Smoothed Games & Potential Games

Thursday, October 12, 2023 11:00 AM

Recall: $PoA\begin{pmatrix}\text{affine}\\ \text{atomic}\\ \text{Routing}\end{pmatrix} \le 5/2.$

$p^* = \arg\min_{q \in P} cost(q)$

P: Pure NE, p^*: OPT.

step-1 (easy)

$$cost(p) = \sum_i C_i(p) \underset{(\because NE)}{\le} \sum_i C_i(p_i^*, p_{-i}) \quad \rightarrow ①$$

step-2

$$\sum_i C_i(p_i^*, p_{-i}) \le \underbrace{5/3}_{\lambda}\, cost(p^*) + \underbrace{1/3}_{\mu}\, cost(p) \rightarrow ②$$

step-3 (easy)

$$①, ② \Rightarrow cost(p) \le 5/3\, cost(p^*) + 1/3\, cost(p)$$

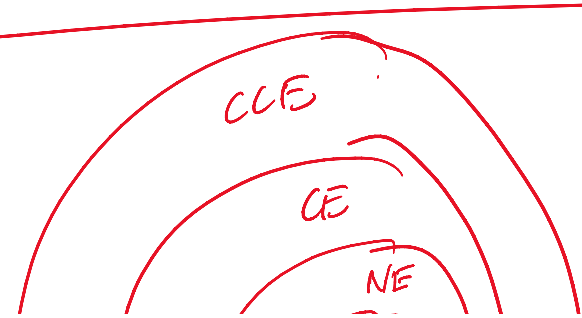
$$\Rightarrow PoA = \frac{cost(p)}{cost(p^*)} \le \frac{5/3}{(1 - 1/3)} = \frac{\lambda}{(1-\mu)}$$

★ (λ, μ) - Smooth Game:

$\forall p \in \times_i P_i$, $\forall p^* \in \times_i P_i$

(typically $p \sim$ equilibrium, p^*: OPT $= \arg\min_{p \in P} cost(p)$)

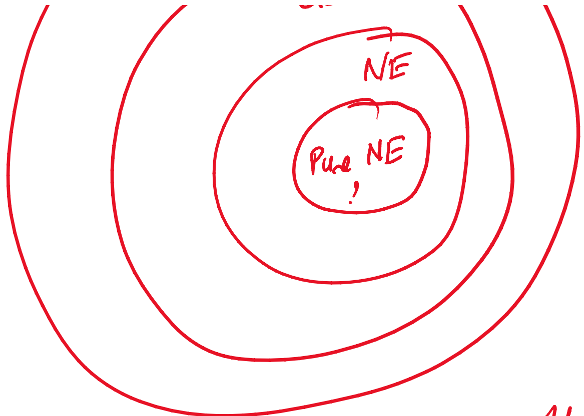
$$\sum_i C_i(p_i^*, p_{-i}) \le \lambda\, cost(p^*) + \mu\, cost(p)$$



$PoA = \frac{\text{worst Eq. cost}}{\text{OPT cost.}}$

$PoA^{PNE} \le PoA^{NE}$

CE



$$PoA^{PNE} \le PoA^{NE} \le PoA^{CE} \le PoA^{CCE}.$$

Thm: If the game is (λ, μ)-smooth then

$$PoA^{CCE} \le \frac{\lambda}{(1-\mu)}$$

pf:

Recall: ε-CCE $\sigma \in \Delta(\times_i P_i)$

$\forall p \in \times_i P_i$, $\sigma(p)$: probability of choosing play p.

$$\forall i \in N, \quad \mathbb{E}_{p\sim\sigma}[c_i(p)] \overset{(1+\varepsilon)}{\le} \mathbb{E}_{p\sim\sigma}[c_i(p_i', p_{-i})] \quad \forall p_i' \in P_i$$

(OPT IN) (OPT out & play p_i')

σ: CCE, p^*: OPT.

$$\text{cost}(\sigma) = \mathbb{E}_{p\sim\sigma}[\text{cost}(p)] = \mathbb{E}_{p\sim\sigma}\left[\sum_i c_i(p)\right]$$

$$= \sum_{i\in N} \mathbb{E}_{p\sim\sigma}[c_i(p)]$$

$$\overset{(1+\varepsilon)}{\le} \sum \mathbb{E}_{p\sim\sigma}[c_i(p_i^*, p_{-i})]$$

$$\le^{(1+\varepsilon)} \sum_{i\in N} \mathbb{E}_{p\sim\sigma}\left[C_i(p_i^*, p_{-i})\right]$$

$(\because \sigma \text{ is } \varepsilon\text{-CCE})$

$$= (1+\varepsilon)\,\mathbb{E}_{p\sim\sigma}\left[\sum_{i\in N} C_i(p_i^*, p_{-i})\right]$$

$$\le (1+\varepsilon)\,\mathbb{E}_{p\sim\sigma}\left[\lambda\, cost(p^*) + \mu\, cost(p)\right]$$

