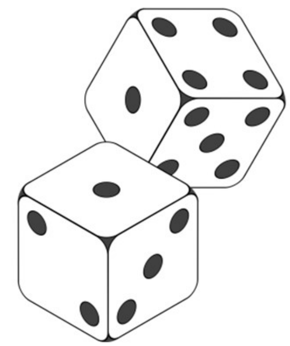


Players





tegies



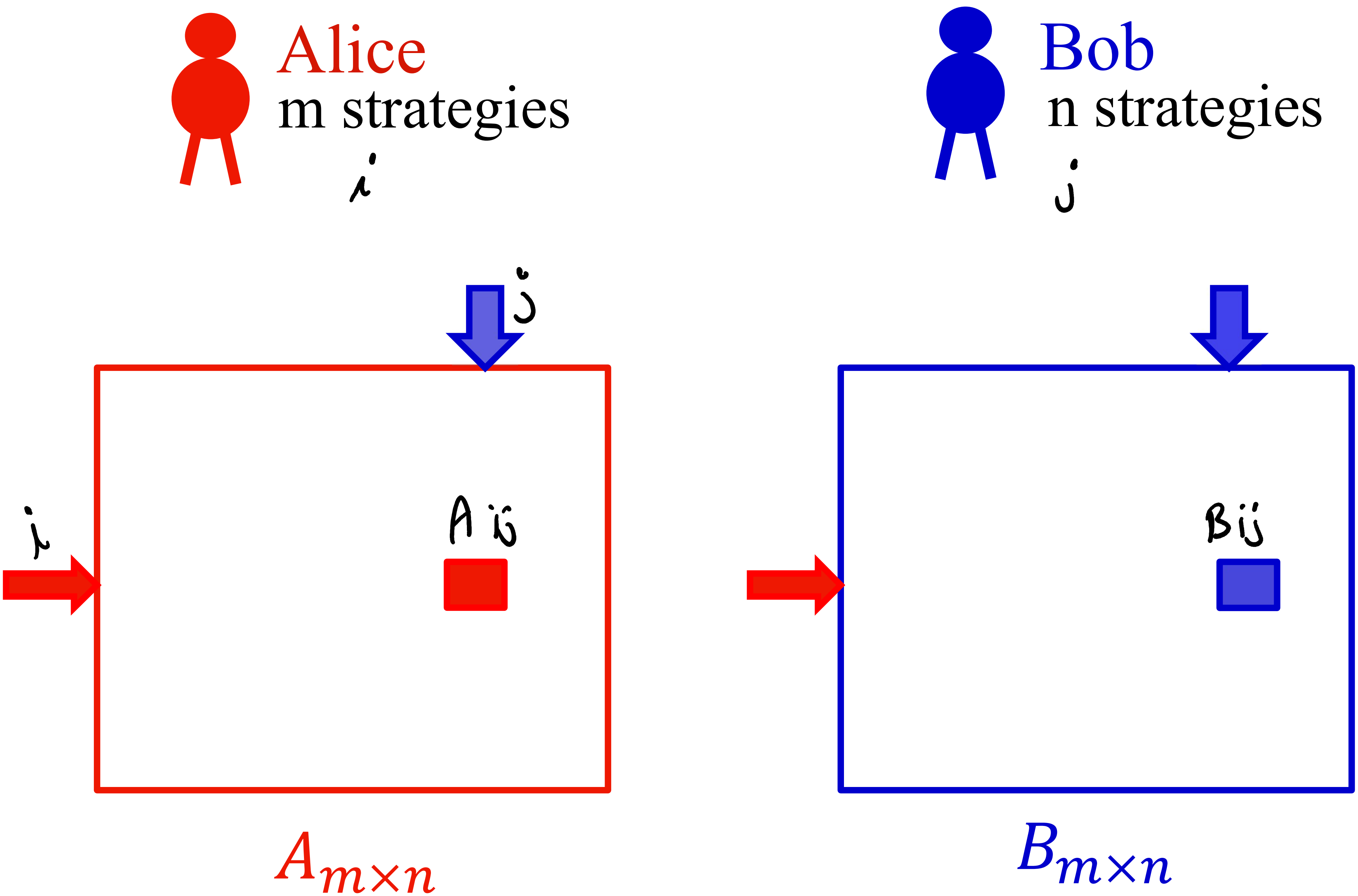
Randomize!

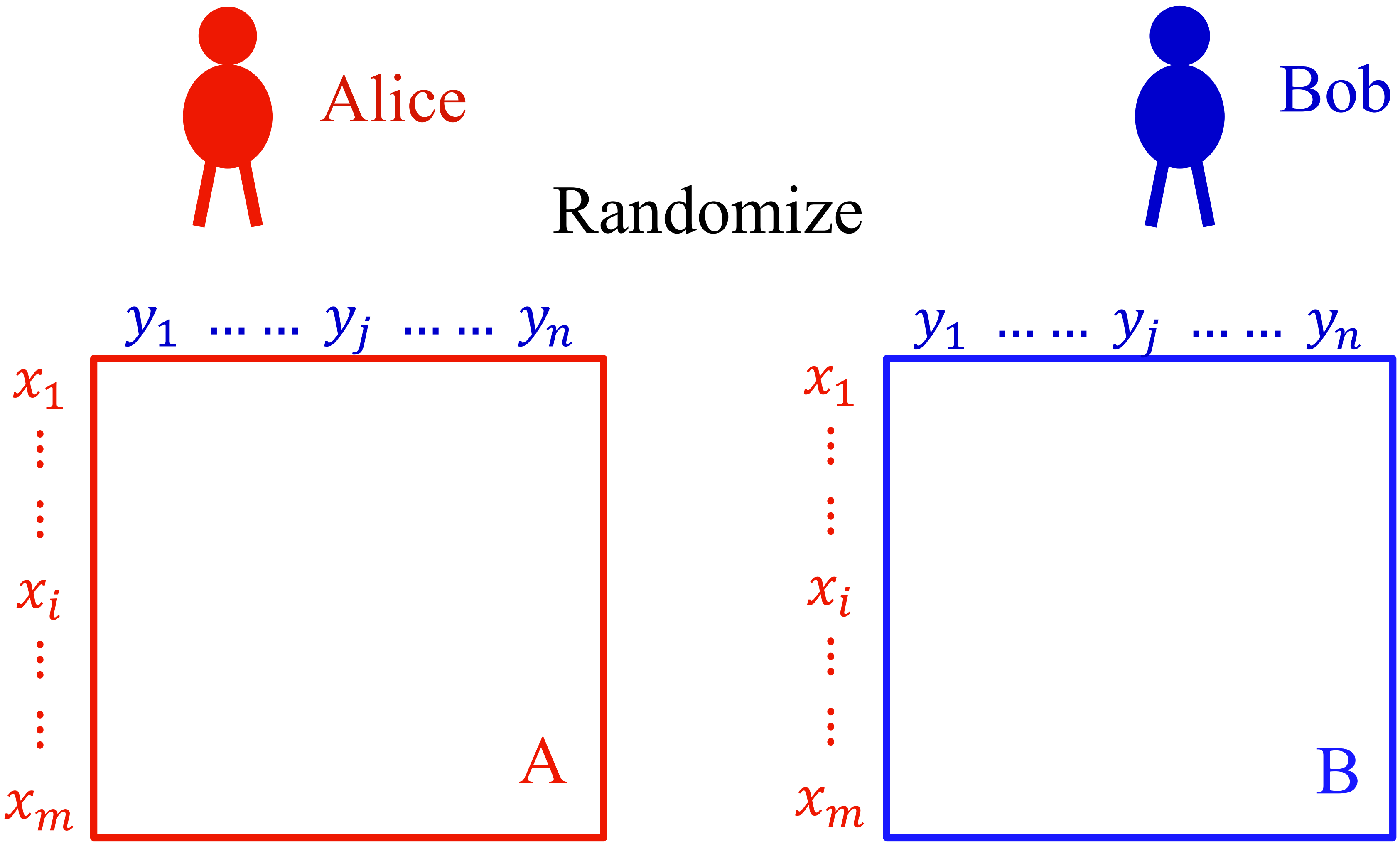
Nash (1950):

There exists a (stable) state where no player gains by unilateral deviation.

