# Fair Division of Indivisible Items (Part II)

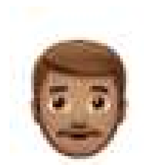
## CS 580

Instructor: Ruta Mehta

ILLINOIS
UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN

Most slides are curtesy Prof. J. Garg

- $N$: set of n agents, 1 ,…, n,
- $M$: set of $m$ indivisible items (like cell phone, painting, etc.)



- Agent $i$ has a valuation function $v_i : 2^m \rightarrow \mathbb{R}$ over subsets of items
  - Monotone: the more the happier

# Last Lecture

- EF: Envy-free, Prop: Proportional
  - Do not exist
- EF1: Envy-free up to one item.
  - Round Robin for additive valuations
  - Envy-cycle elimination for general monotone
- Prop1: Proportional up to one item
  - EF1 implies Prop1 under additive valuations
  - CE + Rounding algorithm for general valuations.
- EFX: Envy-free up to any item
- Open:
  - EF1+PO for submodular valuations
  - EFX with 3 agents. EFX with 4 agents under additive valuations
  - …

# Proportionality

- A set $N$ of $n$ agents, a set $M$ of $m$ indivisible goods

- Proportionality: Allocation $A = (A_1, \ldots, A_n)$ is proportional if each agent gets at least $1/n$ share of all items:

$$v_i(A_i) \geq \frac{v_i(M)}{n}, \qquad \forall i \in N$$

Cut-and-choose?

# Maximin Share (MMS) [B11]

Cut-and-choose.

- Suppose we allow agent $i$ to propose a partition of items into $n$ bundles with the condition that $i$ will choose at the end.
- Clearly, $i$ partitions items in a way that maximizes the value of her least preferred bundle.
- $\mu_i :=$ Maximum value of $i's$ least preferred bundle

# Maximin Share (MMS) [B11]

Cut-and-choose.

- Suppose we allow agent $i$ to propose a partition of items into $n$ bundles with the condition that $i$ will choose at the end
- Clearly, $i$ partitions items in a way that maximizes the value of her least preferred bundle
- $\mu_i :=$ Maximum value of $i's$ least preferred bundle

- $\Pi \coloneqq$ Set of all partitions of items into $n$ bundles
- $\mu_i \coloneqq \max_{(A_1,\ldots,A_n)\in\Pi} \min_{k\in[n]} v_i(A_k)$

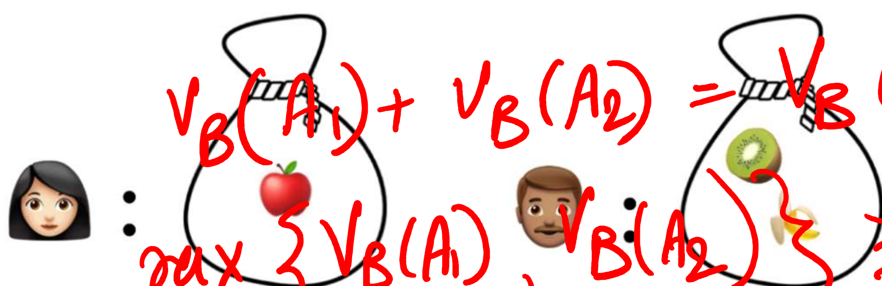
- MMS Allocation: $A$ is called MMS if $v_i(A_i) \geq \mu_i$ , $\forall i$
- Additive valuations: $v_i(A_i) = \sum_{j\in A_i} v_{ij}$

# MMS value/partition/allocation

| Agent\Items | 🍎 | 🍌 | 🥝 |
|---|---|---|---|
| 👩 | 3 | 1 | 2 |
| 👨 | 4 | 4 | 5 |

| 👩 | 🍎 | 🍌🥝 |
|---|---|---|
| Value | 3 | 3 |
| MMS Value | 3 | |

| 👨 | 🍎🍌 | 🥝 |
|---|---|---|
| Value | 8 | 5 |
| MMS Value | 5 | |



$$v_B(A_1) + v_B(A_2) = v_B(M)$$

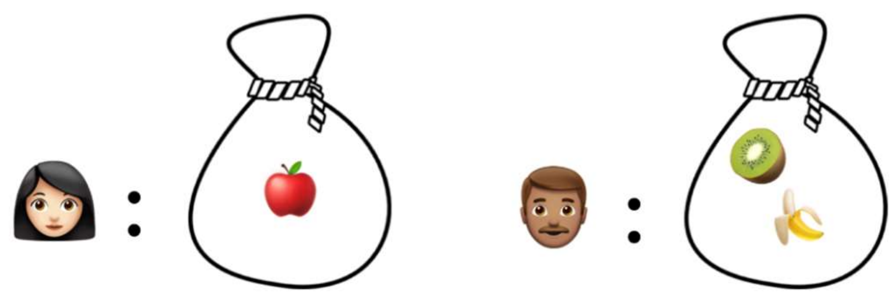
$$\max\{v_B(A_1), v_B(A_2)\} \geq \frac{v_B(M)}{2} \geq \mu_B$$

# MMS value/partition/allocation

| Agent\Items | 🍎 | 🍌 | 🥝 |
|---|---|---|---|
| 👩 | 3 | 1 | 2 |
| 👨 | 4 | 4 | 5 |

| 👩 | 🍎 | 🍌🥝 |
|---|---|---|
| Value | 3 | 3 |
| MMS Value | 3 | |

| 👨 | 🍎🍌 | 🥝 |
|---|---|---|
| Value | 8 | 5 |
| MMS Value | 5 | |



**Finding MMS value is NP-hard!**

# What is Known?

- PTAS for finding MMS value [W97]

Existence (MMS allocation)?

