Advanced Topics in Machine Learning and Algorithmic Game Theory

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Lecture 10: Market Equilibrium

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10.1 Introduction

Today topic is market equilibrium. The basic elements of a market are buyers and product (and there are also potentially sellers). The goal is to find *prices* that clear the market. Namely, under those prices every product is completely sold, and there is no "remaining" demand. In other words, we would like to achieve a state where *demand equals supply*.

This is probably one of the most important conceptual contributions of economic theory to the everyday life. The assumption that if we let the market set the prices then the market will reach prices that will clear the market.

10.2 Fisher Market

we will start by introducing the Fischer market. The Fisher market has the following ingredients:

- a set of n buyer
- A set of m products
- A budget of B_i for buyer i.

The goal of buyer i, given a set of prices p, is to buy a set a products S, such that its cost is at most B_i (no value for leftover budget) and maximizes its utility.

The first step we need to take is to define the utility of a buyer. A few desired features are the following:

- Monotonically non-decreasing in the amount of any product (this is equivalent to the *free disposal* hypothesis).
- Normalize the value of the empty set to 0.

Here are a few poplar utility functions:

Linear $u_i(x) = \sum_{j=1}^m a_{i,j} x_{i,j}$, where $a_{i,j} \geq 0$.

Cobb-Douglas $u_i(x) = \prod_{j=1}^m x_{i,j}^{a_{i,j}}$, such that $\sum_{j=1}^m a_{i,j} = 1$. Namely a geometric average with weights $a_{i,*}$.

Leontief $u_i(x) = \min_j \{a_{i,j} x_{i,j}$.

All the above examples are a special case of what is called Constant Elasticity Substitution (CES), which is define as:

$$u(x) = (\sum_{j=1}^m \alpha_j x_j^\rho)^{1/\rho}$$

For $\rho = 1$ we have the linear utility, and the products are substitutes. For $\rho \to 0$ we have Cobb-Douglas utility, which balances substitutes and complementary. For $\rho \to -\infty$ we have the Leontief.

Without loss of generality we assume that from each product we have a single copy. (The amount that we get from a product is the percentage we get from all the available products.)

Our goal is two fold. Find both **prices** and **allocation** such that: (1) No product is oversold (feasibility) and (2) Each buyer maximizes its utility given its budget and the prices.

We would like the solution to also have a few basic properties:

- Multiplication by a constant of u_i should not change the the allocation.
- If buyer i splits to two buyers with identical utility, the sum of the buyers will get exactly what buyer i gets.

This goals can lead us to try and maximize the average utility, given the feasibility constraint. It turns out that maximizing the the geometric mean is what will work. This is the idea of Eisenberg-Gale.

Let $B = \sum_{i=1}^n B_i$, Then we like to maximize,

$$(\prod_{i=1}^n u_i^{B_i})^{1/B}$$

where u_i is the utility of agent i. This is equivalent to maximizing $\prod_{i=1}^n u_i^{B_i}$, which is equivalent to maximizing $\sum_{i=1}^n B_i \log u_i$. We have the following Eisenberg-Gale convex program for the linear utilities,

$$\begin{aligned} & & \max \sum_{i=1}^{n} B_i \log u_i \\ \forall i \in BUYERS & & u_i = \sum_{j=1}^{m} a_{i,j} x_{i,j} \\ \forall j \in PRODUCTS & & \sum_{i=1}^{n} x_{i,j} \leq 1 \\ \forall i,j, & & x_{i,j} \geq 0 \end{aligned}$$

First let us transform the problem to the standard minimization form,

$$\begin{aligned} & & \min \sum_{i=1}^{n} -B_i \log(\sum_{j=1}^{m} a_{i,j} x_{i,j}) \\ \forall j \in PRODUCTS & & \sum_{i=1}^{n} x_{i,j} - 1 \leq 0 \\ \forall i,j, & & -x_{i,j} \leq 0 \end{aligned}$$

