Cost Sharing Games

Monday, October 16, 2023 1:52 PM

Recall

☆ Cost Sharing Games.

(Routing Games of positive externality)

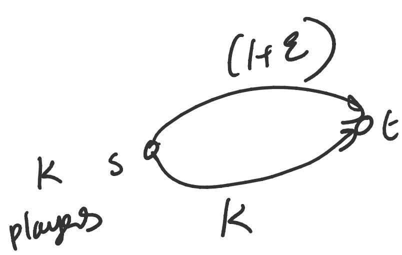
- n/w $G=(V,E)$ $e \in E$, $r_e \geq 0$ cost of building e.
- $i \in N$, want to build an $s_i \rightsquigarrow t_i$ path

$\mathcal{P}_i$: set of $s_i \rightsquigarrow t_i$ paths.

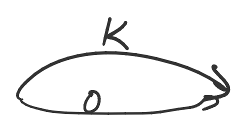
$p_i \in \mathcal{P}_i \rightarrow \bar{P} = (P_1, \dots, P_n) \Rightarrow f_e$: # agents wanting to build edge e.

$$C_i(\bar{P}) = \sum_{e \in P_i} \frac{r_e}{f_e}$$

Q: PoA better or worse than Atomic Routing Games?



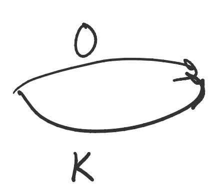
NE-1 = OPT:



cost (OPT) = (1+ε)

cost/person $= \frac{(1+\varepsilon)}{K}$

NE 2:



cost/person $= \frac{K}{K} = 1$

cost(NE2) = K

$$PoA = \frac{K}{1+\varepsilon} \sim K$$

$$PoA = \frac{k}{1+\varepsilon} \sim k$$

$\star$ PoS = Price-of-Stability

$$= \frac{\text{Best NE Cost}}{\text{OPT Cost}}$$

?

Use "default" "options" to steer towards a good NE.

PoS $\sim$ constant? NO!

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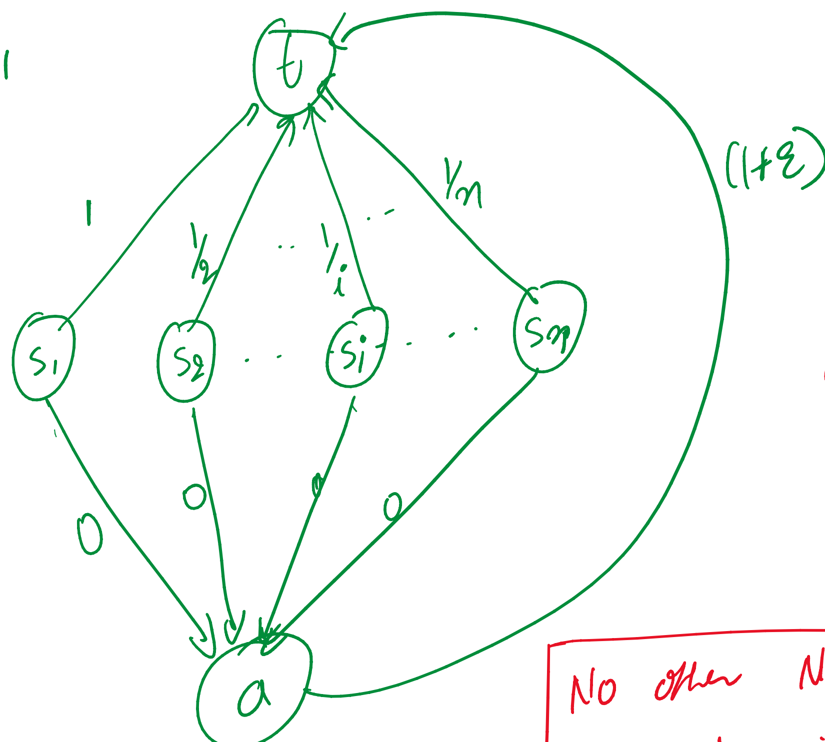
e.g.

