Revenue Maximization

Tuesday, November 7, 2023 10:51 AM

Recall

→ DSIC Indirect Mechanism ⇒ DSIC Direct Mechanism

English, Dutch

↓ solicit bids.

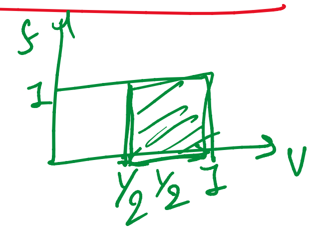
→ Single Parameter

① DSIC ② $\boxed{\text{S.W. maximization}}$ ③ poly-time.

↓

② Revenue maximization.

Example: ① single item, one bidder. $v \sim U[0,1]$



$Rev[\text{second-price Auc}] = 0$

→ Post take-it-or-leave-it (posted) price $p = 1/2$

$$\mathbb{E}[\text{Rev of } \uparrow] = \tfrac{1}{2} \Pr[v \geq \tfrac{1}{2}]$$

$$= \tfrac{1}{2} \cdot \tfrac{1}{2} = \tfrac{1}{4}$$

② Two-buyers $v_1, v_2 \sim U[0,1]$

$$\mathbb{E}[\text{Rev of second price}] = \mathbb{E}_{v_1, v_2 \sim U[0,1]}[\min\{v_1, v_2\}] = 1/3$$

reserve price.

→ Highest bidder wins & pays $\max\{\underbrace{1/2}, \text{second-highest bid}\}$

$$\mathbb{E}[\text{Rev } \uparrow] = \frac{5}{12} > 1/3$$

Goal: Design DSIC, revenue maximizing mechanism.

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★ Single Parameter:

→ $N = \{1, \dots, n\}$

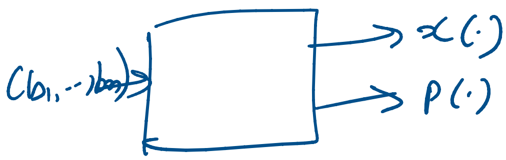
→ $i \in N$, $v_i \sim D_i$

X = set of feasible allocations. $D = \times_{i \in N} D_i$

f_i = density func.

F_i = cumm. density func.

$\Pr[v_i \le a] = F(a)$

Myerson: 

Design Space

(x, p) is DSIC $\Longleftrightarrow$ $\boxed{x(\cdot) \text{ is monotone}}$ &

$p(\cdot)$ is a specific formula wrt $x(\cdot)$

Goal: Design allocation rule $x(\cdot)$ that leads to revenue maximization.

(Proof by reverse engineering; Assume DSIC)

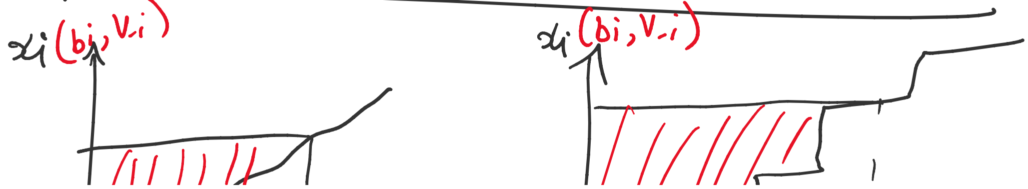
$\forall i,\ b_i = v_i$

$$\max \mathop{\mathbb{E}}_{\substack{v=(v_1,\dots,v_n)\\ \sim D}}[Rev] = \max \mathop{\mathbb{E}}_{v \sim D}\left[\sum_{i=1}^{n} p_i(v)\right]$$