Nash equilibrium (NE)

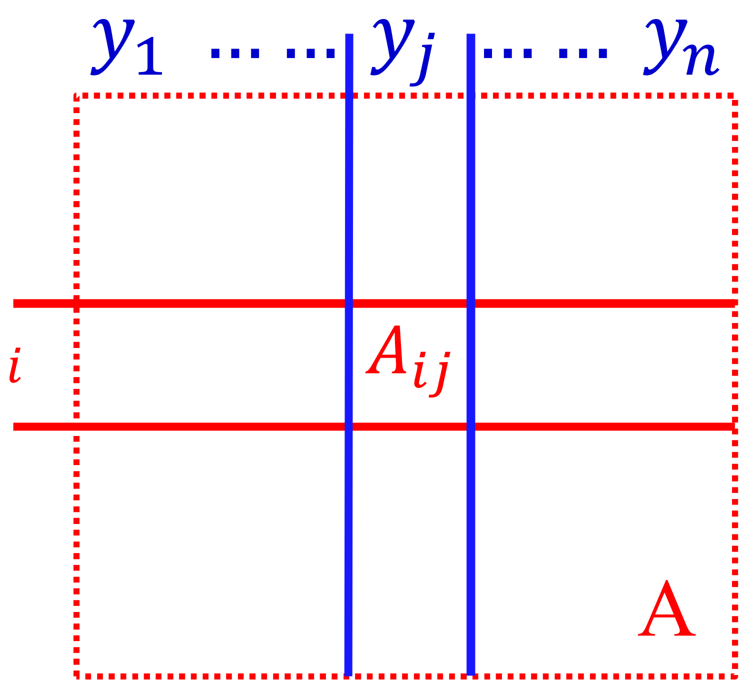
Our focus: Two-player games




Alice
Bob
Randomize
y_1 …… y_j …… y_n
x_1
x_i
x_m
A
y_1 …… y_j …… y_n
x_1
x_i
x_m
B

