Single Parameter Auctions

Tuesday, October 31, 2023 11:01 AM

Single item → First price
Single item → second price

↓

k-items.

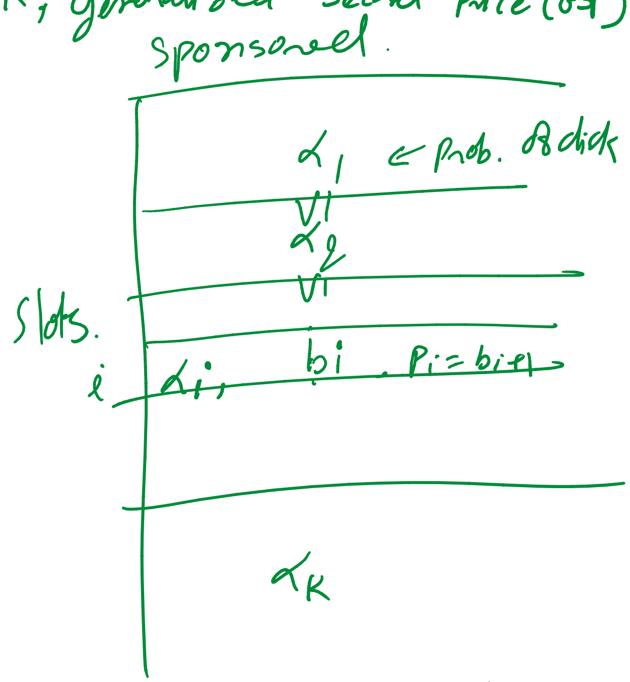
Goal: $\boxed{\text{Awesome Auction}}$

1. Truthful (DSIC)
2. S.W. maximization
3. Easy to implement (poly-time computable)

★ Sponsored search: Pay-per-click, Generalized-Second-Price (GSP)

Google: $\boxed{\text{Pizza}}$

- N: set of advertisers who wants to sell "pizza".



- $i \in N$: v_i value-per-click
  private.

  bids b_i bid-per-click.

$$\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_K.$$

$$b_1 \geq b_2 \geq \cdots \geq b_k \geq b_n$$

x: $b_1 \to$ slot 1, $b_2 \to$ slot 2, $b_k \to$ slot 3.

$$P_i = \text{Payment}_i = b_{i+1} \quad \text{if } i \leq K$$
$$= 0 \quad \text{o.w.}$$

$$\text{Net utility} = \alpha_i (v_i - b_{i+1})$$

★ Single-Parameter:

- Single stuff on auction.

- Single stuff on auction.

N: set of bidders.

$i \in N$, v_i / unit-stuff. (private)

- bid: b_i / unit-stuff (need not be v_i)

$$\boxed{X = \{(x_1, \dots, x_n) \in \mathbb{R}_+^n \mid (x_1, \dots, x_n) \text{ is feasible}\}}.$$

★ Format:

1. Collect bids in "sealed envelops".

$$\bar{b} = (b_1, \dots, b_n)$$

2. Allocation: SW maximization wrt $\bar{b}$

$$x(\bar{b}) \in \underset{x \in X}{\operatorname{argmax}} \sum_i x_i b_i$$

3. Decide payment: $P_i(b)$ for agent i.

$$\boxed{U_i(\bar{b}) = v_i x_i(\bar{b}) - P_i(\bar{b})}$$

Mechanism.

$\forall i \in N$:

$$\bar{b} = (b_1, \dots, b_n) \to \boxed{\text{Mechanism}} \to x_i(\bar{b}),\ P_i(\bar{b})$$

x

$\downarrow$

(x, P) a mechanism.

☆ S.W. maximizing allocation.