$\varepsilon \sim 0.001$



OPT: $P_i = s_i \to a \to t$

$C_i(OPT) = 0 + \frac{1+\varepsilon}{n}$

$cost(OPT) = (1+\varepsilon)$

NE: $P_i = s_i \to t$

$C_i(P_i) = \frac{1}{i}, \ \forall i \in N.$

$cost(NE) = 1 + 1/2 + \dots + 1/n$

$\approx H_n = \ln n$

No other NE! why?

$A \subset N$, $i \in A$, $P_i: s_i \to a \to t$

$i \notin A: P_i: s_i \to t.$

$i^* = \max_{i \in A} i \geq |A|$

$i \in A$: $C_i(P) = \frac{1+\varepsilon}{|A|} \geq \frac{1+\varepsilon}{i^*}$

$$PoS = \frac{cost(NE)}{cost(OPT)} = \frac{H_n}{1+\varepsilon} \sim H_n = \ln n$$

cost ...

$$c_i(P) = \frac{1+\varepsilon}{|A|} \geq \frac{1+\varepsilon}{i^*}$$

$\therefore$ i^* can deviate to $s_{i^*} \to t$ & improve to $\frac{1}{i^*}$

Thm: $PoS \leq H_n$ for any cost-sharing Game.

γ_e, e

Pf:

why $\exists$ Pure NE? $\because$ It is a potential game.

$$\boxed{\phi(P) = \sum_{e \in E} \sum_{k=1}^{f_e} c_e(k)} = \sum_{e \in E} \gamma_e \sum_{k=1}^{f_e} \frac{1}{k} \quad \left(\because c_e(k) = \frac{\gamma_e}{k}\right)$$

(exe): Prove that ϕ is a valid potential func for C.S.G.

$\forall P \in \mathcal{P}$, $\forall i \in N$, $P_i' \in \mathcal{P}_i$

$$\phi(P_i', P_{-i}) - \phi(P_i, P_{-i}) = c_i(P_i', P_{-i}) - c_i(P_i, P_{-i})$$

P^*: OPT $= \operatorname{argmin}_{P \in \mathcal{P}} \text{cost}(P)$

$$\boxed{H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}}$$

P': $\operatorname{argmin}_{P \in \mathcal{P}} \phi(P) \Rightarrow P'$ is a NE.

$\forall P \in \mathcal{P}$:

$$\text{cost}(P) = \sum_{\substack{e \in E \\ f_e \geq 1}} \gamma_e \, 1 \leq \phi(P) = \sum_{\substack{e \in E \\ f_e \geq 1}} \gamma_e \left(\underbrace{\sum_{k=1}^{f_e \leq n} \frac{1}{k}}_{\leq H_n} \right)$$

$$\leq H_n \, \text{cost}(P)$$

$$\boxed{\text{cost}(P) \leq \phi(P) \leq H_n \cdot \text{cost}(P)} \to (1).$$

$$\boxed{cost(P) \le \phi(P) \le H_n \cdot cost(P)} \to (1).$$

$$cost(P') \underset{c:(1)}{\le} \phi(P) \le \phi(P^*) \le H_n \, cost(P^*)$$

$$PoS = \frac{\min_{P:NE} cost(P)}{cost(P^*)} \le \frac{cost(P')}{cost(P^*)} \le H_n$$

Strong NE: P is a strong NE iff "No Beneficial Coalition"

$P_A = (P_i)_{i \in A}$

$-A = N \setminus A$.

~~Def: Beneficial Coalition.~~ wrt $P = (P_1, \dots, P_n)$

$\forall A \subseteq N, \ \nexists q_A \in \bigtimes_{i \in A} \mathcal{P}_i$

$\forall i \in A: \ c_i(q_A, P_{-A}) \le c_i(P_A, P_{-A})$

& at least one strict inequality.

$A: P_A \to q_A$

$A = \{1, 2, \dots, k\}$ $\quad P_A \to (q_1, P_{A \setminus \{1\}})$

$\to (q_1, q_2, P_{A \setminus \{1,2\}})$

$\to \vdots$

$$PoS^{\text{Pure NE}} \le PoS^{\text{Strong NE}} \le PoA^{SNE} \le H_n$$

$\rightarrow$:

Thm: $PoA^{SNE} \le H_n$.