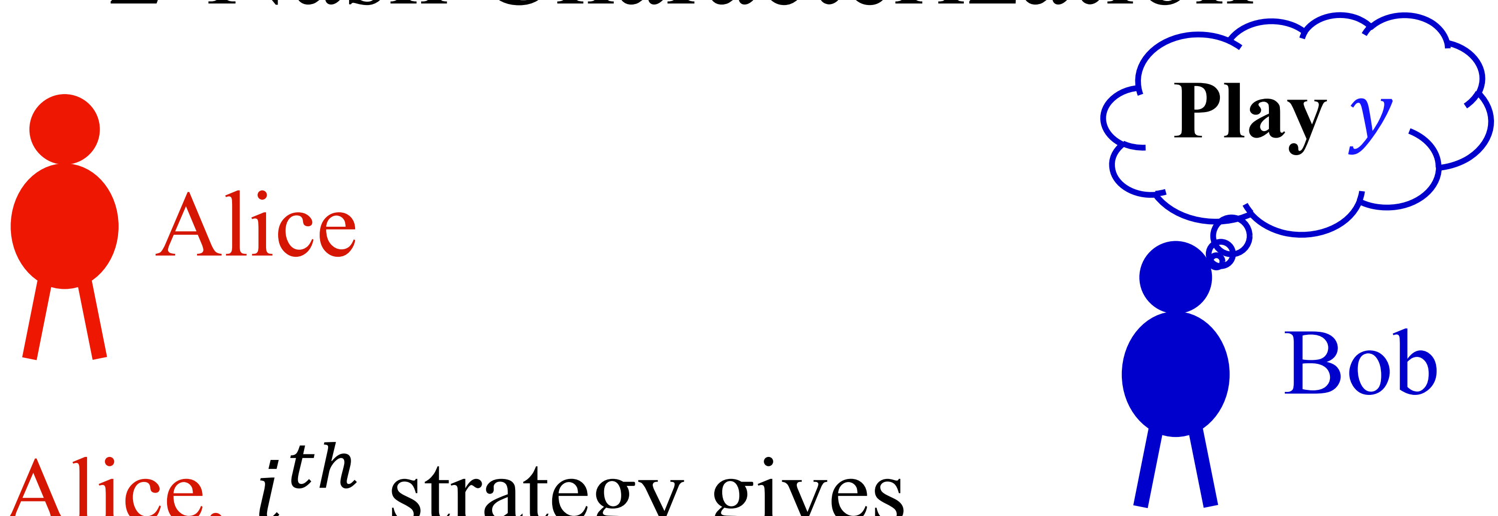
2-Nash Characterization

- For Alice, i^{th} strategy gives

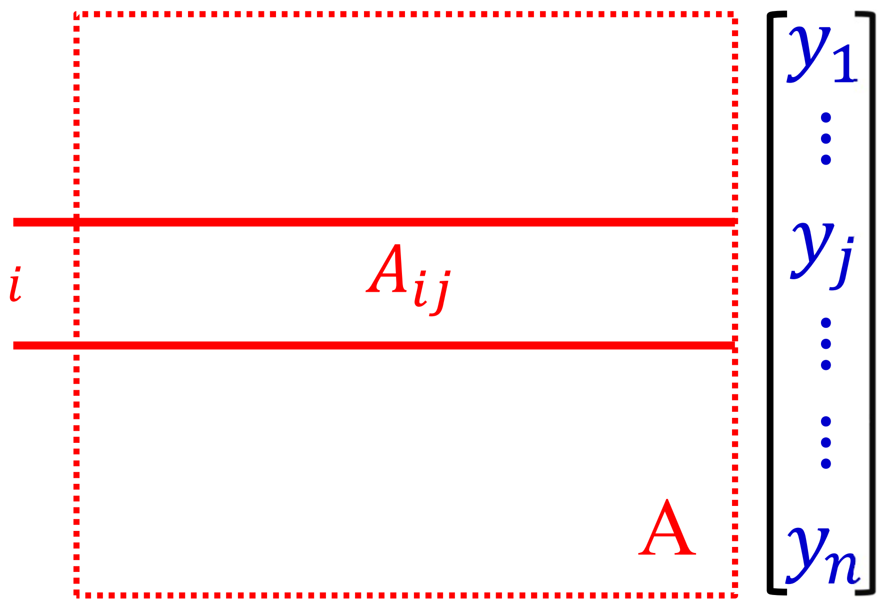


$$\longrightarrow \sum_j A_{ij} y_j$$

2-Nash Characterization

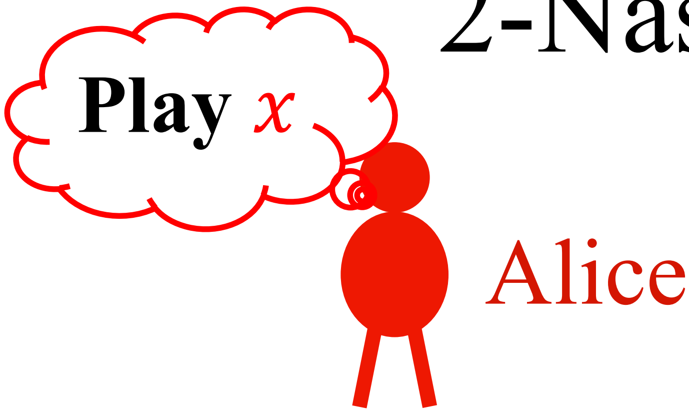


- For Alice, i^{th} strategy gives

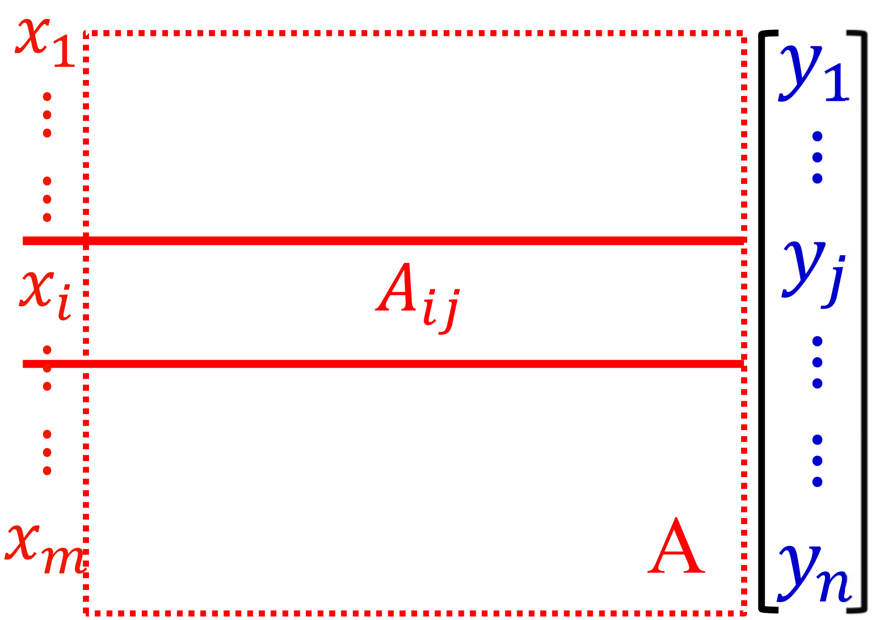


$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

2-Nash Characterization



- Alice's expected payoff is

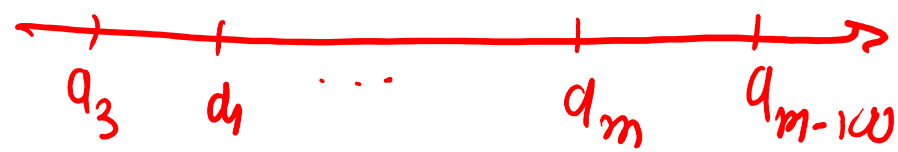


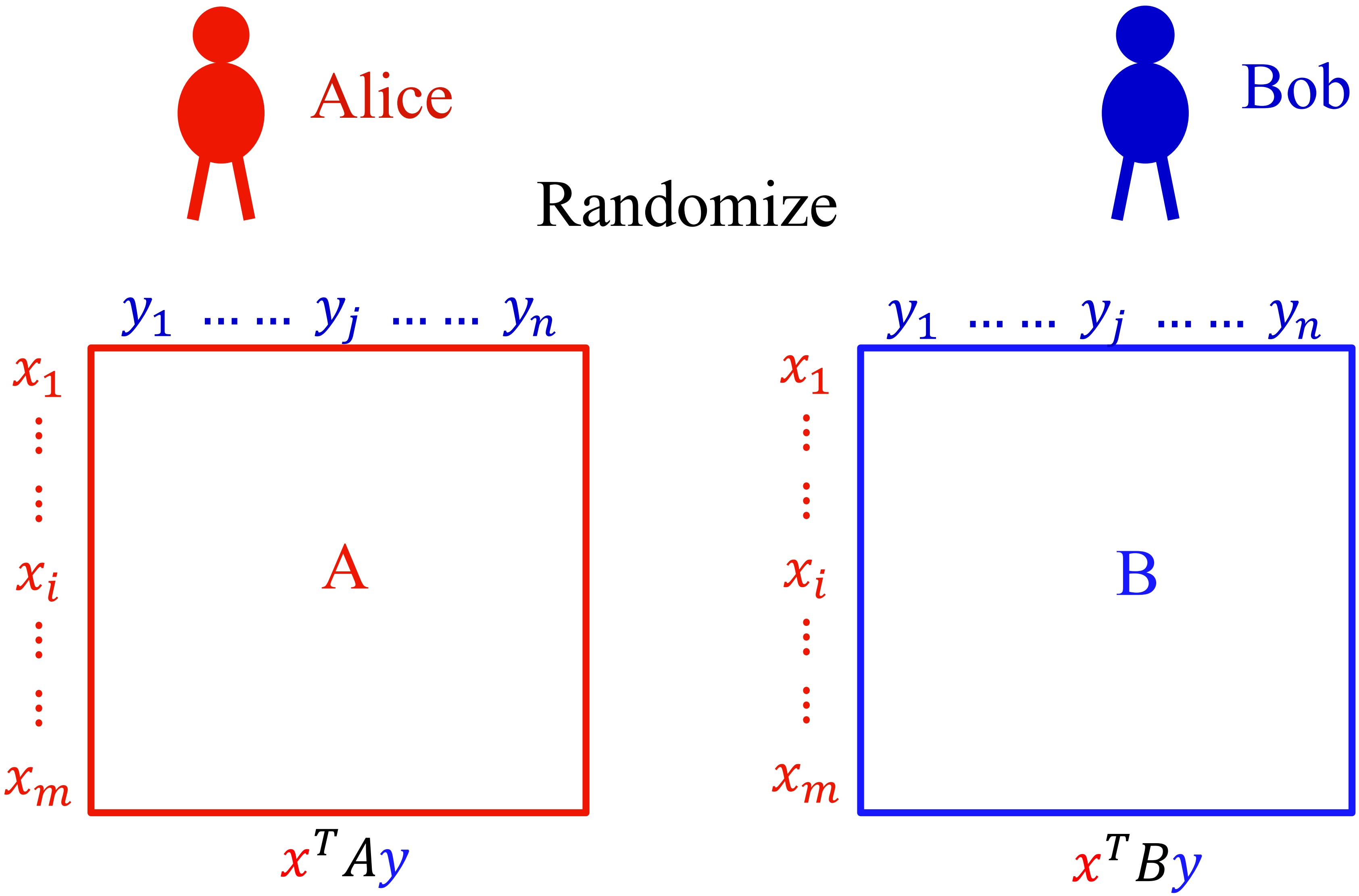
ith row: $x_i (Ay)_i$

Expected payoff $= \sum_{i=1}^{n} x_i$ (payoff from i)

$$\sum_i x_i (Ay)_i = x^T A y$$

$(Ay)_i = a_i$





NE: No unilateral deviation is beneficial

$$x^T A y \geq z^T A y, \qquad \forall z \in \Delta_m$$

$$x^T B y \geq x^T B z, \qquad \forall z \in \Delta_n$$

Example: Matching Pennies

	H	T
H	1 -1	-2 2
T	-2 2	1 -1

$$\begin{array}{c|cc} & H & T \\ & 1/2 & 1/2 \\ \hline H & 1 & -2 \\ T & -2 & 1 \end{array} \quad \begin{array}{l} \rightarrow -0.5 \\ \rightarrow -0.5 \end{array}$$

$$E_A[H] = (1)\,1/3 + (-2)(2/3) = 1/3 - 4/3 = -3/3 = -1$$

$$E_A[T] = (-2)\,1/3 + (1)\,2/3 = -2/3 + 2/3 = 0$$

Nash’s Existence Theorem (1950)