Now we can build the Lagrangian. We have dual variables p_j for each inequality for a product j and $\lambda_{i,j}$ for each inequality of $x_{i,j}$. The Lagrangian is,

$$L = \sum_{i=1}^{n} -B_i \log(\sum_{j=1}^{m} a_{i,j} x_{i,j}) + \sum_{j=1}^{m} p_j(\sum_{i=1}^{n} x_{i,j} - 1) + \sum_{i,j} \lambda_{i,j}(-x_{i,j})$$

We can now consider the KKT condition for optimality. First we have the non-negativity of the dual variables.

$$\begin{aligned} \forall j \in PRODUCTS & & p_j \geq 0 \\ \forall i,j, & & \lambda_{i,j} \geq 0 \end{aligned}$$

Next we have the complementary slackness,

$$\begin{aligned} \forall j \in PRODUCTS & & p_j(\sum_{i=1}^{n} x_{i,j} - 1) = 0 \\ \forall i,j, & & \lambda_{i,j} x_{i,j} = 0 \end{aligned}$$

Finally, we have that the gradient is zero, which implies that,

$$\frac{-B_i a_{i,j}}{\sum_{j=1}^{m} a_{i,j} x_{i,j}} + p_j - \lambda_{i,j} = 0$$

The $\lambda_{i,j}$ are redundant and we can have,

$$p_j \geq \frac{B_i a_{i,j}}{\sum_{j=1}^{m} a_{i,j} x_{i,j}}$$

For the KKT conditions we derive a few interesting consequences. First,

$$p_j > 0 \Rightarrow \sum_{j=1}^{m} x_{i,j} = 1$$

which implies that the products with non-zero price p_j are completely distributed.

We also have that

$$\begin{aligned} \frac{a_{i,j}}{p_j} &\leq \frac{\sum_{j=1}^{m} a_{i,j} x_{i,j}}{B_i} \\ x_{i,j} > 0 \quad \Rightarrow \quad \frac{a_{i,j}}{p_j} &= \frac{\sum_{j=1}^{m} a_{i,j} x_{i,j}}{B_i} \end{aligned}$$

which implies that the utility per dollar is maximized by the buyer's bundle, and each product in the bundle maximizes it.

Theorem 10.1. *For the linear Fisher model, if for each product j some buyer i has $a_{i,j} > 0$ then,*

- *The prices clear the market (Each buyer uses all its budget and each item is completely distributed)*
- *The set of equilibrium allocations is convex.*
- *The prices and utilities are unique (not the allocations)*

Proof. From the KKT conditions, each buyer gets an optimal allocations (given the prices). In addition, for any $x_{i,j} > 0$ we have

$$\frac{B_i a_{i,j} x_{i,j}}{\sum_{j=1}^{m} a_{i,j} x_{ui,j}} = p_j x_{i,j}$$

When we sum over the products, for a given buyer i, we have,

$$\frac{B_i \sum_{j=1}^{m} a_{i,j} x_{i,j}}{\sum_{j=1}^{m} a_{i,j} x_{ui,j}} = \sum_{j=1}^{m} p_j x_{i,j}$$

which implies that

$$B_i = \sum_{j=1}^{m} p_j x_{i,j}$$

therefore buyer i completely uses its budget B_i.

Since every equilibrium is an optimal solutions to the convex program. □

10.3 Arrow-Debrue market

In this section we will discuss the Walrasian market, also known as the Arrow-Debrue model. This model can be viewed as an extension of the Fisher model. The agents rather than having a budget, have an endowment of goods that they start with. They sell the endowment to buy other goods. (An good can be also the ability to work.)

The model has,

- A set of agents A.
- A set of goods G.
- Agent i has an endowment $e_i = (e_{i,1}, \ldots, e_{i,m})$ and a utility function $u_i(x)$. (We will again concentrate on linear utility functions.)

The main result of Arrow-Debrue is that under rather minimalistic assumptions there exists a set of prices that clear the market. We will concentrate more on the computational issue.

First we show that the Fisher market is a special case of the Arrow-Debrue market.

