Voting: Arrow's Theorem

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$O = \{1, \dots, m\}$ candidates / outcomes

$N = \{1, \dots, n\}$ voters.

$O = \{a, b, c\}$

$L = \{(a,b,c), (b,c,a), (a,b), \dots\}$

L = All possible orderings over the candidates in O.

$|L| = m!$

For every agent $i \in N$, $\succ_i \in L$ $\quad a \succ_i c \succ_i b$

Goal:

- Social Choice Func.

$$c : L^n \to O$$

$$(\succ_1, \succ_2, \dots, \succ_n)$$

- Social welfare Func

$$W : L^n \to L$$

$$(\succ_1, \dots, \succ_n) \to \succ_W$$

$O = \{A, B, C\}$

# Voters	2	1	0
499 :	A	B	C
3 :	B	C	A
498 :	C	B	A

- Voting Schemes:

1. Plurality (majority): Most top votes wins $\to$ A.
2. Plurality w/ elimination: Eliminate least top votes one-by-one. $\to$ C
3. Borda Count: j^{th} candidate gets $(m-j)$ points. Most points win.
   - A: $499 \times 2 = 998$
   - B: $499 + 2 \times 3 + 498 = 1003$
   - C: $3 + 2 \times 498 = 1001$
4. Condorcet winner: Winner of every "pairwise match"
   - A vs. B
   - B vs. C
   - A vs. C

☆ Desired Properties: Given $\succ \in L^n$, $\succ_W = W(\succ)$

① Pareto Efficiency (PE):

$\forall i \in N, \quad a \succ_i b \Rightarrow a \succ_W b$

② Independent of Irrelevant Alternative (IIA).

Given $a, b \in O$, Suppose $\succ, \succ' \in L^n$, s.t. $\forall i$, $a \succ_i b \Rightarrow a \succ'_i b$

$b \succ_i a \Rightarrow b \succ'_i a$

Then,

$a \succ_W b \Rightarrow a \succ'_W b$

$b \succ_W a \Rightarrow b \succ'_W a$

③ Non-dictatorial: W does not have a "dictator".

$\nexists i^* \in N, \ \forall a, b \in O: \quad a \succ_{i^*} b \Rightarrow a \succ_W b$

Thm: [Arrow '50]. Any W that is PE & IIA is Dictatorial.

Pf: 4 Claims.

Suppose W satisfies PE & IIA.

Fix $b \in O$.

Claim I: In $\succsim$, if every agent puts b at the top or at the bottom, then in $>_w$ b should be at the top or bottom.

Pf: By contradiction. Suppose, $\boxed{a >_w b >_w c}$, $\exists a, c \in O$.

Construct $\succsim'$ from $\succsim$ by moving c above a for all agents i.

Case I

$$\begin{array}{cc} >_i & >'_i \\ \vdots & \vdots \\ a & c \\ \vdots & \vdots \\ c & a \\ \vdots & \vdots \end{array}$$

Case II

$$\begin{array}{ccc} >_i & & >'_i \\ \vdots & & \vdots \\ c & & c \\ \vdots & = & \vdots \\ a & & a \\ \vdots & & \vdots \end{array}$$

OBS: Relative ordering of $a \& b$, $b \& c$ are same b/w $>$, $>'$.

$$\left.\begin{array}{l} ① \; a >_w b \overset{IIA}{\Longrightarrow} a >'_w b \\ ② \; b >_w c \overset{IIA}{\Longrightarrow} b >'_w c \end{array}\right\} \Longrightarrow a >'_w b >'_w c$$

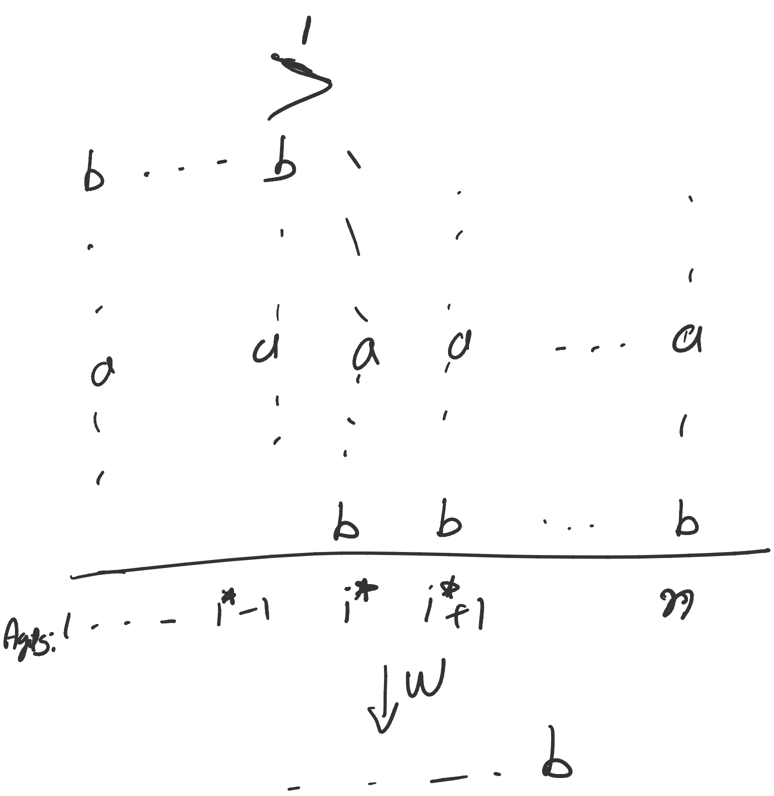
contradicts PE because $c >'_i a$ $\forall i$. ∎

