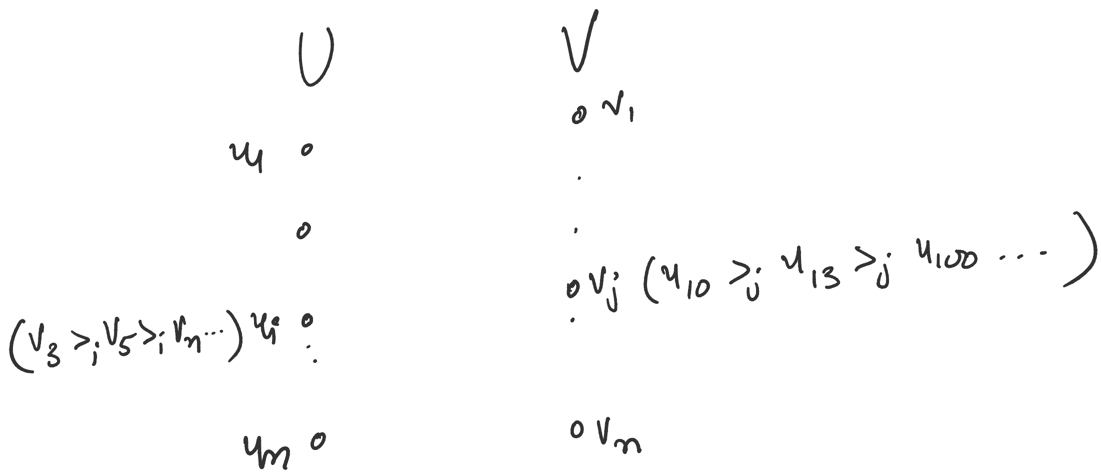
Memchanism Design without Money: Stable Matching, TTC

Thursday, September 7, 2023 10:56 AM

✰ Stable Matching (Marriage):

Appln: hospital-resident matching

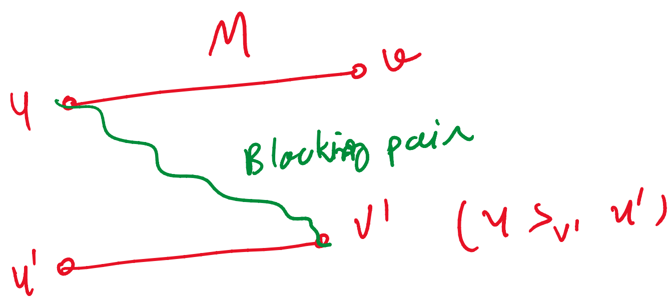
school / course seats assignment...



Goal: Find a "stable matching".

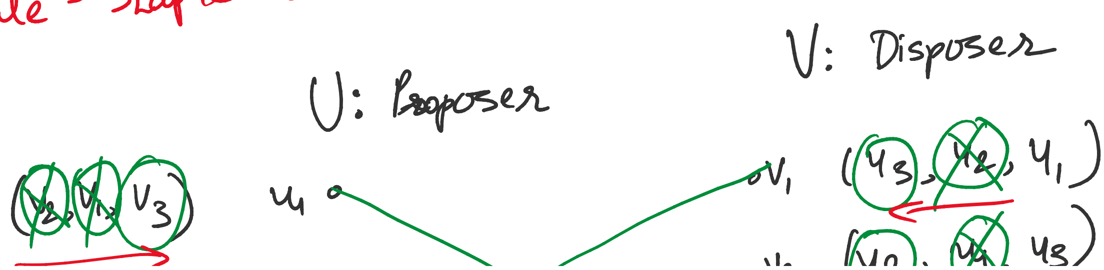
Blocking Pair:

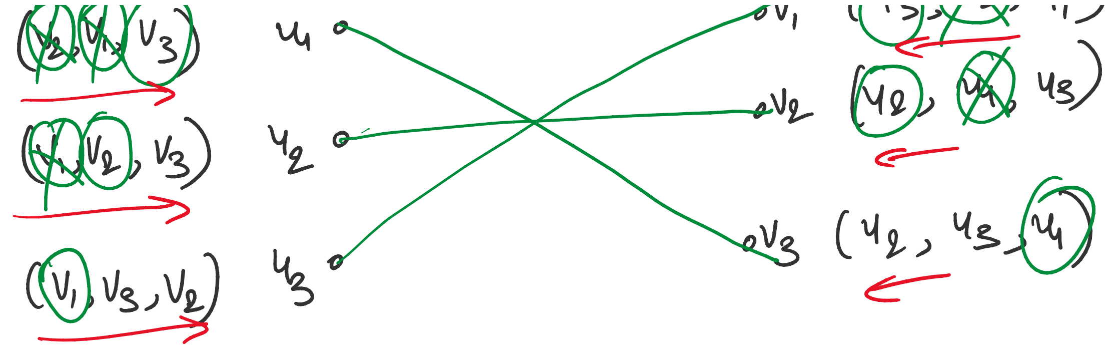
$v' >_u v$



We say that M is stable if there are no blocking pairs.