(1) single item: $\mathcal{X} = \{ x \in \{0,1\}^n \mid \sum_i x_i \le 1 \}$

$$x(b) \in \underset{x \in \mathcal{X}}{\operatorname{argmax}} \sum_i x_i b_i = \text{highest bidder.}$$

(2) K-items : $\mathcal{X} = \{ x \in \{0,1\}^n \mid \sum_i x_i \le K \}$

. . . . - -

(3) Sponsored search : $\mathcal{X} = \{ x \in \{\alpha_1, \alpha_2, \dots, \alpha_K, 0\}^n \mid x_i \ne x_j \ \forall i,j \}$

$\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_K$

$$x(\bar{b}) \in \underset{x \in \mathcal{X}}{\operatorname{argmax}} \sum_i b_i \cdot x_i \rightarrow \text{GSP.}$$

Claim: If $x(\cdot)$ is s.w. maximizing allocation rule $\Rightarrow x(\cdot)$ is monotone.

☆

Def 1: Allocation $x(\bar{b})$ is implementable if $\exists$ paymt $P(\bar{b})$ s.t. (x, P) is DSIC.

Def 2: Allocation $x(\bar{b})$ is monotone if

$\forall i, \quad \forall b_{-i}$

$x_i(b_i, \cdot)$ is monotonic

* (Myerson's Thm):

① $x(\cdot)$ is implementable $\overset{\Rightarrow}{\underline{\text{iff}}}$ it is monotone.

② If $x(\cdot)$ is monotone then $\exists$ unique $p(\cdot)$ s.t.

(x, P) is DSIC.

③ There is an explicit formula to compute $P_i(\cdot)$ $\forall i$

Pf: $\boxed{\text{DSIC}: \forall i, \forall b_{-i}, \; U_i(v_i, b_{-i}) \geq U_i(b_i, b_{-i}), \; \forall b_i \in \mathbb{R}}$

Fix, $\underline{i \in N}$, $\underline{b_{-i}}$

$$\boxed{\begin{aligned} x(b_i) &= x_i(b_i, b_{-i}) \quad \text{Notation.} \\ P(b_i) &= p_i(b_i, b_{-i}) \end{aligned}}$$

$\underline{\text{Claim I}}$: (x, P) DSIC $\Rightarrow$ $x(\cdot)$ is monotone

$\underline{\text{Pf}}$: Let $y \neq z$

DSIC $\Rightarrow$
$$\begin{cases} \overset{b_i = v_i = z}{z\,x(z) - P(z)} \geq \overset{b_i = y,\; v_i = z}{z\,x(y) - \underline{P(y)}} \rightarrow ① \\[2ex] \overset{b_i = v_i = y}{y\,x(y) - P(y)} \geq \overset{b_i = z,\; v_i = y}{y\,x(z) - P(z)} \rightarrow ② \end{cases}$$

#
$$\boxed{z\big(x(y) - x(z)\big) \overset{(\because ①)}{\leq} P(y) - P(z) \overset{(\because ②)}{\leq} y\big(x(y) - x(z)\big)}$$

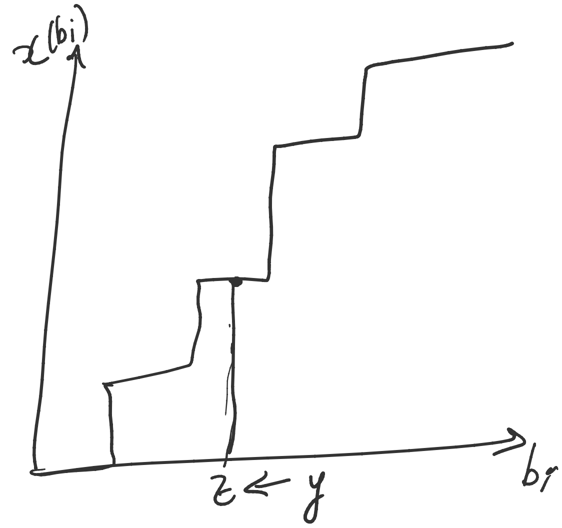
$\Downarrow$

$\ldots \Rightarrow (x(y) - x(z))$

$$0 \le (y-z)\,(x(y)-x(z))$$

$$y \ge z \Leftrightarrow x(y) \ge x(z) \equiv x \text{ is monotone.}$$

Claim 2: $\exists$ <u>unique P</u> s.t. (x, P) is DSIC.