- $n = 2$ : yes 


  $\Longrightarrow$ A PTAS to find $(1 - \epsilon)$-MMS allocation for any $\epsilon > 0$
- $n \geq 3$ : NO [PW14]

# What is Known?

- PTAS for finding MMS value [W97]

Existence (MMS allocation)?

- $n = 2$ : yes 


  $\Longrightarrow$ A PTAS to find $(1 - \epsilon)$-MMS allocation for any $\epsilon > 0$
- $n \geq 3$ : NO [PW14]

- $\alpha$-MMS allocation for $\alpha \in [0,1]$: $v_i(A_i) \geq \alpha . \mu_i$
  - $2/3$-MMS exists [PW14, AMNS17, BK17, KPW18, GMT18]
  - $3/4$-MMS exists [GHSSY18]
  - $(3/4 + O(1))$-MMS exists [AG23]
  - $39/40$-MMS does not exist [Feige et al. 2020]

# Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$ WHY?

MMS partition of agent i.

$A_1$ . . . $A_n$

$a_k = v_i(A_k)$ $\quad \sum_{k=1}^{n} a_k = v_i(M) = \sum_{j \in M} v_{ij} = n$

$$\mu_i = \min_{k=1}^{n} a_k \leq \frac{\sum_{k=1}^{n} a_k}{n} = \frac{v_i(M)}{n} = \frac{\sum_{j \in M} v_{ij}}{n} = \frac{n}{n} = 1$$

# Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \leq 1$
- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

# Properties

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \leq 1$
- Ordered Instance: We can assume that agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

| | | | | | |
|---|---|---|---|---|---|
| | 3 | 1 | 2 | 5 | 4 |
| | 4 | 4 | 5 | 3 | 2 |

→

| | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| | 5 | 4 | 3 | 2 | 1 |
| | 5 | 4 | 4 | 3 | 2 |

# Challenge

- Allocation of high-value items!
- If for all $i \in N$
  - $v_i(M) = n \Rightarrow \mu_i \leq 1$
  - $\boldsymbol{v_{ij} \leq \epsilon, \forall i, j}$

Goal: $(1-\epsilon)$-MMS allocation.

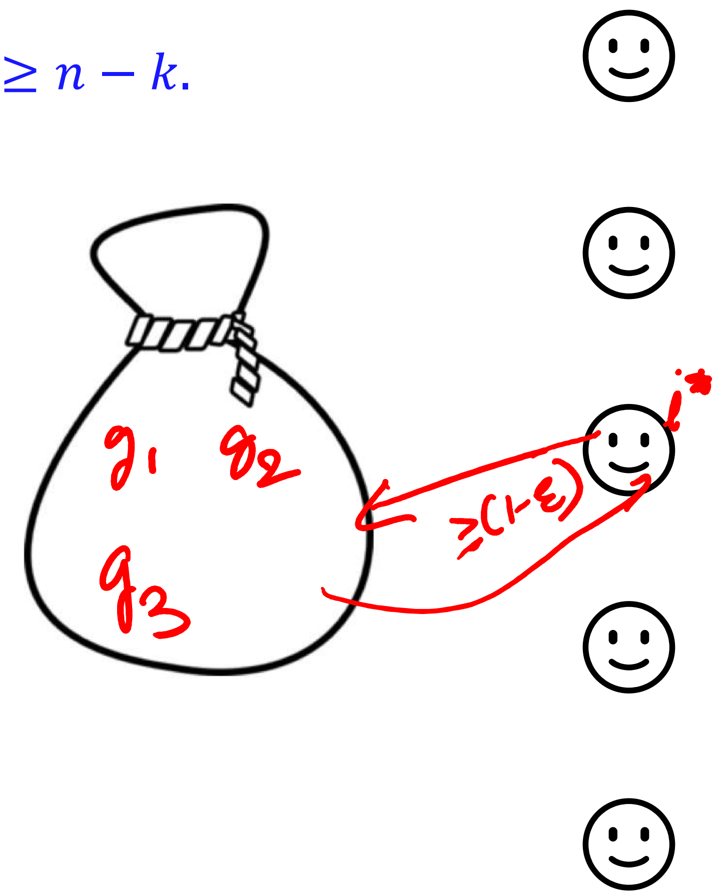
$v_{ij} \leq \epsilon, \forall i, j$

Claim: After round k, if $i$ remains then $v_i$(remaining goods) $\geq n - k$.

$v_i(g_1, g_2, g_3, g_4) < (1-\varepsilon) + \varepsilon$
$\leq 1$.