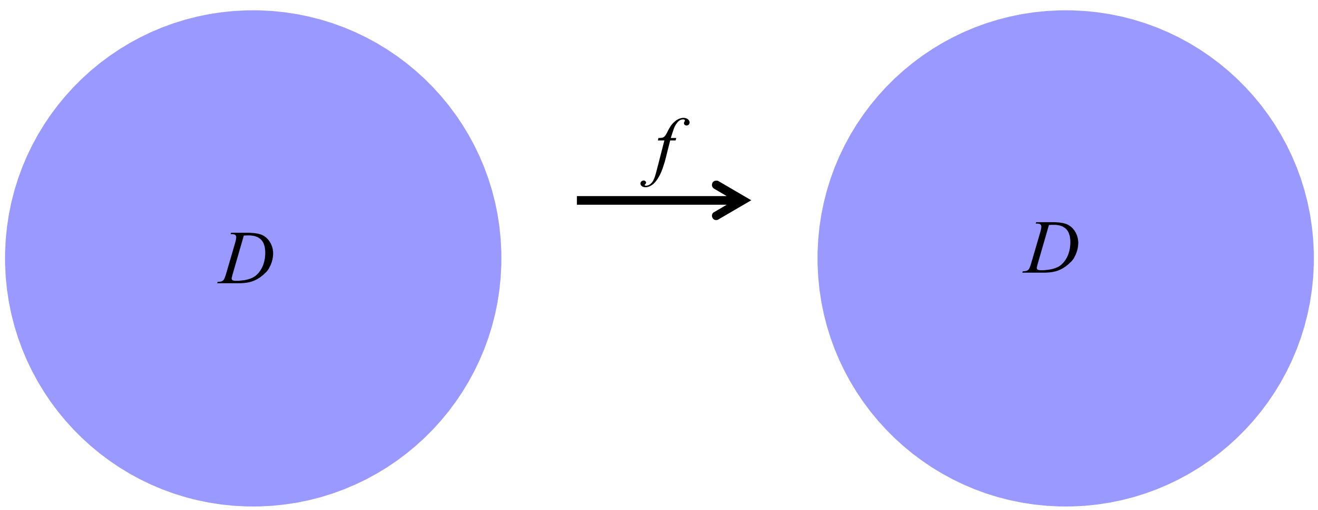
Brouwer's Fixed Point Theorem

[Brouwer 1910]: Let $f : D \rightarrow D$ be a continuous function from a convex and compact subset D of the Euclidean space to itself.

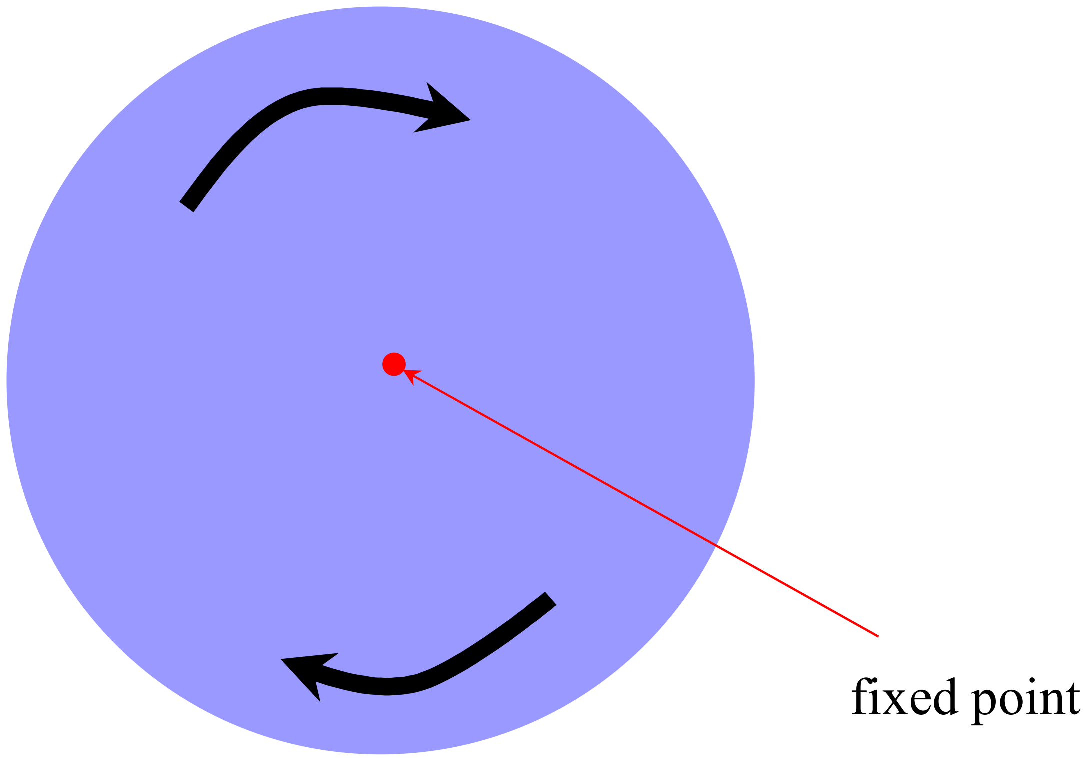
Then there exists an $x \in D$ s.t. $x = f(x)$.