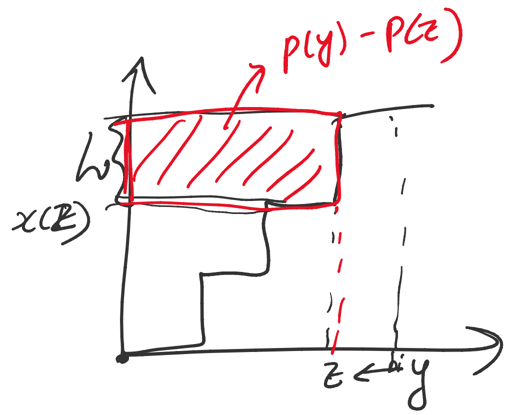
Pf:

Case I: z is in middle of a segment.

$$\lim_{y \to z^+} x(y) = x(z)$$

$$(\#) \Rightarrow 0 \le P(y) - P(z) \le 0$$

$$\Rightarrow P(y) = P(z)$$



Case II: z at a break point

$$\lim_{y \to z^+} x(y) = x(z) + h$$

$$x(y) - x(z) = h$$

$$(\#) \Rightarrow z * h \le P(y) - P(z) \le z * h$$

$$\Rightarrow P(y) - P(z) = z \cdot h$$

If $x(z) = 0$ then
$P(z) = 0$.
I.R.

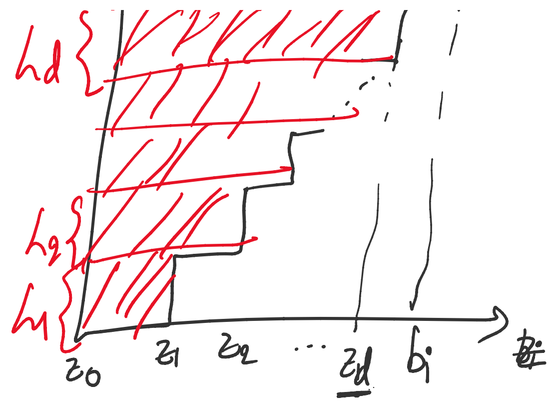
$$P(b_i) = \underbrace{P(b_i) - P(z_d)}_{= z_d \cdot h_d} + P(z_d) - P(z_{d-1})$$

$$+\; P(z_d) - P(z_{d-1})$$

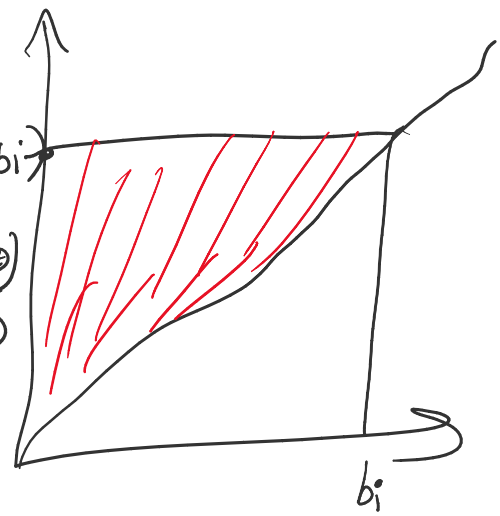
$$+\; \vdots$$

$$+\; P(z_1) - P(z_0)$$

$$\underset{0}{P(z_0)} \qquad z_1 \cdot h_1$$



$$= \sum_{k=1}^{d} z_k \cdot h_k$$



$$\lim_{y \to z^+} \frac{z\,(x(y) - x(z))}{(y - z)} \le \frac{p(y) - p(z)}{(y - z)} \le \frac{y\,(x(y) - x(z))}{(y - z)}$$

$$z \cdot x'(z) \le p'(z) \le z\, x'(z)$$

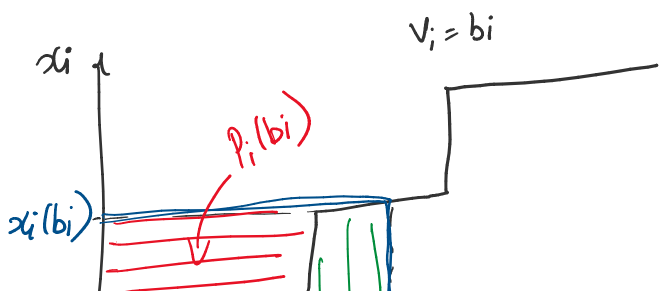
$$\Rightarrow \quad p'(z) = z \cdot x'(z)$$