$S = \{g_1, g_2, g_3, g_4\} \rightarrow i^*$

$v_i(M \setminus S) \geq (n-1)$



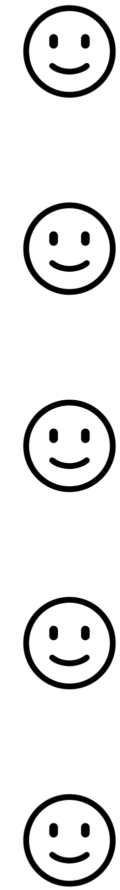

Bag Filling Algorithm:

Repeat until every agent is assigned a bag

- Start with an empty bag $B$
- Keep adding items to $B$ until some agent $i$ values it $\geq (1 - \epsilon)$
- Assign $B$ to $i$ and remove both

$v_{ij} \leq \epsilon, \forall i, j$

Claim: After round k, if $i$ remains then $v_i$(remaining goods) $\geq n - k$.



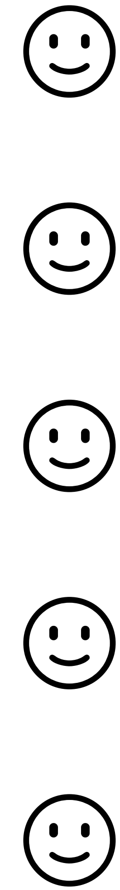
## Bag Filling Algorithm:

Repeat until every agent is assigned a bag

- Start with an empty bag $B$
- Keep adding items to $B$ until some agent $i$ values it $\geq (1-\epsilon)$
- Assign $B$ to $i$ and remove them both

$v_{ij} \leq \epsilon, \forall i, j$

Thm: Every agent gets at least $(1 - \epsilon)$.



Bag Filling Algorithm:

Repeat until every agent is assigned a bag

- Start with an empty bag $B$
- Keep adding items to $B$ until some agent $i$ values it $\geq (1 - \epsilon)$
- Assign $B$ to $i$ and remove them both

# Warm Up: 1/2-MMS Allocation

$(1-\varepsilon) = 1/2$ - MMS

Goal: $V_i(A_i) \geq \frac{\mu_i}{2}$

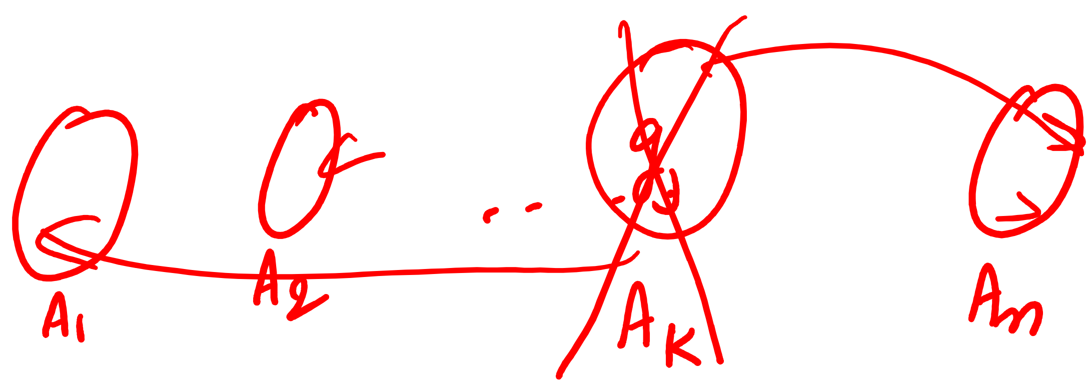
- If all $v_{ij} \leq 1/2$ then?
  - Done, using bag filling.

$\frac{\mu_i}{2} \leq 1/2$

- What if some $v_{i^*j} > \frac{1}{2}$?

$\{g_j\} \rightarrow i^*$  Reduced instance: $[n] \setminus \{i^*\}$, $M \setminus \{g_j\} \rightarrow \mu_i'$

$i \neq i^*$  MMS partition in the original instance  $V_i$  $\mu_i$



# Valid Reductions

- Normalized valuations
  - Scale free: $v_{ij} \leftarrow c.v_{ij}, \forall j \in M$
  - $\sum_j v_{ij} = n \quad \Rightarrow \quad \mu_i \leq 1$
- Ordered Instance: Agents' order of preferences for items is same: $v_{i1} \geq v_{i2} \geq \cdots v_{im}, \forall i \in N$

- Valid Reduction ($\alpha$-MMS): If there exists $S \subseteq M$ and $i^* \in N$
  - $i^*$gets $\alpha$-MMS value from S $(v_{i^*}(S) \geq \alpha.\mu_{i^*}^n(M))$
  - Once we give $S$ to $i^*$, and remove both, the MMS value of the remaining agents does not decrease. $\mu_i^{n-1}(M \setminus S) \geq \mu_i^n(M), \forall i \neq i^*$