✰ Gale - Shapley (GS) algo : Differed Acceptance Algo.





Algo: In every round an un-matched vertex on U-side proposes to their "next most preferred" vertex on V-side

OBS1: Has to terminate because in each round one of the u_i's goes down in their list.

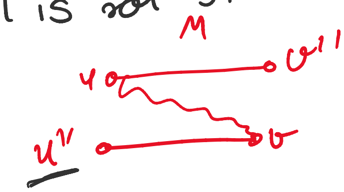
Running Time: $O(n^2)$

OBS2: Algo o/p Perfect Matching.

Thm: GS Algo o/p a Stable Perfect Matching.

Pf: M: o/p of GS Algo. Suppose M is not stable.

$\exists\ (u,v) \notin M$ & is a blocking pair.

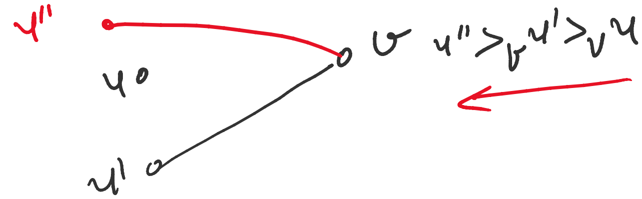


Case I: u never proposed to v during the algo.

$\Rightarrow v'' > v \Rightarrow (u,v)$ is not blocking.

Case II: u did propose to v, but v rejected u for some other u'. $\Rightarrow u' >_v u$

Case II: ... u for some other u'. $\Rightarrow u' >_v u$



$\Rightarrow (u,v)$ is not a blocking pair ∎

Fun Fact:

$u \in U$: $OPT(u)$ = Most pref of $\{v \mid (u,v)$ is in some stable matching$\}$ by u.

$v \in V$: $OPT(v)$ = Most pref from $\{u \mid (u,v)$ in some stable matching$\}$ by v.

U-opt: M^u: $\{(u, OPT(u)) \mid u \in U\}$ why matching?

V-opt: M^v: $\{(OPT(v), v) \mid v \in V\}$ " "

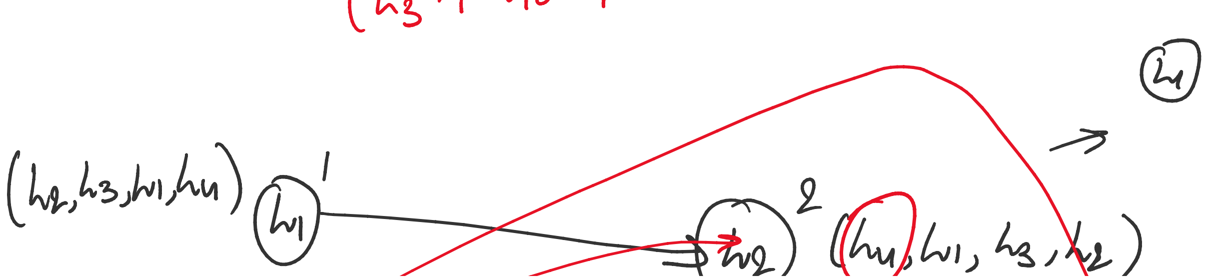
GS Algo o/ps M^u.

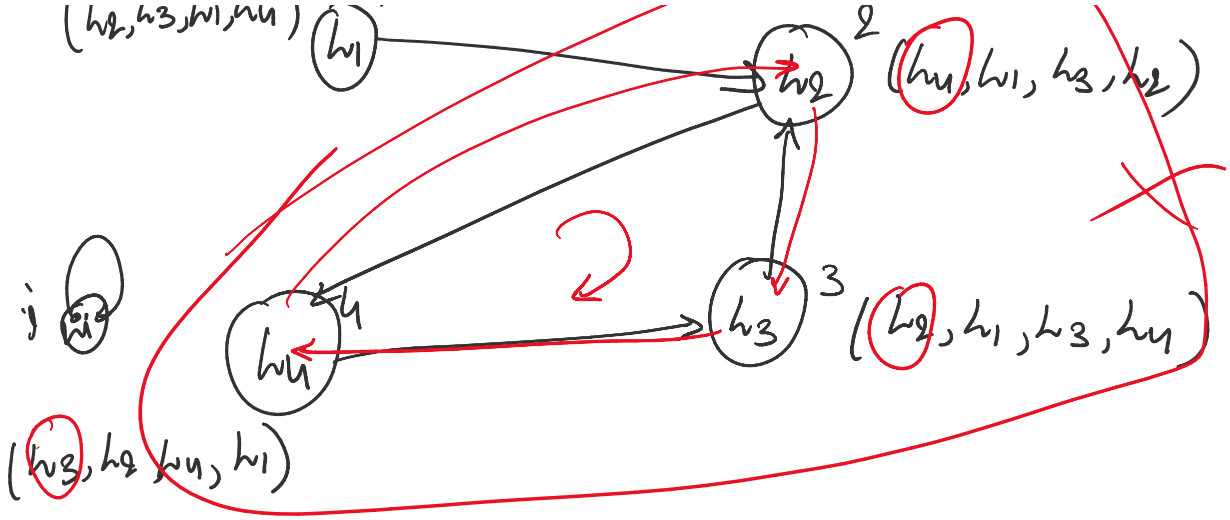
$\boxed{\text{Message: Be the Proposer.}}$