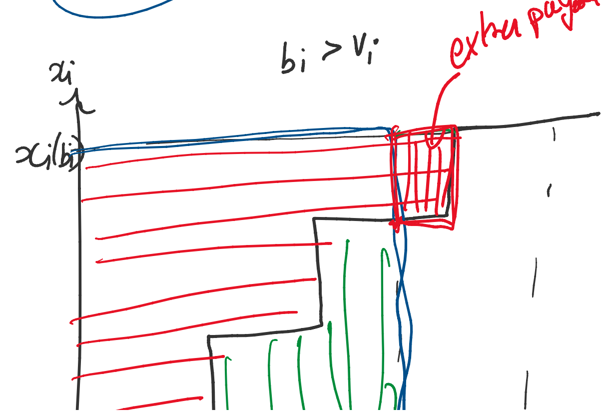
$$\Rightarrow \quad p(b_i) = \int_0^{b_i} p'(z)\, dz = \int_0^{b_i} z \cdot x'(z)\, dz$$

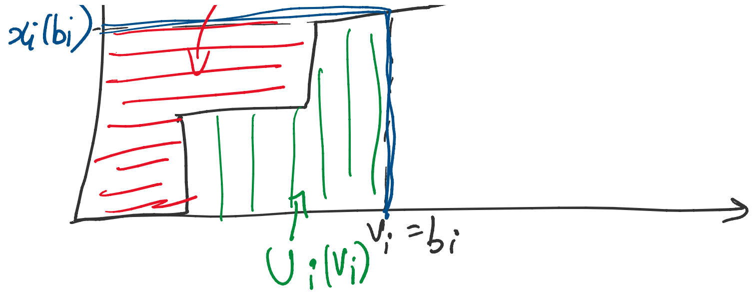
Claim 3: (x, P) is DSIC.

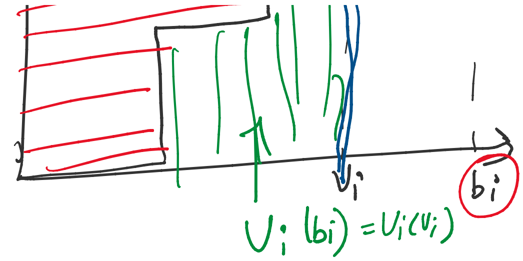
Pf: (by picture).

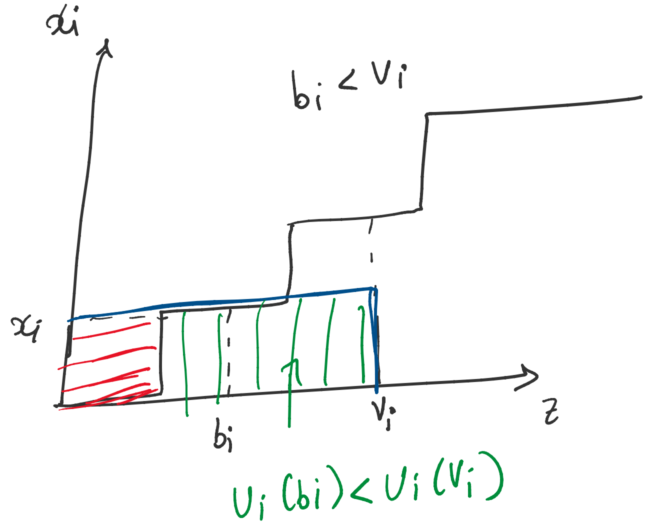
$$U_i(b_i) = v_i\, x_i(b_i) - P_i(b_i)$$






$x_i(b_i)$
$v_i = b_i$
$U_i(v_i)$


v_i
b_i
$U_i(b_i) = U_i(v_i)$


x_i
$b_i < v_i$
x_i
b_i
v_i
z
$U_i(b_i) < U_i(v_i)$