$$= \max \sum_{i=1}^{n} \mathop{\mathbb{E}}_{v \sim D}[p_i(v)]$$

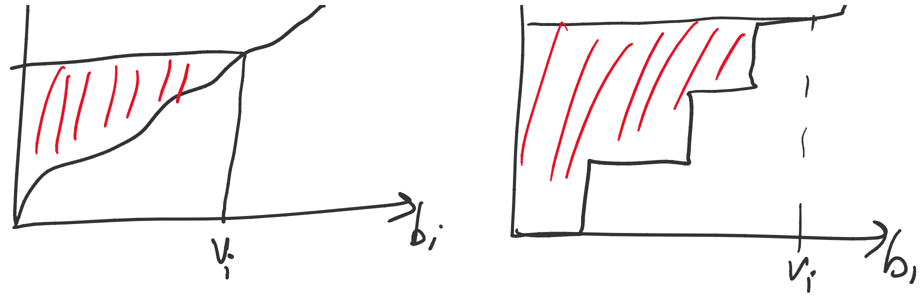
$$= \max \sum_{i=1}^{n} \mathop{\mathbb{E}}_{v_{-i} \sim D_{-i}} \boxed{\mathop{\mathbb{E}}_{v_i \sim D_i}[p_i(v_i, v_{-i})]} \rightarrow (1)$$

Myerson:



Myerson...

$p_i(v_i, v_{-i}) = \int_0^{v_i} z\, x_i'(z, v_{-i})\, dz$



Fix $i \in N$, $v_{-i} \sim D_{-i}$. Let $x(\cdot) = x_i(\cdot)$, $p(\cdot) = p_i(\cdot)$, $f(\cdot) = f_i(\cdot)$, $F(\cdot) = F_i(\cdot)$

$$(1) = \mathbb{E}_{v_i \sim D_i}\left[p_i(v_i, v_{-i})\right] = \int_0^{v_{max}} p_i(v_i, v_{-i}) \cdot f(v_i)\, dv_i$$

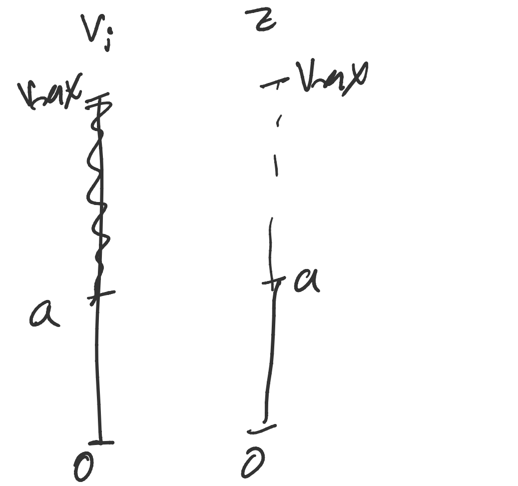
$$\underset{(\because \text{Myerson})}{=} \int_0^{v_{max}} \left(\int_0^{v_i} z\, x_i'(z, v_{-i})\, dz \right) f(v_i)\, dv_i$$

$$\underset{(\text{swap sums})}{=} \int_0^{v_{max}} \underbrace{\left(\int_z^{v_{max}} f(v_i)\, dv_i \right)}_{(1-F(z))} z\, x_i'(z, v_{-i})\, dz$$

$$= \int_0^{v_{max}} \underbrace{(1-F(z)) \cdot z}_{f} \cdot \underbrace{x_i'(z, v_{-i})}_{g'}\, dz$$

$$= \Big[(1-F(z)) \cdot z \cdot x_i(z, v_{-i}) \Big]_0^{v_{max}} - \int_0^{v_{max}} (1 - F(z) - z f(z))\, x_i(z, v_{-i})\, dz$$

v_{max}



For what all v_i's $z = a$ occurs

$$\int f g' = fg - \int f' g$$

$$= 0 + \int_0^{V_{max}} \left(z f(z) - \left(1-F(z)\right)\right) x_i(z, v_{-i})\, dz$$

$$\underset{v_i := z}{=} \int_0^{V_{max}} \frac{\left[v_i f(v_i) - \left(1-F(v_i)\right)\right]}{f(v_i)} x_i(v_i, v_{-i})\, dv_i \cdot f(v_i)$$

$$= \int_0^{V_{max}} \boxed{\left[v_i - \underbrace{\frac{\left(1-F(v_i)\right)}{f(v_i)}}\right]} x_i(v_i, v_{-i}) \cdot f(v_i)\, dv_i$$

$\phi_i(v_i)$ = Virtual value of agent i.