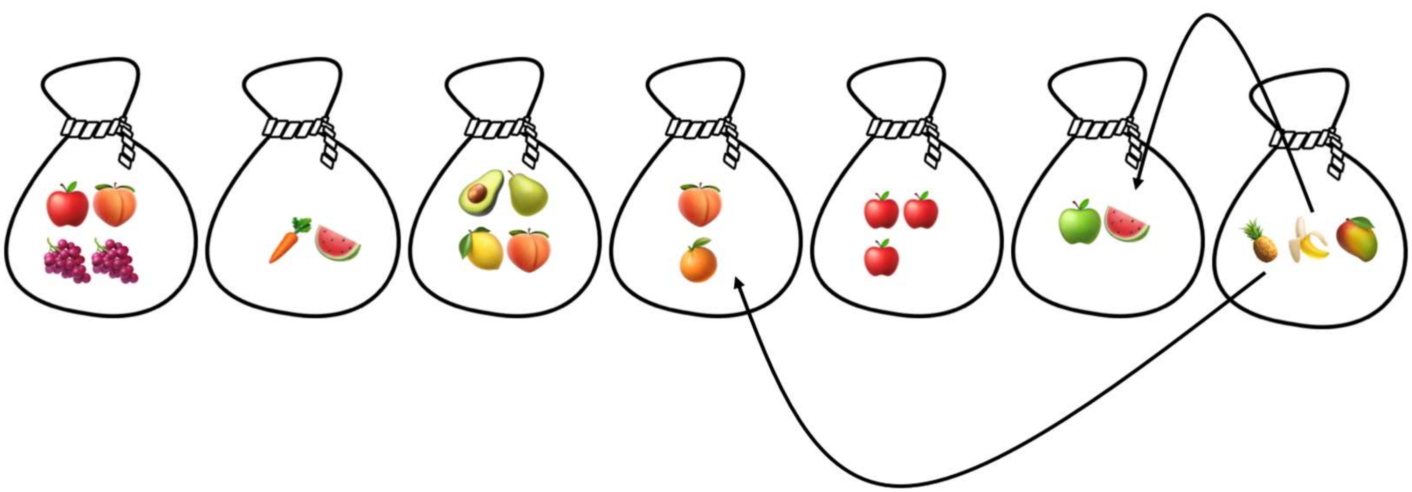
$\Rightarrow$ reduce the instance size!

**Claim.** Suppose agent i $\neq i^*$ gets $A_i$ in an $\alpha$-MMS allocation of $M \setminus S$ to agents $N \setminus \{i^*\}$, then $(A_1, \ldots, A_{i^*-1}, S, A_{i^*+1}, \ldots, A_n)$ is an $\alpha$-MMS allocation in the original instance.

# 1/2-MMS Allocation

**Step 1:** Valid Reductions

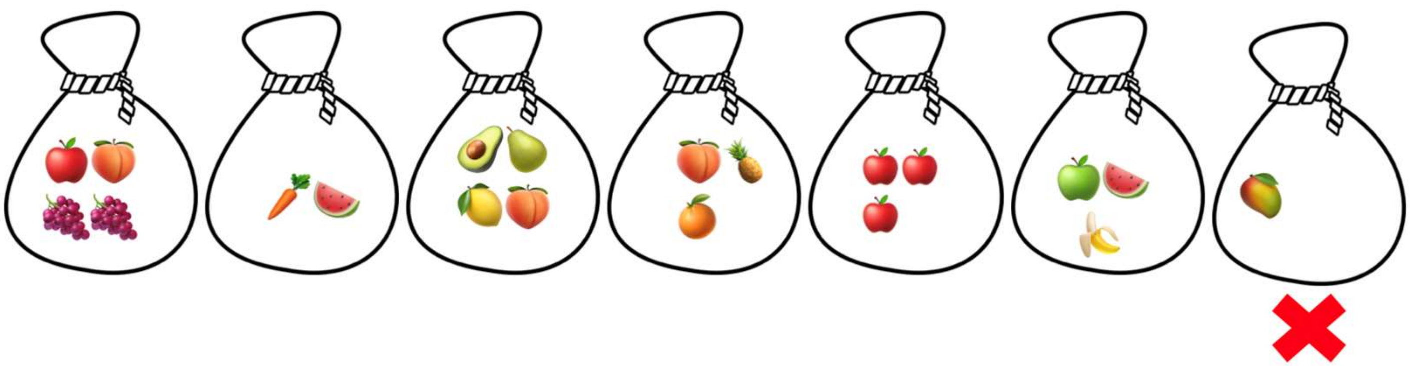
- If $v_{i^*1} \geq 1/2$ then assign item 1 to $i^*$



# 1/2-MMS Allocation

**Step 1:** Valid Reductions

- If $v_{i^*1} \geq 1/2$ then assign item 1 to $i^*$



# 1/2-MMS Allocation

- Re-normalization

Step 0: Normalized Valuations: $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$

Step 1: Valid Reductions

- If $v_{i^*1} \geq 1/2$ then assign item 1 to $i^*$. Remove good 1 and agent $i^*$
- After every valid reduction, normalize valuations

