N/w Overprovisioning (Case Study) & Atomic Routing Games

Tuesday, October 10, 2023 9:50 AM

Case Study: within ISPs $\rightarrow$ n/w over provisioning.

Provide "more capacity than needed"

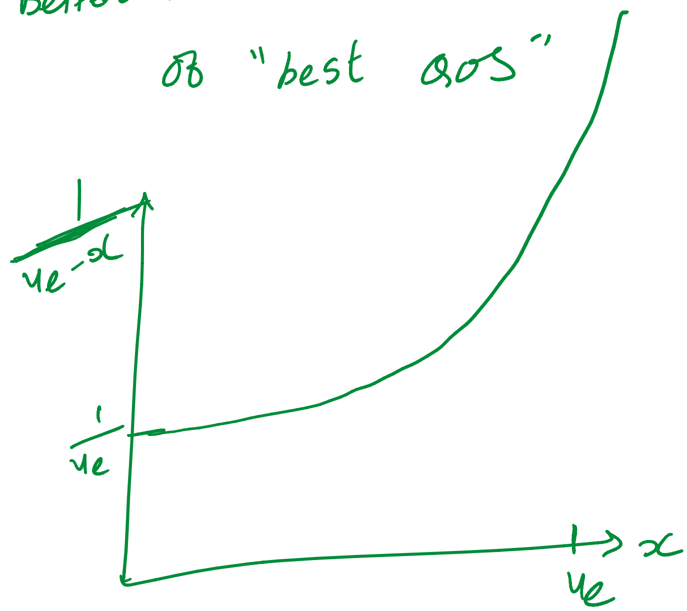
↳ Reduces the overall delays

↳ Better than design & implementation of "best QoS"

$$c_e(x) = \frac{1}{u_e - x} \quad \text{if } x < u_e$$

$$= \infty \quad \text{if } x = u_e.$$

u_e ↑ capacity of the edge e.



✲ γ- over provisioned:

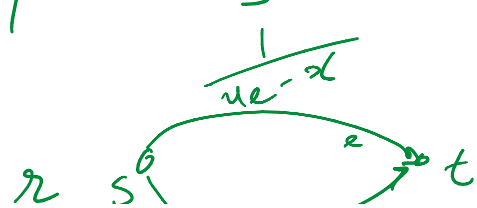
if at "NE" $f_e \leq (1-\gamma)u_e$

(γ fraction of capacity is unused).

$$\boxed{\text{Recall: } PoA(G, \mathcal{C}) \leq \alpha(\mathcal{C}) = PoA(\text{Pigou}, \mathcal{C}).}$$

$$\mathcal{C} = \left\{ \frac{1}{u_e - x} \;\middle|\; u > 0 \right\}.$$

exe. ←

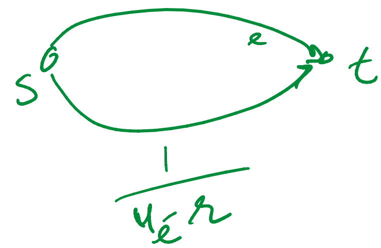


[illegible]

$PoA = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\gamma}}\right) \longleftarrow$ exe.

r

$= (1-\gamma)u_e$



if $\gamma = 1$ $\quad = 1$

if $\gamma = 0$ $\quad = \infty$

if $\gamma = 0.25$ $\quad = 3/2$

if $\gamma = 0.1$ $\quad \sim 2$.

Thm: Given $\mathcal{C}$, $G = (V, E)$, $\forall e$, $c_e \in \mathcal{C}$.

$s, t \in V$, $r \geq 0$ units flow.

f: NE flow $s \to t$ of r units $=$

f^*: OPT flow $s \to t$ of $2r$ units.

$\exists$ SP

NE w/ u_e capacity

OPT w/ $\frac{u_e}{2}$ capacity

$$\text{cost}(f) \leq \text{cost}(f^*)$$

$\mathcal{P}$: set of $s \to t$ paths

pf: $L = \min_{p \in \mathcal{P}} c_p(f) \Rightarrow L \leq c_p(f) \quad \forall p \in \mathcal{P}$.

f is NE: $\quad f_p > 0 \Rightarrow c_p(f) = L$.

$$\text{cost}(f) = \sum_{p \in \mathcal{P}} f_p \cdot c_p(f) = L \underbrace{\left(\sum_{p \in \mathcal{P}} f_p\right)}_{= r} = L \cdot r.$$

$$\text{cost}\left(\begin{array}{l} f^* \text{ w/ cost on} \\ \text{each edge } e \\ \text{fixed to } c_e(f_e) \end{array}\right) = \sum_{p \in \mathcal{P}} f^*_p\, c_p(f) \geq L\left(\sum_{p \in \mathcal{P}} f^*_p\right) = L \cdot (2r)$$

ach edge e fixed to $c_e(f_e)$
OR
on each path P fixed to $c_P(f)$

$$- \sum_{P \in \mathcal{P}}$$

$$\text{cost}(f) = L \cdot r = 2Lr - Lr \quad \boxed{\le \sum_{P \in \mathcal{P}} f^*_P c_P(f) - \sum_{P \in \mathcal{P}} f_P c_P(f) \le \text{cost}(f^*) = \sum_{P \in \mathcal{P}} f^*_P c_P(f^*)}$$