Information rent of auctioneer for not knowing exact v_i & only " D_i

$$= \int_0^{V_{max}} \underbrace{\phi_i(v_i)\, x_i(v_i, v_{-i})}_{\text{Virtual welfare of agent } i.} \cdot f(v_i)\, dv_i$$

$$\mathop{\mathbb{E}}_{v_i \sim D_i}\left[p_i(v_i, v_{-i})\right] = \mathop{\mathbb{E}}_{v_i \sim D_i}\left[\phi_i(v_i)\, x_i(v_i, v_{-i})\right]$$

$$\max \mathbb{E}[Rev] = \max \mathop{\mathbb{E}}_{v \sim D}\left[\sum_{i=1}^{n} p_i(v)\right]$$

max each.

$= \max \underset{v\sim D}{\mathbb{E}} \left[\underbrace{\sum_{i=1}^{n} \phi_i(v_i)\, x_i(v_i, v_{-i})}_{\text{virtual s.w.}} \right]$

$VSW(v)\,P(v) + VSW(v')\,P(v') + VSW(v'')\,P(v'') + \cdots$

$$\boxed{\begin{array}{c} \underset{(\because \text{DSIC})}{x^*(b) = x^*(v)} = \underset{x\in X}{\operatorname{argmax}} \sum_{i=1}^{n} \phi_i(v_i)\, x_i \\ \text{Then, Myerson's lemma fixes } p^*(\cdot) \end{array}}$$

(x^*, p^*) DSIC $\Longleftrightarrow$ x^* is monoton

$\Uparrow$

if $\phi_i(v_i) = v_i - \dfrac{1-F_i(v_i)}{f_i(v_i)}$

is monotone increasing in v_i

$\Updownarrow$

D_i is regular.

Example: single item, n bidders.

(Assume DSIC: $\forall i,\ b_i = v_i$)

$X = \left\{ x \in \{0,1\}^n \;\middle|\; \sum_{i=1}^{n} x_i \le 1 \right\}$

$x^*(v) = \operatorname{argmax} \sum^{n} \phi_i(b_i)\, x_i = \sum_{i=1}^{n} \phi_i(v_i)\, x_i$

$\left(v_i - \dfrac{1-F_i(v_i)}{\quad} \right.$

$$x^*(b) = \underset{x \in X}{\arg\max} \sum_{i=1}^{n} \phi_i(b_i)\, x_i = \sum_{i=1} \left(V_i - \frac{1 - F_i(V_i)}{f_i(V_i)} \right)$$

= give item to the highest agent of highest $\phi_i(V_i)$

if $\max_i \phi_i(V_i) \geq 0$

$i^* = \arg\max_i \phi_i(V_i)$

Then i^* gets the item if $\phi_{i^*}(V_{i^*}) \geq 0$

$$\equiv \quad V_{i^*} \geq \underbrace{\phi_{i^*}^{-1}(0)}_{\pi}$$

reserve price for agent i.

① suppose $D_i = D$ regular

$\Rightarrow \phi_i = \phi$ monotone

$$i^* = \underset{i \in N}{\arg\max}\ \phi(V_i) = \arg\max_i V_i$$

i^* wins if

① $\phi(V_{i^*}) \geq 0 \Rightarrow \overset{b_{i^*} =}{V_{i^*}} \geq \phi^{-1}(0)$

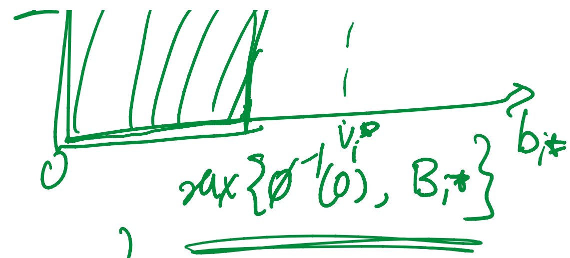
② $\phi(V_{i^*}) \geq \phi(V_i) \quad \forall i \neq i^*$

$\Leftrightarrow b_{i^*} = V_{i^*} \geq V_i \quad \forall i \neq i^*$

$\geq B_{i^*}$

$B_{i^*} = \max V_i$

$$B_{i^*} = \max_{i \neq i^*} v_i$$



$$p_{i^*} = \max\Big\{\underbrace{\phi^{-1}(0)}_{\text{reserve-price.}},\ \text{second-highest bid}\Big\}$$

ex. reserve-price $\quad D = U[0,1]$!