closed and bounded

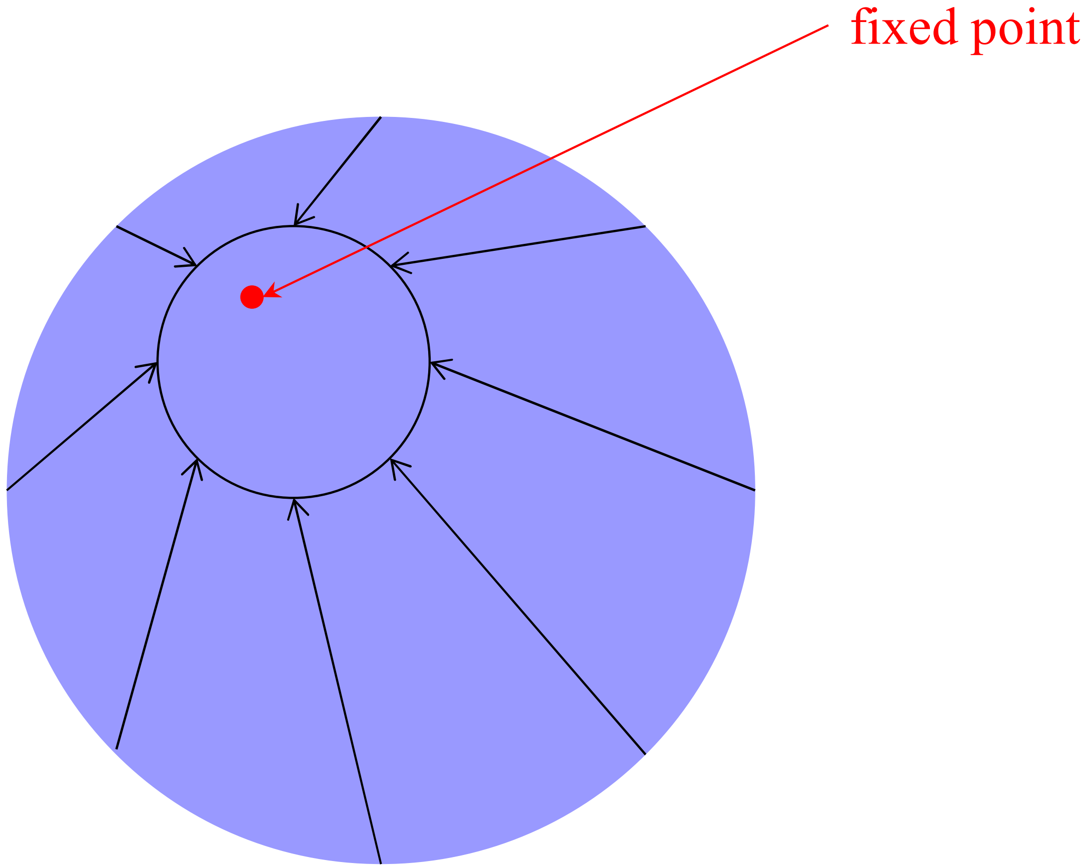
A few examples, when D is the 2-dimensional disk.



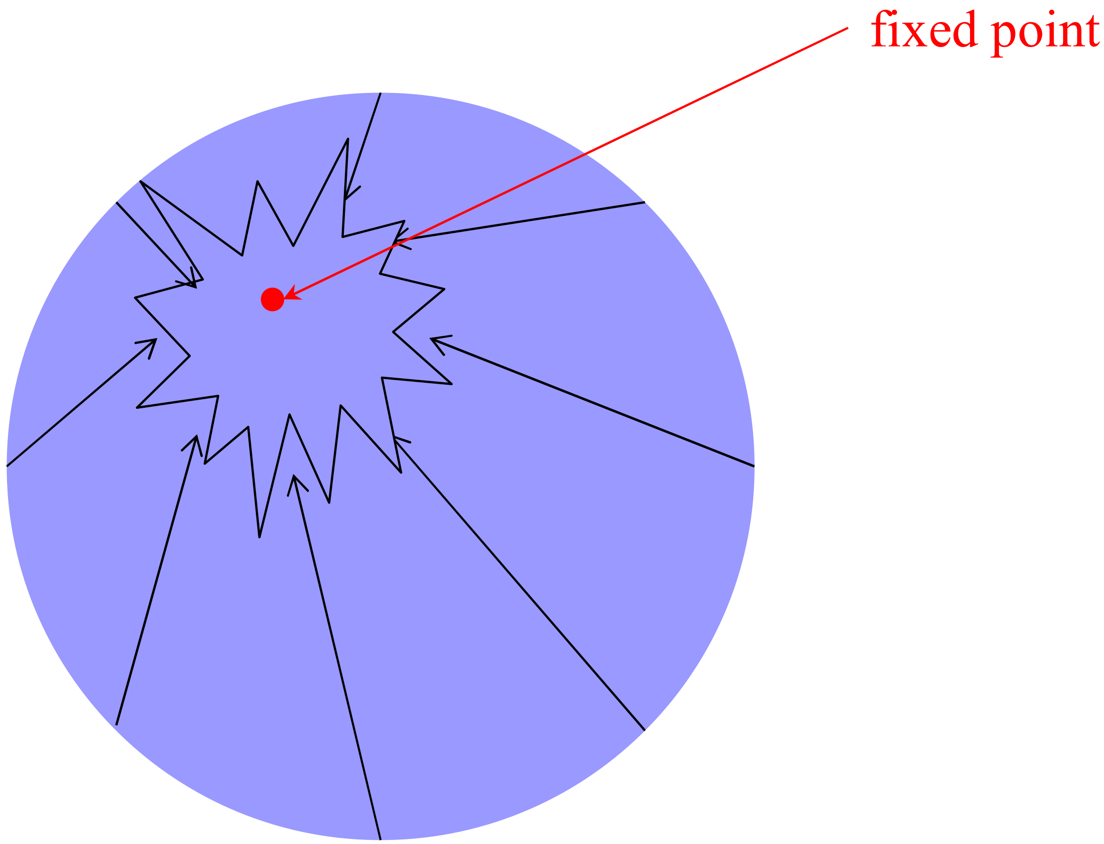
Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem



Brouwer's Fixed Point Theorem



$$Brouwer \Rightarrow Nash$$

(Nash'51)

Nash's Proof:

$f: \Delta_m \times \Delta_n \to \Delta_m \times \Delta_\text{n}$, $(x', y') = f(x, y)$ = (x, y)

(x, y) → (x', y')

(x, y) is a F.P.
⇕
(x', y') = (x, y)
⇒ (x, y) is a NE

$\forall i, \quad \delta_i = \max\{(Ay)_i - x^T Ay, 0\},$

$\forall i, \quad x_i' = \dfrac{x_i + \delta_i}{\sum_{k=1}^{n}(x_k + \delta_k)}$

Lemma. If $x' = x$ then x is best for Alice against y
If y' = y " y " " " Bob " x
≡ If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

$\forall i, \qquad \delta_i = \max\{(Ay)_i - x^T Ay, 0\}; \geq 0 \quad \leq \max_i (Ay)_i = (Ay)_{\bar{k}}$

$$\forall i, \qquad x_i' = \frac{x_i + \delta_i}{\sum_{k=1}^{n}(x_k + \delta_k)}$$

Lemma. If $x^T Ay < z^T Ay$ for some $z \in \Delta_m$ then $x' \neq x$.

Pf:

$\bar{k} \in \operatorname{argmax}_{i=1}^{m} (Ay)_i$

$\underline{k} \in \operatorname{argmin}_{i=1}^{m} (Ay)_i$

$(Ay)_{\underline{k}} \leq x^T Ay < (Ay)_{\bar{k}}$

$\delta_{\bar{k}} > 0 \rightarrow ①$

$\delta_{\underline{k}} = 0 \rightarrow ②$

$$x'_{\underline{k}} = \frac{x_{\underline{k}} + \delta_{\underline{k}}}{\sum_k x_k + \sum_k \delta_k} < x_{\underline{k}}$$

($\delta_{\underline{k}} = 0$ ∵ ②; $\sum_k x_k = 1$; $\sum_k \delta_k > 0$ ∵ ①) ∎

Nash's Proof

$f: \Delta_m \times \Delta_n \to \Delta_m \times \Delta_n$, $(x', y') = f(x, y)$

$(x, y) \longrightarrow (x', y')$

$\forall j, \quad \tau_j = \max\left\{\left(x^T B\right)_j - x^T B y, 0\right\},$