$(\because (\lambda,\mu)\text{-smooth})$

$$= (1+\varepsilon)\lambda\left(\mathbb{E}_{p\sim\sigma}\left[cost(p^*)\right]\right) + (1+\varepsilon)\mu\left(\underbrace{\mathbb{E}_{p\sim\sigma}\left[cost(p)\right]}\right)$$

$$= (1+\varepsilon)\lambda\; cost(p^*) + (1+\varepsilon)\mu\; cost(\sigma)$$

$$\therefore\; cost(\sigma) \le (1+\varepsilon)\lambda\; cost(p^*) + (1+\varepsilon)\mu\; cost(\sigma)$$

$$\Rightarrow\; \frac{cost(\sigma)}{cost(p^*)} \le \frac{\lambda(1+\varepsilon)}{(1-\mu(1+\varepsilon))}$$

Payoff / Utilities (example : location Game).

$\forall s, s^* \in \times_i S_i$ $\quad U(S) = \sum_i u_i(S) \ge \underbrace{\sum_i u_i(s_i^*, s_{-i})}$

$\downarrow$

NE

$$\ldots \ge \lambda U(s^*) - \mu\, U(S)$$

$$\sum_i u_i(s_i^*, s_{-i}) \geq \lambda\, U(s^*) - \mu\, U(s)$$

$$\Downarrow$$

$$1 \geq PoA = \frac{U(s)}{U(s^*)} \geq \frac{\lambda}{1+\mu}$$

OR.

$$1 \leq PoA = \frac{U(s^*)}{U(s)} \leq \frac{1+\mu}{\lambda}$$

Q: Why Pure NE exist in Routing Games?

☆ Potential Games (Rosenthal '70s)

- $N = \{1, \dots, n\}$ player
- $i \in N$, $\mathcal{P}_i$: ~~set~~ set of moves/strategies of player i.
- $P_i \in \mathcal{P}_i$ $\quad \bar{P} = (P_1, \dots, P_n)$. $\quad C_i(\bar{P})$ = cost of player i at $\bar{P}$.

[NE: $\forall i \in N \quad C_i(P_i, P_{-i}) \leq C_i(P_i', P_{-i})$, $\forall P_i' \in \mathcal{P}_i$]

Potential Fun^c:

$$\phi : \times_i P_i \rightarrow \mathbb{R}$$

$$\forall p \in \times_i P_i \quad , \quad \forall i \in N, \quad \forall p_i' \in P_i$$

$$\phi(p_i', \tilde{p}_{-i}) - \phi(\tilde{p}_i, \tilde{p}_{-i}) = c_i(p_i', \hat{p}_{-i}) - c_i(\tilde{p}_i, \tilde{p}_{-i})$$

$\geq 0 \qquad \qquad \geq 0$

$\Downarrow$

existance of pure NE. $\Downarrow$

$$\tilde{p} = \arg\min_{p \in P} \phi(p)$$

$\forall i \in N$ $\forall p_i' \in P_i$ $c_i(p_i', \tilde{p}_{-i}) \geq c_i(\tilde{p}_i, \tilde{p}_{-i})$

$\Downarrow$

Claim: $\tilde{p}$ is a pure NE. $\tilde{p}$ is a NE.

Consequences of ϕ:

① existence of Pure NE ←

② Implies simple "local-search" algo = BR Dynamics

converges to a Pure NE.

$\Downarrow$

③ Finding Pure NE $\in$ PLS complexity class "Polynomial local-search"

... local-min of ϕ can be computed in ...

(4) ε-approx local min of ϕ can be computed in

$$\text{poly}\left(\frac{1}{\varepsilon}, |N|, \max_i |\mathcal{P}_i|\right)$$

Thm: Atomic Routing Games are Potential Games.

Pf: Recall: $G=(V,E)$, $e \in E$, $c_e : \{0,1,\ldots,n\} \to \mathbb{R}$

$\mathcal{P}_i$: set of paths $s_i \rightsquigarrow t_i$

$P_i \in \mathcal{P}_i$, $\bar{P} = (P_1,\ldots,P_n) \to f_e$: # agents taking e as per $\bar{P}$

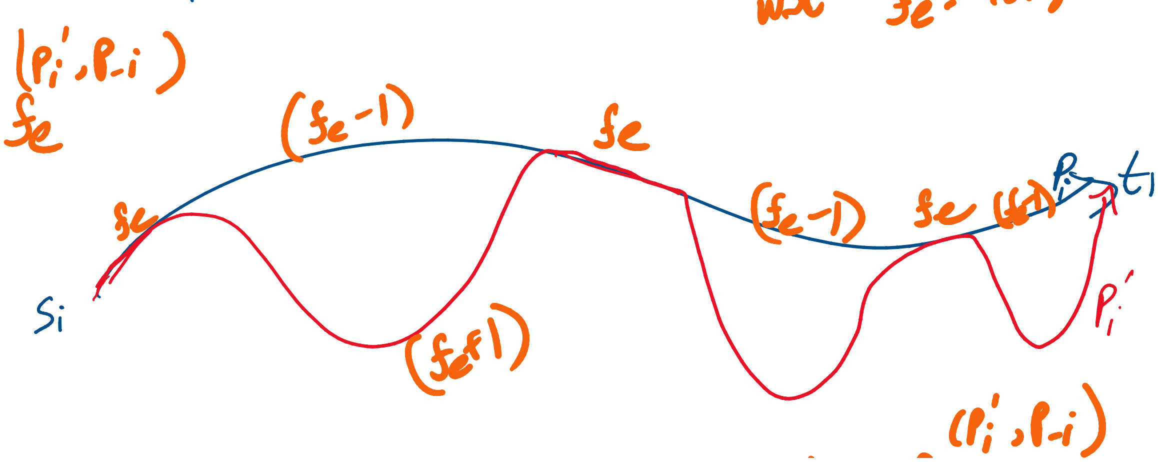
$$C_i(\bar{P}) = \sum_{e \in P_i} c_e(f_e)$$