Step 2: Bag Filling

# 2/3-MMS Allocation [GMT19]

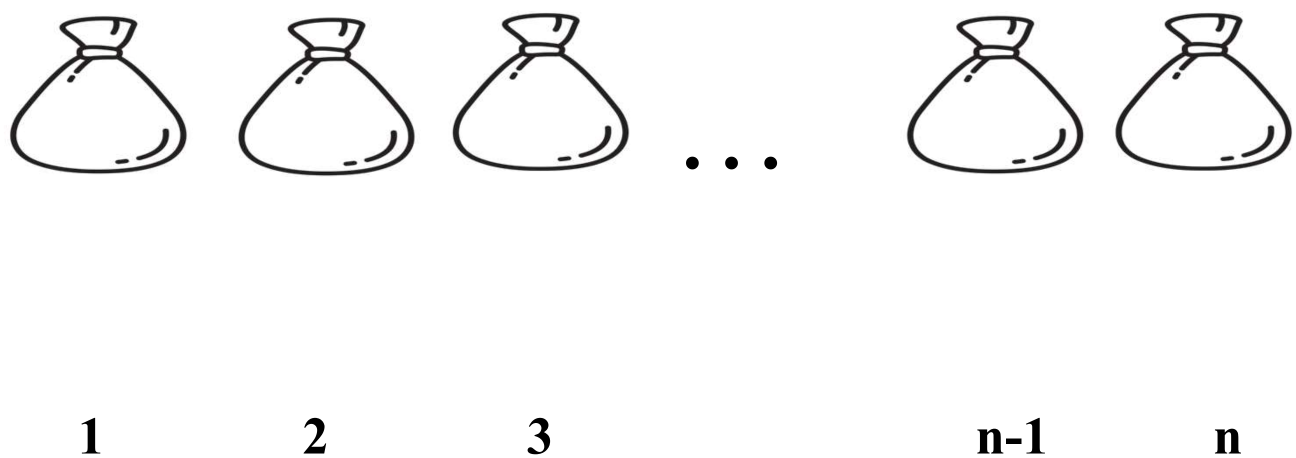
- If all $v_{ij} \leq 1/3$ then?

$= \varepsilon$

$(1-\varepsilon) = \frac{2}{3}$ - MMS

**Step 1:** Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to $i^*$



# 2/3-MMS Allocation [GMT19]

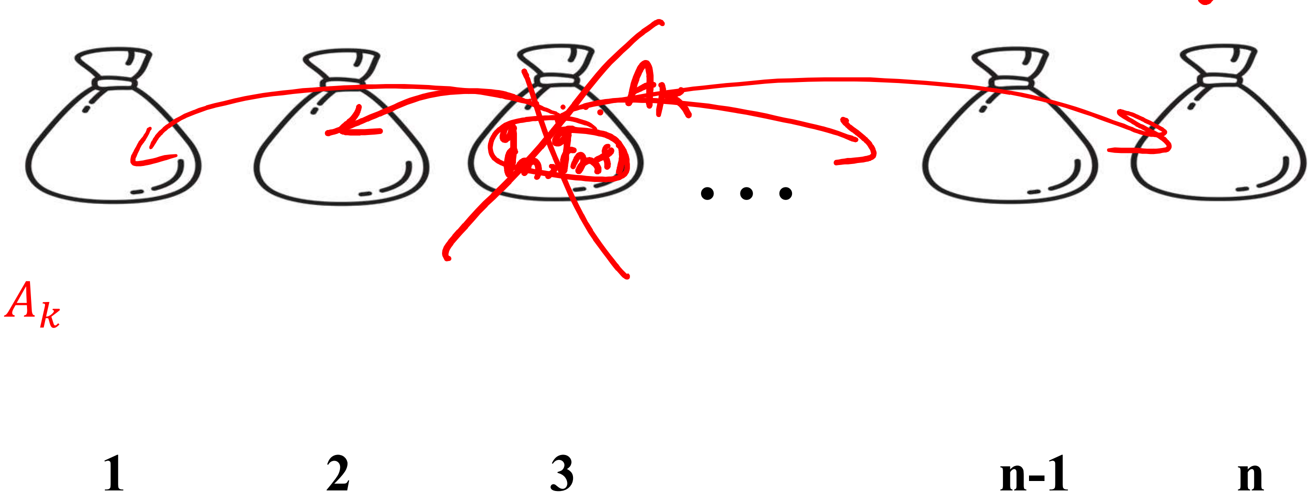
**Step 1:** Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to $i^*$
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to $i^*$

$g_1 \ldots \ldots g_{n+1}$

Q: why valid reduction?

For agent $i \neq i^*$, let the MMS defining partition be




Case I:
$n, n+1 \in A_k$

# 2/3-MMS Allocation [GMT19]

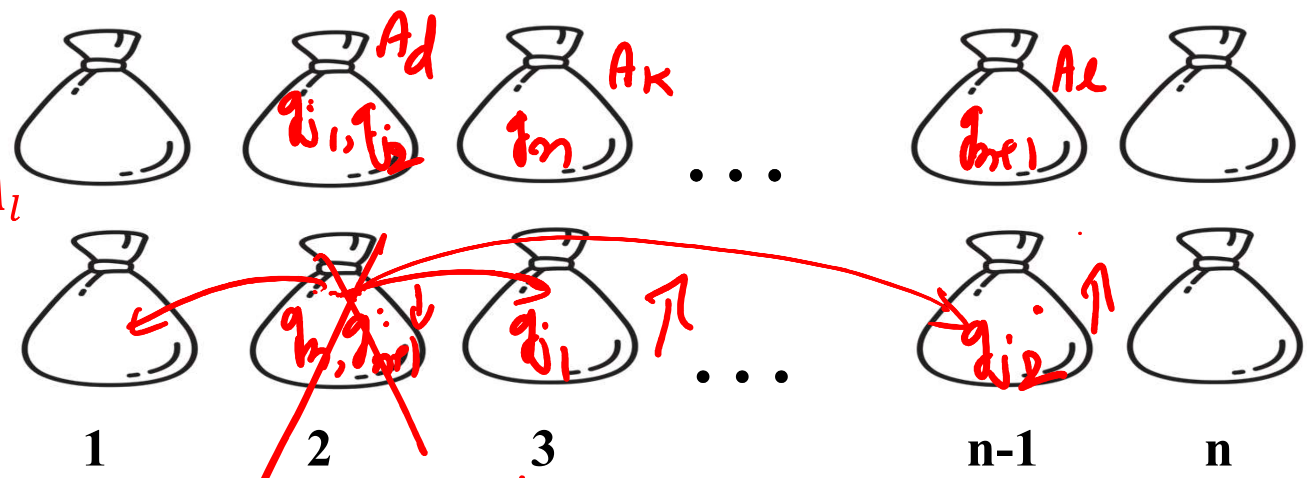
**Step 1:** Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to $i^*$
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to $i^*$