Pf: p^*: OPT, $\quad p$: strong NE.

Q_n: Why $A_n = \{1, \dots, n\}$ does not want to deviate to p^*?

$\exists i \in A_n$ s.t. $c_i(p^*) \ge c_i(p)$. Let this agent be n.

$Q_{(n-1)}$: Why $A_{(n-1)} = \{1, \dots, n-1\}$ does " " " p^*?

$\exists i \in A_{(n-1)}$ s.t. $c_i(p^*_{A_{n-1}}, p_n) \ge c_i(p)$. Let this agent be $(n-1)$.

$\vdots$

Q_k: Why $A_k = \{1, \dots, k\}$ " " " p^*?

$\exists i \in A_k$ s.t. $c_i(p^*_{A_k}, p_{-A_k}) \ge c_i(p)$. Let this agent be k.

$\vdots$

$$\text{cost}(p) = \sum_{k \in N} c_k(p) \le \sum_{k \in N} c_k(p^*_{A_k}, p_{-A_k})$$

$$\le \sum_{k \in N} c_k(p^*_{A_k})$$

↳ Remove agents in $N \setminus A_k$

$\leq \sum_{k \in N}$ ↳ Resolve agents in N1 ... & A_k plays $P^*_{A_k}$

Def: f_e^k be the # agents building e as per $P^*_{A_k}$ from A_k.

$$C_k(P^*_{A_k}) = \sum_{e \in P^*_k} \frac{r_e}{f_e^k} = \phi(P^*_{A_k}) - \phi(P^*_{A_{k-1}}) \quad \underline{(eke)}$$

$$\sum_{k \in N} C_k(P^*_{A_k}) = \sum_{k \in N} \phi(P^*_{A_k}) - \phi(P^*_{A_{k-1}})$$

$$= \cancel{\phi(P^*_{A_1})} - \phi(P_{A_0}) = 0$$

$$+ \cancel{\phi(P^*_{A_2})} - \cancel{\phi(P^*_{A_1})}$$

$$+ \phi(P^*_{A_3}) - \cancel{\phi(P^*_{A_2})}$$

$$+ \dots$$

$$+ \phi(P^*_{A_n}) - \cancel{\phi(P^*_{A_{n-1}})} = \phi(P^*_{A_n}) = \phi(P^*) \underset{(\because (1))}{\leq} H_n \cdot \mathrm{cost}(P^*)$$

$$\Rightarrow PoA^{SNE} = \frac{\mathrm{cost}(P)}{\mathrm{cost}(P^*)} \leq H_n$$

☆ Game Dynamics ☆.

★ Best Response Dynamics:

"If there exist an agent who can deviate & gain then they do deviate",

Q: what about zero-sum?

e.g.

$$\begin{array}{c|cc} & H & T \\ \hline H & (1,-1) & -1,1 \\ T & -1,1 & 1,-1 \end{array}$$

NO point-wise convergence of the play:

$$\text{Avg play} = \frac{1}{T}\left(\#\text{ times } H \text{ played},\ \#\text{ times } T \text{ played}\right)$$

$$= \frac{1}{T}\left(\sim T/2,\ \sim T/2\right)$$

$$= \sim\left(\frac{1}{2}, \frac{1}{2}\right).$$

[Robinson '51]

$\approx$ In zero-sum games avg play of "Fictitious Play" ($\equiv$ BR to time avg Payoff) converges to a NE.

★ Any other games?

① In potential games, pointwise convergence when agents change one-at-a-time.

② If the game has D.S.E.

★ In general not even time avg. may converge:

★ In general not even time avg. may converge.

$$\begin{array}{c|cc} & T & F \\ \hline T & 1,2 & 0,0 \\ F & 0,0 & 2,1 \end{array}$$

Avg $= \frac{1}{T}\left(\frac{T}{2}, \frac{T}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right)$

is not a NE.

★ NO-Regret.