Theorem 10.2. *The Fisher market is a special case of the Arrow-Debrue market.*

Proof. We add another product which is money, and another agent which initial will have all the goods, and is interested only in money. The initial endowments

$$\begin{array}{ll} 0, \ldots & , 0, B_1 \\ 0, \ldots & , 0, B_2 \\ \vdots & \\ 0, \ldots & , 0, B_m \\ 1, \ldots & , 1, 0 \end{array}$$

The utility of the agents are as follows:

$1 \le i \le m$, has zero utility for money and the previous utility to the other products.

$i = m + 1$ has only utility for money and no utility for the other products, i.e., $u(x_1, \ldots, x_{m+1}) = x_{m+1}$.

It is easy to see that this is identical to the Fisher market we had. □

A simplifying assumption is that each agent has one complete product. For simplicity, we will assume that $a_{i,j} > 0$ for all products j and agents i.

We can write the following non-convex program:

$$\begin{aligned}
&\forall i && \sum_{i=1}^{n} x_{i,j} = 1 \\
&\forall i,j, && x_{i,j} \geq 0 \\
&\forall i,j && \frac{a_{i,j}}{p_j} \leq \frac{\sum_{k=1}^{m} a_{i,k}x_{i,k}}{p_i} \\
&\forall j, && p_j \geq 0
\end{aligned}$$

The first two conditions say that the solution is feasible. The third is due to maximizing the agent utility. The fourth is that the prices are positive (there are no zero prices since we assumed that $a_{i,j} > 0$).

Theorem 10.3. *The solutions of the non-convex program are market equilibrium and include all market equilibria.*

Proof. It is clear that every market equilibrium obeys all the requirements. We will show that any solution to the program is a market equilibrium. From the third line we have, after multiplying by $x_{i,j}p_j$,

$$\forall i,j \qquad a_{i,j}x_{i,j} \leq \frac{\sum_{k=1}^{m} a_{i,k}x_{i,k}}{p_i} x_{i,j}p_j$$

we can sum over the product,

$$\forall i,j \qquad \sum_{j=1}^{m} a_{i,j}x_{i,j} \leq \frac{\sum_{k=1}^{m} a_{i,k}x_{i,k}}{p_i} \sum_{j=1}^{m} x_{i,j}p_j$$

The assumption that $a_{i,j} > 0$ implies that $\sum_{j=1}^{m} a_{i,j}x_{i,j} > 0$ Therefore,

$$\forall i \qquad p_i \leq \sum_{j=1}^{m} x_{i,j}p_j$$

summing over all the agnets we have

$$\sum_{i=1}^{n} p_i = \sum_{i=1}^{n}\sum_{j=1}^{m} x_{i,j}p_j = \sum_{j=1}^{m} p_j \sum_{i=1}^{n} x_{i,j} = \sum_{j=1}^{m} p_j$$

This implies that we have

$$\forall i \qquad p_i = \sum_{j=1}^{m} x_{i,j}p_j$$

□

We would like to turn the program to be convex. The problem is the requirement,

$$\forall i,j \quad \frac{a_{i,j}}{p_j} \le \frac{\sum_{k=1}^{m} a_{i,k}x_{i,k}}{p_i}$$

This is important in the case that $a_{i,j} > 0$ (otherwise it is trivial). This is equivalent to,

$$\forall i,j \quad \frac{p_i}{p_j} \le \frac{\sum_{k=1}^{m} a_{i,k}x_{i,k}}{a_{i,j}}$$

Taking the logarithm we have,

$$\forall i,j \qquad \log(p_i) - \log(p_j) \le \log(\sum_{k=1}^{m} a_{i,k}x_{i,k}) - \log(a_{i,j})$$

We can now replace $\log(p_i)$ by new variables $LOGp_i$ and have

$$\forall i,j \qquad LOPp_i - LOGp_j \le \log(\sum_{k=1}^{m} a_{i,k}x_{i,k}) - \log(a_{i,j})$$

In the new variables we have a convex program.