★ Top Trading Cycle (TTC). (Exchange)

1. N: set of agents $\{1, \dots, n\}$.
2. Each agent $i \in N$ owns a house, say h_i. And has a complete preference over set $\{h_1, \dots, h_n\}$

$(h_3 >_i h_{10} >_i h_n >_i h_{n-100} >_i \dots >_i h_i >_i \dots)$





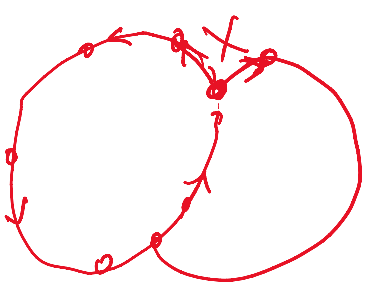
☆ TTC Algo:

① $A = \{1, \dots, n\}$, $r = 1$

② While $A \neq \emptyset$

- 2.1 Each $i \in A$ points to their most preferred available house (houses of agents in A). $\to G$
- 2.2 Until $\exists$ cycle, say C, in G (no sink in $G \Rightarrow G$ not a DAG $\Rightarrow G$ has a cycle)
  - ↳ Exchange houses along C. $A = A \setminus$ agents in C.
- 2.3 $r = r + 1$

Obs:



Every node is part of at most one cycle in G ∵ every node has exactly one outgoing edge.

Claim 0: No agent i gets a house worse than h_i.

Claim 0: No agent i gets a house

DSIC: Dominant Strategy Incentive Compatible.
For each agent it is best to report (or act as per) their true preference list, no matter what others do.

Thm: TTC is DSIC.
Pf: (Induction)
N_r: set of agents who are assigned houses in round r & removed.

Base case: $r=1$. Each $i \in N_1$ gets her most prefered house & hence cannot improve any more by strategizing

For $r \geq 2$:
Inductive Hypotheses: True for agents in $N_1, \ldots, N_{r-1}$

Let $N' = N_1 \cup \ldots \cup N_{r-1}$

Induction: Let $i \in N_r$. Let h_k be the house assigned to i.
Let $h_1 >_i h_2 >_i \ldots >_i h_{k-1} >_i h_k >_i \ldots$

Q: Can i strategize & get one of $h_1, \ldots, h_{k-1}$?
NO!

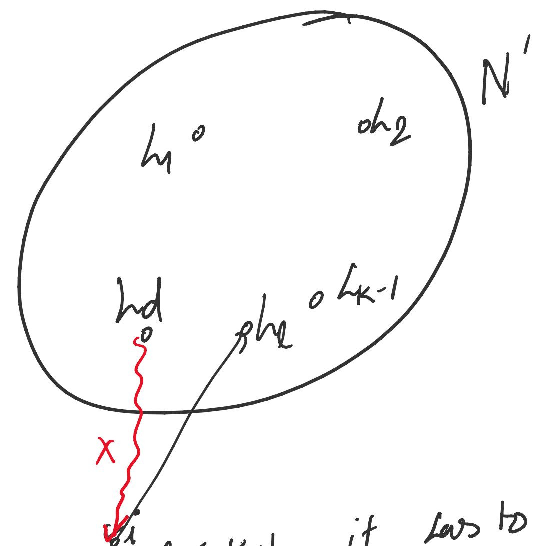
Because,

Obs 1: agent i can only change her outgoing edges in

Obs 1: agent i can only change

every round.

Obs 2:



To get h_ℓ for $\ell \leq k-1$, it has to be part of a cycle in one of the rounds $1, \ldots, (r-1)$.

This requires a new "in-coming" edge to i. A contradiction. ∎