Q: Why valid reduction?

For agent $i \neq i^*$, let the MMS defining partition be

Case II:
$n \in A_k$
$(n+1) \in A_l$




$\exists A_d$, with items $j_1 < j_2 \leq (n+1)$. Then, swap items $j_1$ and n, and items $j_2$ and (n+1). This may only increase $v_i(A_k)$ & $v_i(A_l)$ because $v_i(j_1) \geq v_i(n)$ & $v_i(j_2) \geq v_i(n+1)$.

# 2/3-MMS Allocation [GMT19]

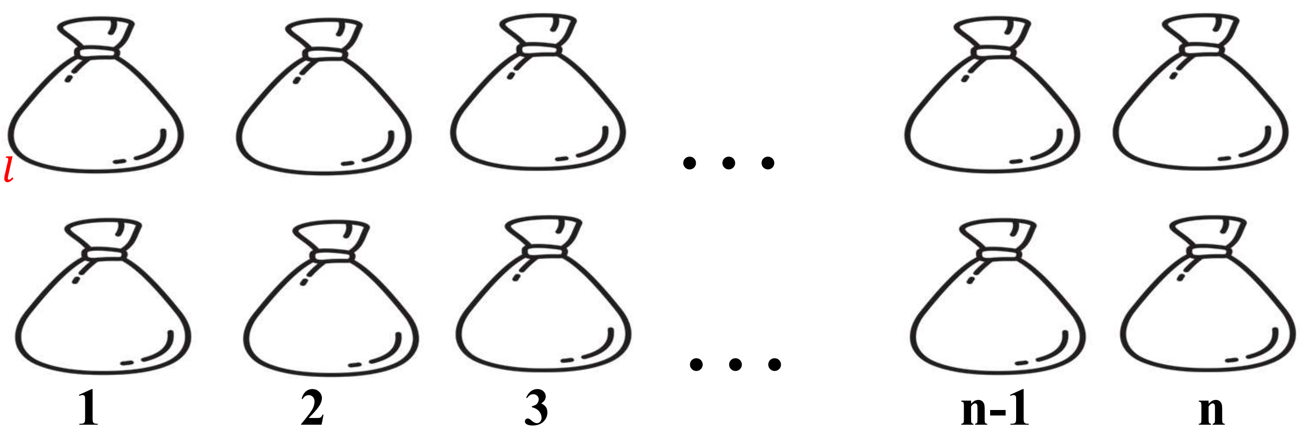
**Step 1:** Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to $i^*$
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to $i^*$

Q: Why valid reduction?

For agent $i \neq i^*$, let the MMS defining partition be

Case II:
$n \in A_k$
$(n+1) \in A_l$




$\exists A_d$, with items $j_1 < j_2 \leq (n+1)$.
Then, swap items $j_1$ and n, and items $j_2$ and (n+1).
Move remaining items of $A_d$ to other bundles and remove $A_d$.

# 2/3-MMS Allocation [GMT19]

**Step 1:** Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to $i^*$
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to $i^*$

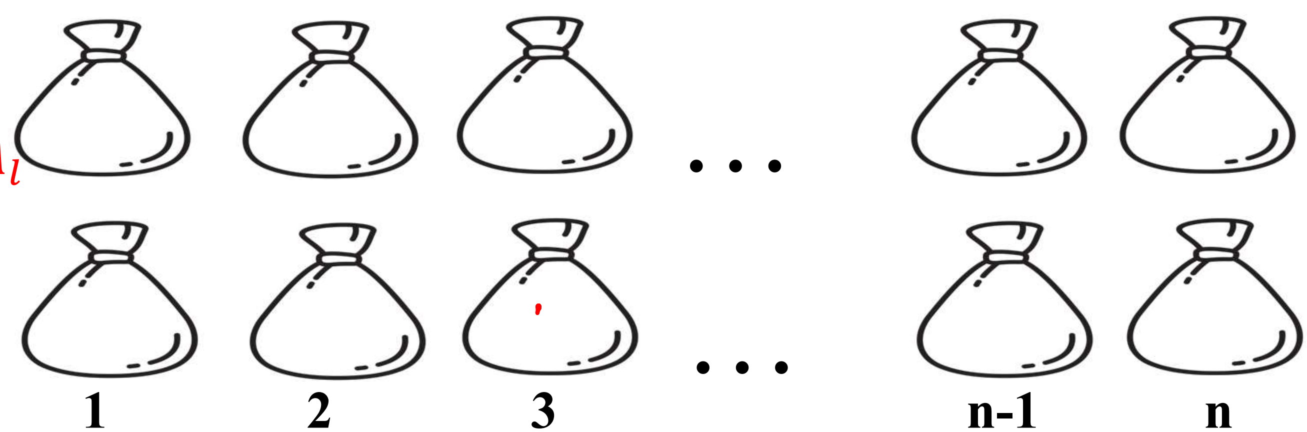
Q: Why valid reduction?