$$\phi(\bar{P}) = \sum_{e \in E} \sum_{k=1}^{f_e} c_e(k)$$

$\forall i \in N$, $\forall P_i' \in \mathcal{P}_i$

TPT. $\phi(P_i', P_{-i}) - \phi(P_i, P_{-i}) = C_i(P_i', P_{-i}) - C_i(P_i, P_{-i})$

w.r.t f_e: congestion at $\bar{P}$



congestion on e at $(P_i', P_{-i}) = f_e^{(P_i', P_{-i})}$

$= f_e$ if $e \in (P_i \cap P_i')$, $e \in E \setminus (P_i \cup P_i')$ $\times$

$= f_e + 1$ if $e \in \underline{P_i' \setminus P_i}$

$= f_e - 1$ if $e \in \underline{P_i \setminus P_i'}$

$$LHS = \phi(P_i', P_{-i}) - \phi(P_i, P_{-i})$$

$$= \sum_{e \in P_i' \setminus P_i} \left(\sum_{k=1}^{(f_e+1)} c_e(k) \right) + \sum_{e \in P_i \setminus P_i'} \left(\sum_{k=1}^{(f_e-1)} c_e(k) \right)$$

$$- \left(\sum_{e \in P_i' \setminus P_e} \sum_{k=1}^{f_e} c_e(k) + \sum_{e \in P_i \setminus P_i'} \sum_{k=1}^{f_e} c_e(k) \right)$$

$$= \sum_{e \in P_i' \setminus P_i} c_e(f_e+1) - \sum_{e \in P_i \setminus P_i'} c_e(f_e)$$

$$RHS = C_i(P_i', P_{-i}) - C_i(P_i, P_{-i})$$

$$= \left(\sum_{e \in P_i' \setminus P_i} c_e(f_e+1) + \cancel{\sum_{e \in P_i' \cap P_i} c_e(f_e)} \right)$$

$$\Big(\sum_{e\in P_i'\setminus P_i} \qquad \cancel{e\in P_i'\cap P_i} \Big)$$

$$- \Big(\sum_{e\in P_i\setminus P_i'} c_e(f_e) + \cancel{\sum_{e\in P_i'\cap P_i} c_e(f_e)} \Big]$$

$$= \sum_{e\in P_i'\setminus P_i} c_e(f_e+1) - \sum_{e\in P_i\setminus P_i'} c_e(f_e). = LHS.$$

☆ Cost Sharing Games.

(Routing Game of positive externality)

- n/w $G=(V,E)$ $\quad e\in E$, $r_e \geq 0$ cost of building e.
- $i\in N$, build $s_i \rightsquigarrow t_i$ path

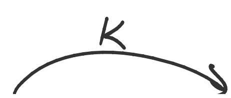
$\mathcal{P}_i$: set of $s_i \to t_i$ paths.

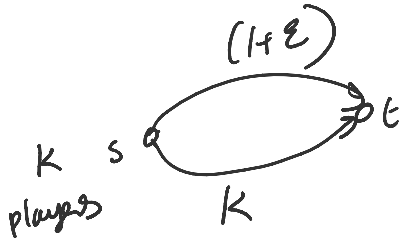
$P_i \in \mathcal{P}_i \to \bar{P} = (P_1, \dots, P_n) \Rightarrow f_e$: # agents want to build edge e.

$$C_i(\bar{P}) = \sum_{e\in P_i} \frac{r_e}{f_e}$$

Q: PoA better or worse than A.R.G.?

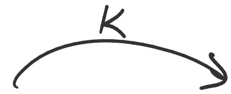
PoA:





$PoS = 1$

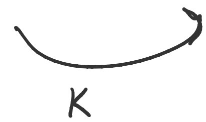
$NE1 = OPT$:



$cost(OPT) = (1+\varepsilon)$

$cost/person = \frac{(1+\varepsilon)}{K}$

$NE2$



$cost/person = \frac{K}{K} = 1$

$cost(NE2) = K$

$$PoA = \frac{K}{1+\varepsilon} \sim K$$

★ PoS = Price-of-Stability

$$= \frac{\text{Best NE Cost}}{\text{OPT Cost}}$$

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