$\forall j, \quad y'_j = \dfrac{y_j + \tau_j}{\sum_{k=1}^{n}(y_k + \tau_k)}$

EXE

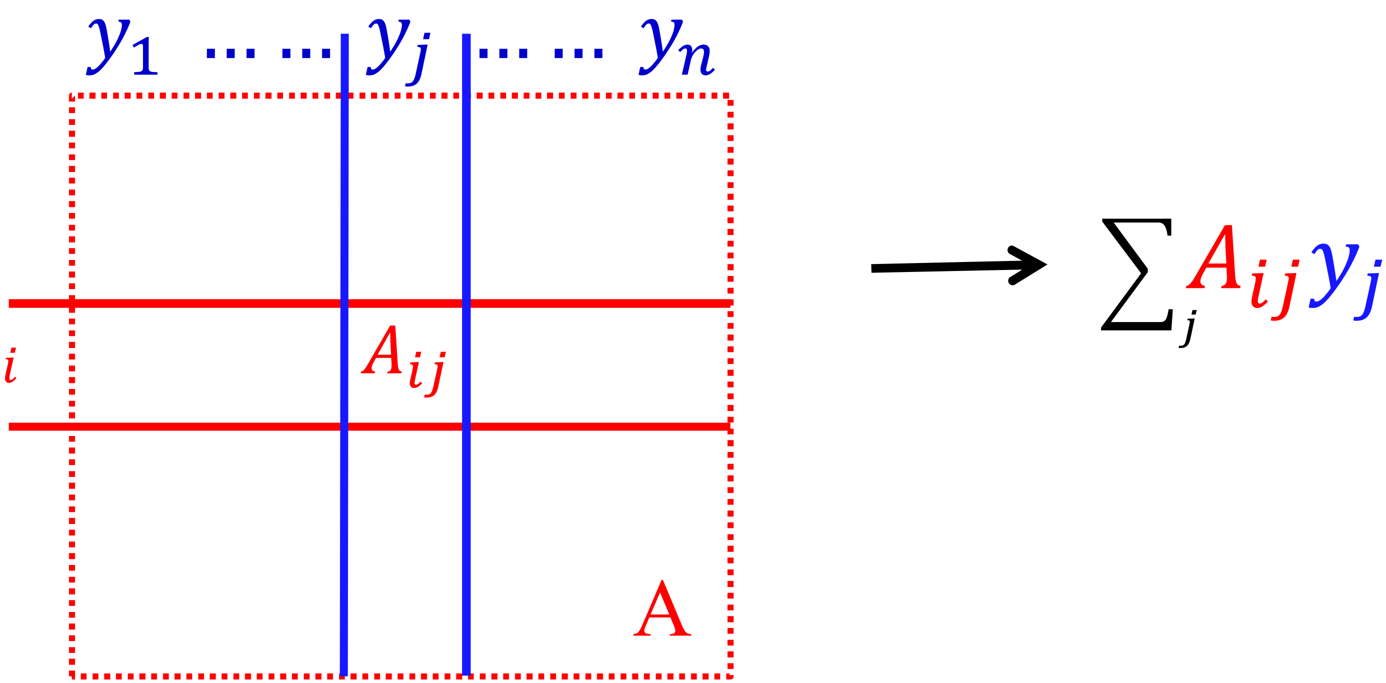
Lemma. If $y' = y$ then y is best for Bob against x

$\equiv$ If $x^T B y < x^T B z$ for some $z \in \Delta_n$ then $y' \neq y$.

2-Nash Characterization

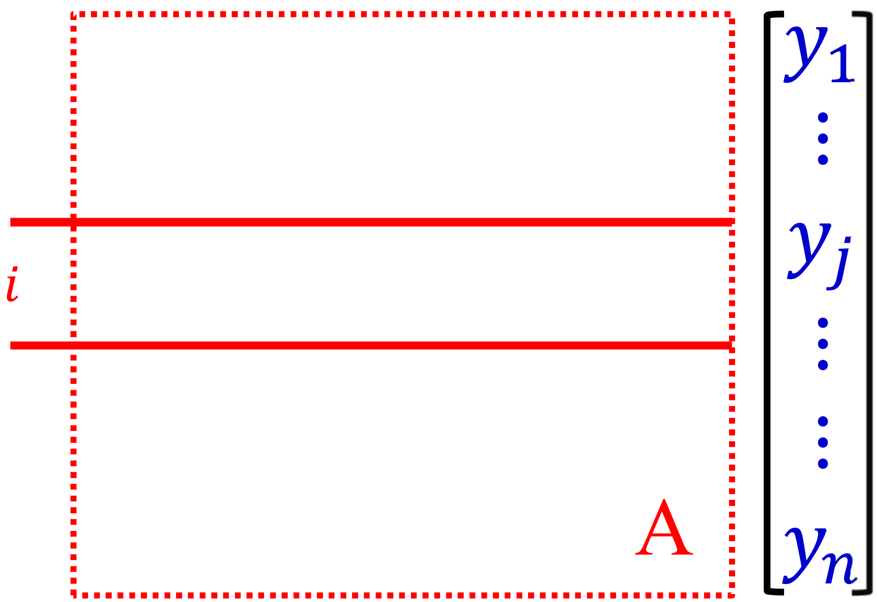
2-Nash Characterization

- For Alice, i^{th} strategy gives



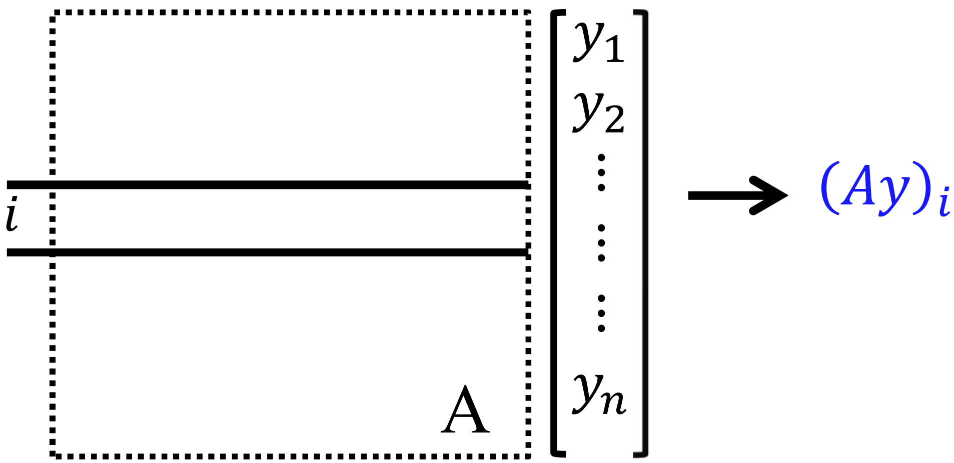
2-Nash Characterization

- For Alice, i^{th} strategy gives



$$\rightarrow \sum_j A_{ij} y_j = (Ay)_i$$

- i^{th} strategy gives Alice



- Max possible payoff: $\max_i (Ay)_i$

- x achieves max payoff iff

$$x^T Ay \geq (Ay)_i, \qquad \forall i$$

$$\equiv$$

$$\forall k, \qquad x_k > 0 \Rightarrow k \in \underset{i}{\operatorname{argmax}}\,(Ay)_i$$