10.4 Linear Arrow-Debrue- approximate equilibrium with auction

The approximation:

- The market clears (exactly).
- Every agent gets a bundle of products whose utility is at least $(1-\epsilon)^2$ from the utility of the optimal bundle.

The algorithm is build from iterations,each iteration is build from rounds. In each round we go over all agents in a round-robin order.

Initially, all the prices of the goods are 1, i.e., $p_j = 1$. This implies that the total money of the agents is n. During a round, when we reach agent i, we check if it has available money. If not we continue to the next agent. Otherwise, we check which bundle it would like to buy at prices $p_j(1+\epsilon)$. (Each product has agents holding it at p_j and $p_j(1+\epsilon)$). If there is such a bundle, we buy the products back from the relevant agents, at the old price, and sell it to agent i. The process completes when agent i exhausts its money, and we continue to agent $i+1$. If there is no more agents with price p_j (all have price $p_j(1+\epsilon)$) then we complete an iteration. At the end of an iteration we update the money of the agents (given their initial endowment). If at some point (at the end of a round) the total money is at most ϵs_{min},where $s_{min} = \min_i \sum_{j=1}^{m} e_{i,j}$, we stop the entire process. At termination we distribute the remaining products arbitrarily.

Let p_{max} be the maximum price.

Lemma 10.4.1. *The number of rounds in an iteration is bounded by* $O(\frac{1}{\epsilon}\log(\frac{mp_{max}}{\epsilon s_{min}}))$

Proof. In every exchange we buy at $p_j(1+\epsilon)$ and sell at p_j. This implies that the total amount of money decreases by a factor of $1+\epsilon$ in every completed round. The total money at the start of an iteration is at most mp_{max}. The iteration ends if we have that the money is less than ϵs_{min}.This bounds the number of rounds is at most $\log_{1+\epsilon}(\frac{mp_{max}}{\epsilon s_{min}})$. □

Lemma 10.4.2. *The number of iteration is bounded by* $O(\frac{m}{\epsilon}\log p_{max})$

Proof. Each iteration increases the price of some product by $1+\epsilon$, so the number of iteration is at most $O(m \log_{1+\epsilon} p_{max})$. □

Lemma 10.4.3. *Relative to the final prices,each agent gets a bundle whose utility is at least* $(1-\epsilon)^2$ *from the optimal bundle.*

Proof. The algorithm always sells the agent an optimal bundle at the current prices (the agent might pay $p_j(1+\epsilon)$).

The sub-optimality can have two sources. First, at the end, agent i has a remaining money. Let M be the value of i's endowment at the final prices and M_1 be the value of the items it bought. Since the money left is at most ϵs_{min}, we have that

$$M_1 \geq (1-\epsilon)M$$

(We have that $M \leq s_{min}$ since prices start at 1 and $s_{min} \leq \sum_{j=1}^{m} e_{i,j}$.)

Second, agent i buy products at price $p_j(1+\epsilon)$ and not p_j. Therefore,

$$\frac{M_1}{1+\epsilon} \geq \frac{(1-\epsilon)M}{1+\epsilon} \geq (1-\epsilon)^2 M$$

□

Let $a_{min} = \min_{i,j} a_{i,j}$ and $a_{max} = \max_{i,j} a_{i,j}$.

Theorem 10.4. *The algorithms finds an* $(1-\epsilon)^2$ *market equilibrium,in time*

$$O(\frac{mn}{\epsilon^2}\log\frac{ma_{max}}{\epsilon s_{min} a_{min}}\log\frac{a_{max}}{a_{min}}).$$

Proof. Every product whose price is above the initial price of 1 is completely sold.

The total money of the agents is the value of the products at the final prices. The requirement that the surplus of money is at most ϵs_{min} has to be reached before all products are sold completely. On termination, the ratio of the maximum to minimum price is at most a_{max}/a_{min}, and this is a bound on p_{max}.

Each round requires $O(n)$ time. □