$g_1 \ldots g_n$ $g_{n+1} \ldots g_{2n}$

$\leq 2/3$ $\leq 1/3$

For agent $i \neq i^*$, let the MMS defining partition be

Case II:
$n \in A_k$
$(n+1) \in A_l$




$\exists A_d$, with items $j_1 < j_2 \leq (n+1)$. Then, swap items $j_1$ and n, and items $j_2$ and (n+1). Move remaining items of $A_d$ to other bundles and remove $A_d$.

Again, value of none of the remaining bundles has decreased.
$\Rightarrow$ MMS value of agent $i$ has only increased in the reduced instance.

## Step 1: Valid Reductions

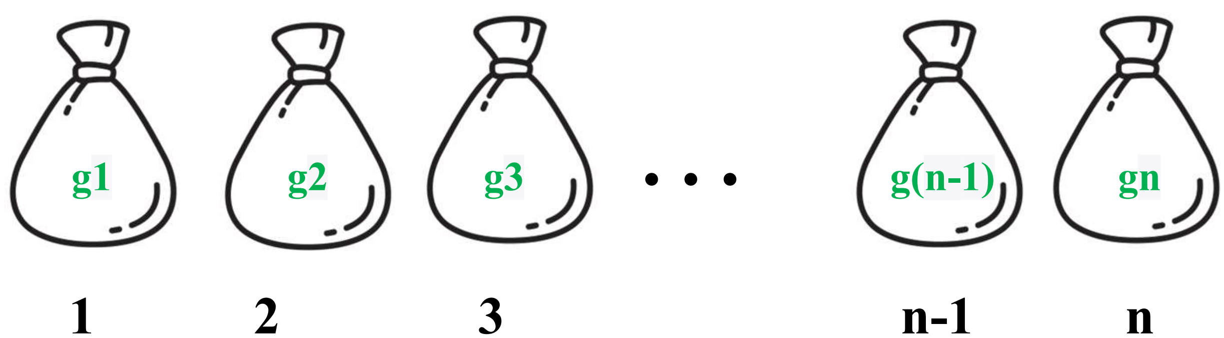
- If $v_{i^*1} \geq 2/3$ then assign item 1 to $i^*$
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to $i^*$

## Step 2: Generalized Bag Filling with $\epsilon = \frac{1}{3}$

- Initialize $n$ bags $\{B_1, \ldots B_n\}$ with $B_k = \{k\}, \forall k$.
- Assign items starting from $(n+1)$th to the first available bag, and give it to the first agent who shouts (values it at least $2/3 = (1-\epsilon)$).

After Step 1,
For each agent $i$,
$v_{ij} < \frac{2}{3}, \forall j \leq n$
$v_{ij} < \frac{1}{3}, \forall j > n$

**Claim.** If agent $i^*$ is the first to shout, then for any agent $i \neq i^*$ the bag is of value at most 1.



# 2/3-MMS Allocation [GMT19]

- (Re)normalization

Step 0: Normalized Valuations: $\sum_j v_{ij} = n \Rightarrow \mu_i \leq 1$

Step 1: Valid Reductions

- If $v_{i^*1} \geq 2/3$ then assign item 1 to $i^*$
- If $v_{i^*n} + v_{i^*(n+1)} \geq 2/3$ then assign $\{n, n+1\}$ to $i^*$
- After every valid reduction, normalize valuations

Step 2: Generalized Bag Filling with $\epsilon = \frac{1}{3}$

- Initialize $n$ bags $\{B_1, \ldots B_n\}$ with $B_k = \{k\}, \forall k$

# Chores

- $N$: set of $n$ agents, 1 ,…, n,
- $M$: set of $m$ indivisible chores


- Agent $i$ has a disutility function $d_i : 2^m \rightarrow \mathbb{R}_+$ over subsets of items
  - Monotone: the more the **un-**happier
- Additive: $d_i(S) = \sum_{j \in S} d_{ij}$, for any subset $S \subseteq M$

- $N$: set of $n$ agents, 1 ,…, n,
- $M$: set of $m$ indivisible chores
- Agent $i$ has a disutility function $d_i : 2^m \rightarrow \mathbb{R}_-$ over subsets of items
  - Additive: $d_i(S) = \sum_{j \in S} d_{ij}$, for any subset $S \subseteq M$

$$\text{Allocation } A = (A_1, \dots, A_n)$$

**EF1: No agent envies another after removing one of her chores.**

$$\forall i, k \in N, \qquad d_i(A_i \backslash c) \leq d_i(A_k), \qquad \exists c \in A_i$$

# EF1: Algorithms

## Round Robin

1. Order agents arbitrarily.
2. Let them pick their best chore (least painful chore), one-at-a-time, in that order.

**Observations:**

- If agent $k$ picks the last chore, then agent $(k + 1)$ does not envy anyone. Why?