Complementarity

- Max possible payoff: $\max_i e_i A y$
- x achieves max payoff iff

$$\forall i, \qquad x^T A y \geq (Ay)_i$$

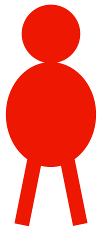
$$\equiv$$

$$\forall k, \qquad x_k > 0 \Rightarrow (Ay)_k = \max_i (Ay)_i$$

Complementarity

	H ½	T ½	
H ½	1 -1	-2 2	→ -0.5
T ½	-2 2	1 -1	→ -0.5

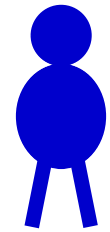
Polyhedra



max-payoff $\leq \pi_A$

P

$$\forall i, (Ay)_i \leq \pi_A$$
$$y \in \Delta_n$$



max-payoff $\leq \pi_B$

Q

$$\forall j, \left(x^T B\right)_j \leq \pi_B$$
$$x \in \Delta_m$$

P

$$\forall i, (Ay)_i \le \pi_A$$
$$y \in \Delta_n$$

Q

$$\forall j, \left(x^T B\right)_j \le \pi_B$$
$$x \in \Delta_m$$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Sum of payoffs

At least the sum of max payoffs

$x^T A y + x^T B y$ =

$$\underbrace{x^T (A + B) y}_{} - \underbrace{(\pi_A + \pi_B)}_{} \le 0$$

P $\forall i, (Ay)_i \le \pi_A$, $y \in \Delta_n$

Q $\forall j, (x^T B)_j \le \pi_B$, $x \in \Delta_m$

$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$

Sum of payoffs: $x^T(A+B)y$

At least the sum of max payoffs: $(\pi_A + \pi_B)$

$$x^T(A+B)y - (\pi_A + \pi_B) = \mathbf{0}$$

≤ 0

exe.

easy ⇑ ⇕ ⇓ ! Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

P
$$\forall i, (Ay)_i \le \pi_A$$
$$y \in \Delta_n$$

Q
$$\forall j, \left(x^T B\right)_j \le \pi_B$$
$$x \in \Delta_m$$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T(A + B)y - (\pi_A + \pi_B) \le 0.$

(ii) $x^T(A + B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

P $\boxed{\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}}$ Q $\boxed{\begin{array}{c} \forall j, \left(x^T B\right)_j \leq \pi_B \\ x \in \Delta_m \end{array}}$

Claim. For $(y, \pi_A) \in P$, $(x, \pi_B) \in Q$

(i) $x^T(A+B)y - (\pi_A + \pi_B) \leq 0$.

(ii) $x^T(A+B)y - (\pi_A + \pi_B) = 0$ if and only if (x, y) is a NE.

P $\boxed{\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n}$ Q $\boxed{\forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m}$

$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$

2-Nash

Sum of payoffs

At least the sum of max payoffs

$$\text{max: } x^T(A+B)y - (\pi_A + \pi_B) \quad \boxed{= 0}$$

$$\text{s.t. } (y, \pi_A) \in P, (x, \pi_B) \in Q$$

⇕

Complementarity

1. (x, y) is a NE
2. π_A and π_B are the max payoffs

Zero-sum Games

Von Neuman’s maxmin theorem (1928) = LP-duality

P

$$\forall i, (Ay)_i \le \pi_A$$
$$y \in \Delta_n$$

Q

$$\forall j, \left(x^T B\right)_j \le \pi_B$$
$$x \in \Delta_m$$

$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$A_{ij} + B_{ij} = 0$

0

$$\text{max: } x^T(A+B)y - (\pi_A + \pi_B)$$

s.t. $(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$ Q $\boxed{\begin{array}{c}\forall j, \left(x^T B\right)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

$$(y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

$$\max: -(\pi_A + \pi_B)$$

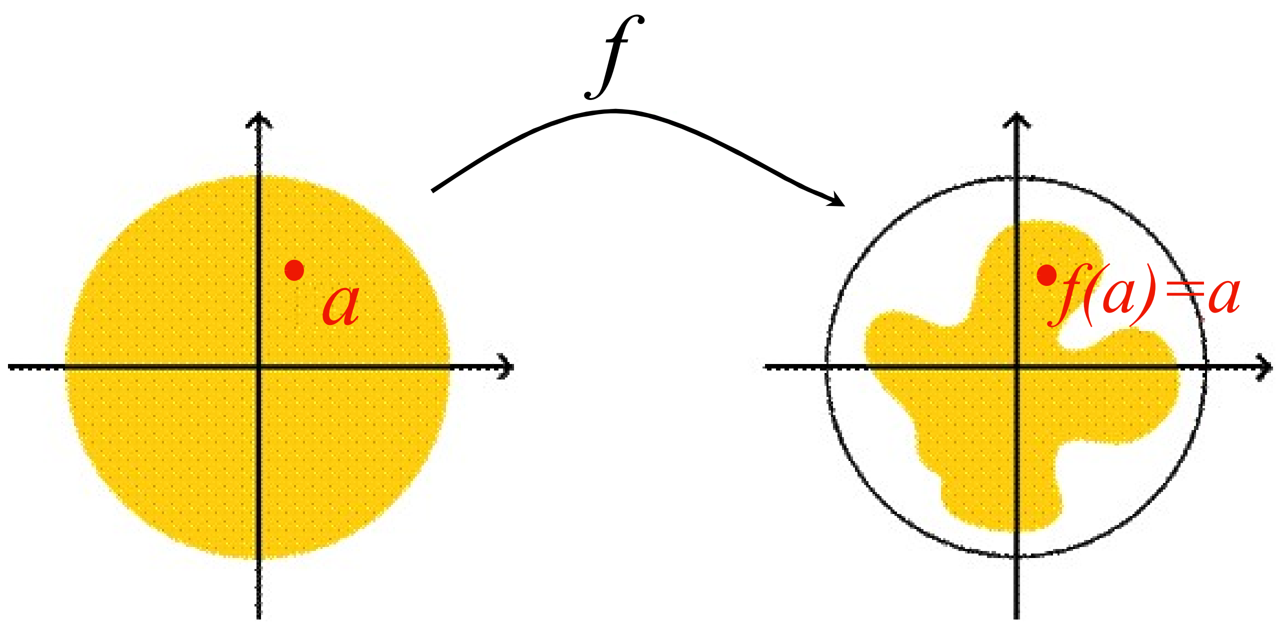
$$\text{s.t. } (y, \pi_A) \in P, \qquad (x, \pi_B) \in Q$$

Theorem. [von Neumann'28] (max-min = min-max) Game $(A, -A)$.
Wrt A, Alice is a maximizer and Bob minimizer. Then,

$$\max_x \min_y x^T A y = \min_y \max_x x^T A y$$ & the max-min is NE.

Computation in general?

NE existence via fixed-point theorem.