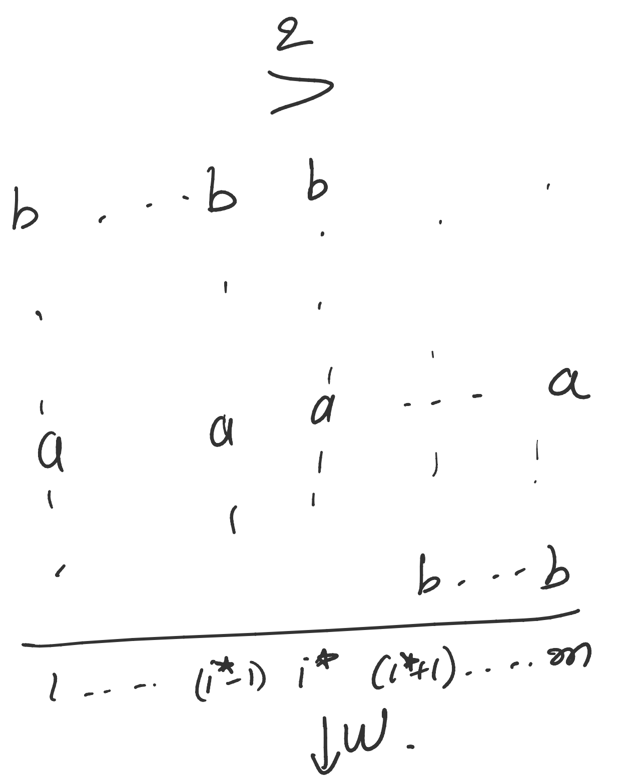
Claim 2: $\exists\, i^* \in N$ that is extremely pivotal. That is, it can move b from bottom to top.

Pf: Consider $\succsim \in L^n$, where each $i \in N$ puts b at the bottom. $\Rightarrow$ in $\succsim_w$ b is at the bottom.

Now move b from bottom to top one agent at a time. For each of these b is at the top or at the bottom in the outcome by Claim 1. Recall, b started at the bottom.

At some point b will switch from bottom to top in the outcome. Let i^* be the corresponding agent.





$\vee$

. . . . — . b $\qquad\qquad$ b - - -

<u>Goal:</u> For any given $\succsim^F \in L^n$, no matter others preference ordering, i^* decides the outcome.

That is, $d \succsim^F_{i^*} e \iff d \succsim^F_{\omega} e \quad \forall d,e \in O$.

<u>Claim 3:</u> i^* is a dictator for any pair not involving b.

<u>Pf:</u> Pick $a,c \in O$, $a,c \neq b$. Let $a \succ^F_{i^*} c$.
(TST: $a \succsim^F_{\omega} c$).

Construct $\succsim^3$ from $\succsim^2$ as follows
(match relative ordering of a & c in $\succsim^F$ for all agents)

① move a to the top of $\succ_{i^*}$

$$a \succsim^3_{i^*} b \succsim^3_{i^*} \cdots \succsim^3_{i^*} c \succsim^3_{i^*} \cdots$$

② $\forall i \neq i^*$ move a & c to match with $\succsim^F_i$

if $a \succsim^F_i c$, $c \succsim^2_i a$
$\quad c \succsim^F_i a$, $a \succsim^2_i c$ $\Rightarrow$ swap a & c in $\succsim^2_i$ to construct $\succsim^3_i$

For all agents.

. . . $\succsim$ . . . of a & b

OBS1: For all agents. Bet $\succ^1$ & $\succ^3$ ordering of a & b is the same.

$$a \succ^1_w b \xRightarrow{IIA} a \succ^3_w b$$

OBS2: $\forall i \in N$, Bet^n $\succ^2$ & $\succ^3$ relative ordering of b & c are unchanged.

$$b \succ^2_w c \xRightarrow{IIA} b \succ^3_w c$$

$$\Rightarrow a \succ^3_w c \xRightarrow[IIA]{} a \succ^F_w c$$

$$(\because \forall i \in N: a \succ^3_i c \iff a \succ^F_i c)$$

<u>Claim 4</u>: i^* decides ordering of (a,b) for any $a \in O$.
That is, $a \succ^F_{i^*} b \Rightarrow a \succ^F_w b$
$b \succ^F_{i^*} a \Rightarrow b \succ^F_w a$

<u>Pf</u>: Consider any $c \neq a, b$. Apply claim 2 for c instead of b $\Rightarrow$ some i^{**} will be pivotal for c.
This implies, by claim 3, that i^{**} decides relative ordering of a & b $W(\succ)$ for any $\succ \in L^n$.

This implies ... ordering of a & b $W(\succ)$ for any $\succ \in L$.
But, for $\stackrel{1}{\succ}$ & $\stackrel{2}{\succ}$ we know that i^* decides position of b & thereby relative ordering of a&b in the corresponding outcome. Hence it must be that $i^{**} = i^*$ ∎