# EF1: Algorithms

## Envy-cycle-elimination

1. $A = (\emptyset, \dots, \emptyset)$
2. While there are unassigned chores
   1. Construct envy-graph of A and remove any cycles.
   2. Give an unassigned chore to ….. ??

Observations:

- Cycle elimination does not increase any agent's disutility.
- Giving a chore to sink maintains EF1. Why?

# MMS

- $N$: set of $n$ agents, 1 ,…, n,
- $M$: set of $m$ indivisible chores
- Agent $i$ has a disutility function $d_i : 2^m \to \mathbb{R}_-$ over subsets of items
  - Additive: $d_i(S) = \sum_{j \in S} d_{ij}$, for any subset $S \subseteq M$

- $\Pi :=$ Set of all partitions of items into $n$ bundles

**MMS value:** $\mathrm{MMS_i} = \mu_i = \min_{A \in \Pi} \max_{A_k \in A} d_i(A_k)$

$\alpha$-MMS allocation for $\alpha \geq 1$: $\forall i,\ d_i(A_i) \leq \alpha \mu_i$

1-MMS allocation may not exist!

# EF1 to $\alpha$-MMS

Claim. If $(A_1, \dots, An)$ is EF1 then it is 2-MMS

Observations: $\mu_i \geq \frac{d_i(M)}{n}$ and $\mu_i \geq \max_{j \in M} d_{ij}$

Proof.

# Summary

## Covered

- Additive Valuations:
  - ½-MMS allocation (poly-time algorithm)
  - 2/3-MMS allocation (polynomial-time algorithm)

## State-of-the-art

- $\left(\frac{3}{4}+\right)$-MMS allocation [GT20]
- More general valuations
  - MMS [GHSSY18]
- Groupwise-MMS [BBKN18]
- Chores: 11/9-MMS [HL19]

## Major Open Questions (additive)

- $c$-MMS + PO: polynomial-time algorithm for a constant $c > 0$
- Existence of 4/5-MMS allocation? For 5 agents?

# References (Indivisible Case).


[AMNS17] Georgios Amanatidis, Evangelos Markakis, Afshin Nikzad, and Amin Saberi. “Approximation algorithms for computing maximin share allocations”. In: *ACM Trans. Algorithms 13.4 (2017)*

[BBKN18] Siddharth Barman, Arpita Biswas, Sanath Kumar Krishnamurthy, and Y. Narahari. “Groupwise maximin fair allocation of indivisible goods”. In: *AAAI 2018*

[BK17] Siddharth Barman and Sanath Kumar Krishna Murthy. “Approximation algorithms for maximin fair division”. In *EC 2017*

[BK19] Siddharth Barman and Sanath Kumar Krishnamurthy. “On the Proximity of Markets with Integral Equilibria” In *AAAI 2019*

[BKV18] Siddharth Barman, Sanath Kumar Krishnamurthy, and Rohit Vaish. Finding fair and efficient allocations. In: *EC 2018*

[B11] Eric Budish. “The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes”. In: *J. Political Economy* 119.6 (2011)

[CKMPSW14] Ioannis Caragiannis, David Kurokawa, Herve Moulin, Ariel Procaccia, Nisarg Shah, and Junxing Wang. “The Unreasonable Fairness of Maximum Nash Welfare”. In: *EC* 2016

[GMT19] Jugal Garg, Peter McGlaughlin, and Setareh Taki. “Approximating Maximin Share Allocations”. In: *SOSA@SODA 2019*

[GT20] Jugal Garg and Setareh Taki. “An Improved Approximation Algorithm for Maximin Shares”. In: *EC 2020*

[GHSSY18] Mohammad Ghodsi, MohammadTaghi HajiAghayi, Masoud Seddighin, Saeed Seddighin, and Hadi Yami. “Fair allocation of indivisible goods: Improvement and generalization”. In *EC 2018*

[HL19] Xin Huang and Pinyan Lu. “An algorithmic framework for approximating maximin share allocation of chores”. In: *arxiv:1907.04505*

[KBKZ09] Bart de Keijzer, Sylvain Bouveret, Tomas Klos, and Yingqian Zhang. “On the Complexity of Efficiency and Envy-Freeness in Fair Division of Indivisible Goods with Additive Preferences”. In: *Algorithmic Decision Theory (ADT)*. 2009

[KPW18] David Kurokawa, Ariel D. Procaccia, and Junxing Wang. “Fair Enough: Guaranteeing Approximate Maximin Shares”. In: *J. ACM 65.2 (2018), 8:1–8:27*

[PW14] Ariel D Procaccia and Junxing Wang. “Fair enough: Guaranteeing approximate maximin shares”. In *EC 2014*

[W97] Gerhard J Woeginger. “A polynomial-time approximation scheme for maximizing the minimum machine completion time”. In: *Operations Research Letters 20.4 (1997)*