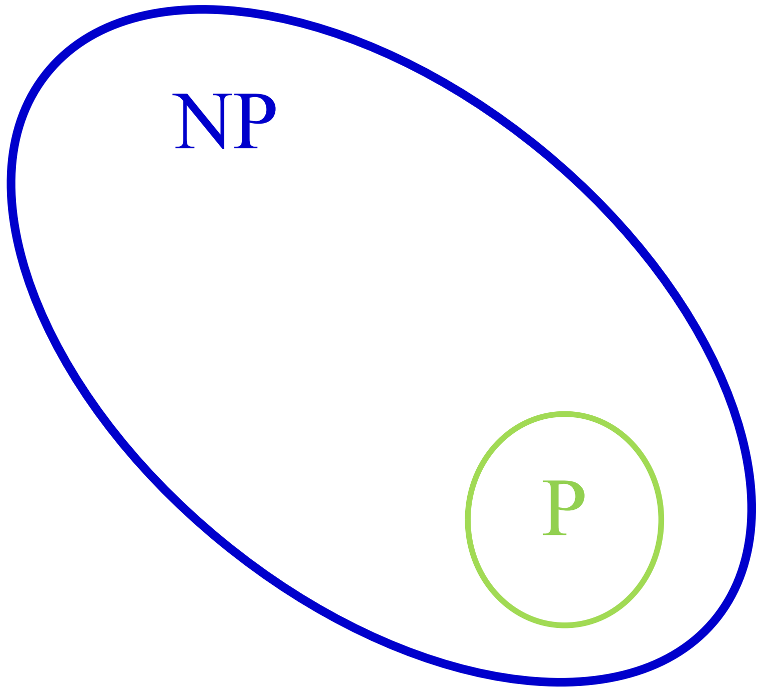
Computation? (in Econ)

- Special cases: Dantzig'51, Lemke-Howson'64, Elzen-Talman'88, Govindan-Wilson'03, …
- Scarf'67: Approximate fixed-point.
  - Numerical instability
  - Not efficient!
- …

Computation? (in CS)

Not easy!

∃ solution?



What if solution always exists, like Nash Eq.?

Computation? (in CS)

Megiddo and Papadimitriou'91 :

Nash is NP-hard $\Rightarrow$ NP=Co-NP

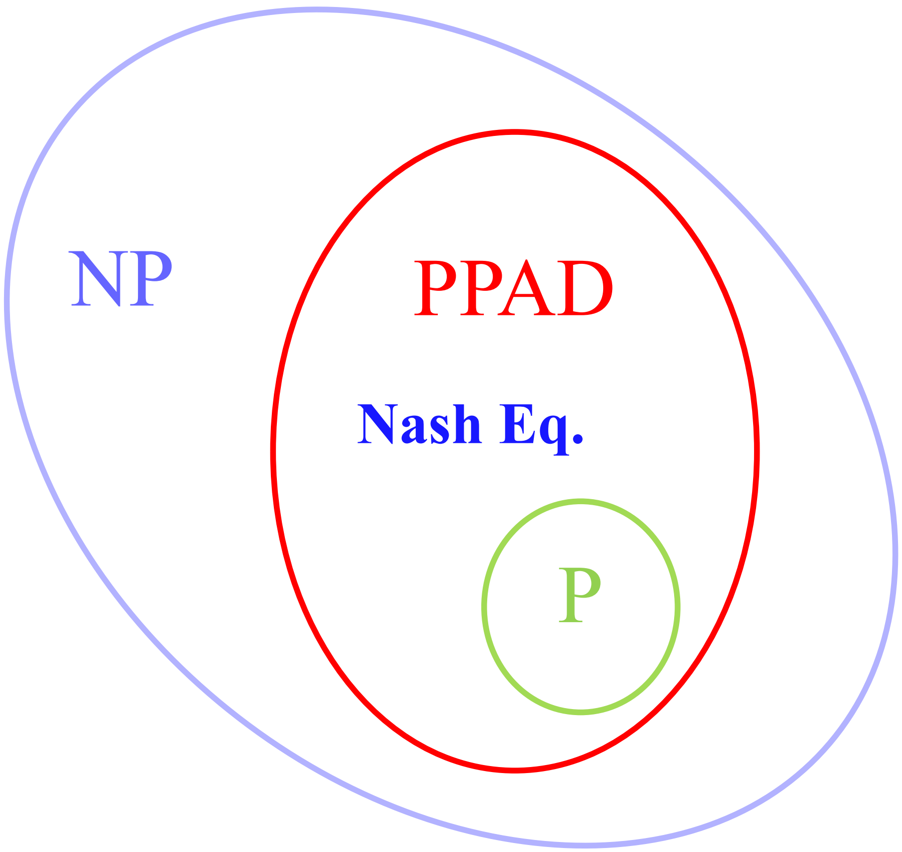
NP-hardness is ruled out!

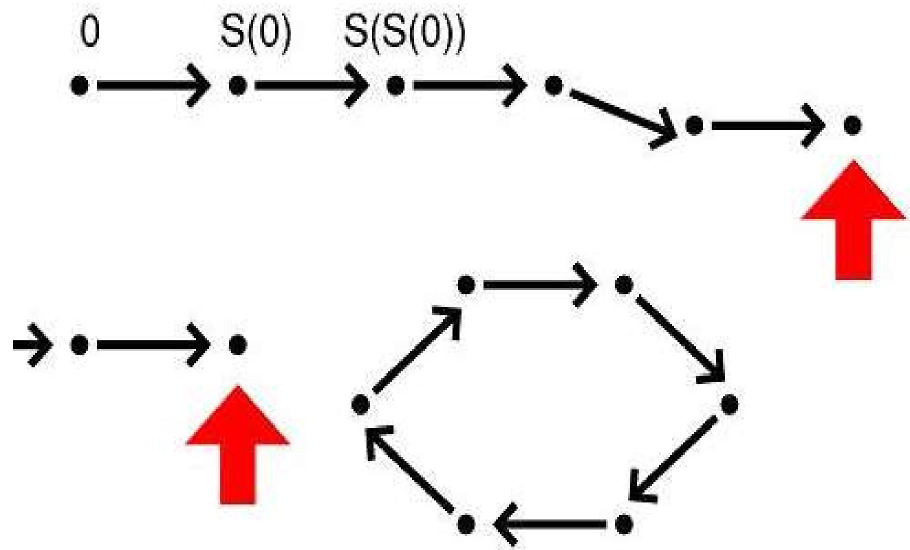
Complexity Classes

2-Nash is PPAD-complete!
[DGP'06, CDT'06]

Papadimitriou'94

PPAD   Polynomial Parity Argument for Directed graph





Find an end

Brute-force Algorithm?

P $\forall i, (Ay)_i \leq \pi_A$
$y \in \Delta_n$

Q $\forall j, \left(x^T B\right)_j \leq \pi_B$
$x \in \Delta_m$

Let (x, y) be a NE. Suppose we know supp(x) and supp(y).

Now can we find a NE?

Can we do better than “brute-force”?

Not so far. And may be never!

It is one of the hardest problems in PPAD.

What about special cases/approximation?

- Rank(A) or rank(B) is constant
- O(1)-approximate NE: quasi-polynomial time algorithm
- Constant rank games: rank(A+B) is a constant
  - FPTAS

P $\boxed{\begin{array}{c}\forall i, (Ay)_i \le \pi_A \\ y \in \Delta_n\end{array}}$

Q $\boxed{\begin{array}{c}\forall j, (x^T B)_j \le \pi_B \\ x \in \Delta_m\end{array}}$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

max: $-(\pi_A + \pi_B)$

s.t. $(y, \pi_A, x, \pi_B) \in P \times Q$

Rank of a game: rank(A+B)

Zero-sum ≡ Rank-0 games

P $\begin{array}{c} \forall i, (Ay)_i \leq \pi_A \\ y \in \Delta_n \end{array}$

Q $\begin{array}{c} \forall j, (x^T B)_j \leq \pi_B \\ x \in \Delta_m \end{array}$

$$(y, \pi_A, x, \pi_B) \in P \times Q$$

Theorem. If (A, B) is zero-sum, i.e., $A + B = 0$, then

2-Nash → linear programming

Rank of a game: rank(A+B)

Poly-time approximation for constant rank games [KT'03].

Poly-time exact for rank-1 games [AGMS'11].

Exact for rank > 2 is PPAD-hard [M'13].