TPT.

$$\text{TPT: } \sum_{P \in \mathcal{P}} f^*_P c_P(f) - \sum_{P \in \mathcal{P}} f_P c_P(f) \le \sum_{P \in \mathcal{P}} f^*_P c_P(f^*)$$

$$\iff \sum_P f^*_P c_P(f) - \sum_{P \in \mathcal{P}} f^*_P c_P(f^*) \le \sum_P f_P c_P(f)$$

$$\iff \sum_e \underline{f^*_e} \left(c_e(f_e) - c_e(f^*_e) \right) \le \sum_e f_e c_e(f_e)$$

$$\Leftarrow \forall e: \ f^*_e \left(c_e(f_e) - c_e(f^*_e) \right) \le f_e c_e(f_e)$$

c_e: non-neg, non-dec

$\Leftarrow$ case I: $0 \le f^*_e \le f_e$

$$0 \le c_e(f_e) - c_e(f^*_e) \le c_e(f_e)$$

$$\Rightarrow f^*_e \left(c_e(f_e) - c_e(f^*_e) \right) \le f_e c_e(f_e).$$

case II: $f^*_e > f_e \Rightarrow c_e(f_e) - c_e(f^*_e) \le 0$ ∎

☆ Atomic Routing Games:

$\to$ $G = (V, E)$ directed n/w.

$\to$ $N = \{1, \ldots, n\}$ agents / players.

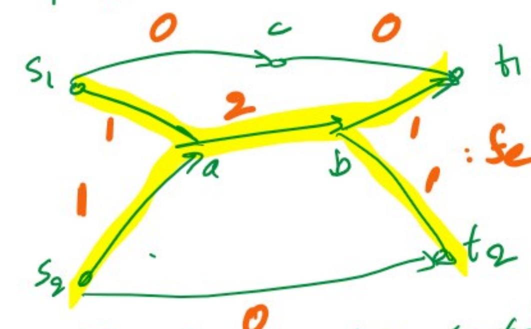
$\to$ $i \in N$, wants to go from s_i to t_i

$s_i, t_i \in V$

$\mathcal{P}_i$: set of $s_i \to t_i$ path.

$p_i \in \mathcal{P}_i$: pure strategy of player i

$\mathcal{P}_1 = \{s_1 - c - t_1, s_1 - a - b - t_1\}$



$\mathcal{P}_2 = \{s_2 - a - b - t_2, s_2 - t_2\}$

$p = (p_1, \ldots, p_n)$

$\forall e \in E$:

$c_e : \{0, 1, \ldots, n\} \to \mathbb{R}_+$ non-dec non-neg.

$f_e^p = f_e =$ # agents taking edge e in p.

$$c_i(p) = \sum_{e \in p_i} c_e(f_e) \; , \qquad \boxed{\text{cost}(p) = \sum_{i \in N} c_i(p)}$$

$$= \sum_{i \in N} \sum_{e \in p_i} c_e(f_e)$$

$$= \sum_{e \in E} c_e(f_e) \underbrace{\sum_{i \in N : e \in p_i} 1}_{f_e}$$

$$\boxed{\text{cost}(p) = \sum_{e \in E} f_e \cdot c_e(f_e)}$$

$$\boxed{p \text{ is NE iff } \forall i : \; c_i(p_i, p_{-i}) \leq c_i(q_i, p_{-i}) \; : \forall q_i \in \mathcal{P}_i}$$

$$\mathcal{C} = \{ ax+b \mid a,b \geq 0 \}$$ affine/linear functions.

Thm: $PoA(\text{affine}) \leq 5/2$

$(\forall e \in E, \quad c_e(x) = a_e x + b_e, \quad a_e, b_e \geq 0)$.

Pf: P is a NE $\longrightarrow$ f_e # agents taking e in P.

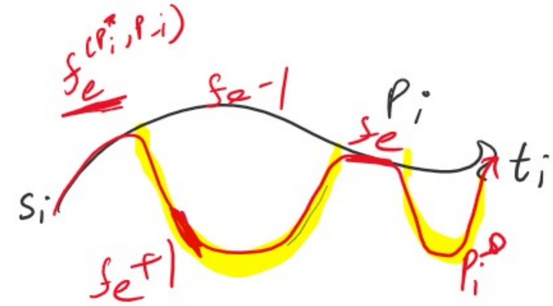
P^* is OPT $\qquad$ f_e^* : " " " P^*

$\boxed{\text{step-1:}}$ (trivial) $\quad$ P is NE: $\forall i \; C_i(P_i, P_{-i}) \leq C_i(P_i^*, P_{-i})$

$$\Rightarrow \boxed{cost(P) = \sum_i C_i(P_i, P_{-i}) \leq \sum_i C_i(P_i^*, P_{-i})} \rightarrow (1)$$

$$C_i(P_i^*, P_{-i}) = \sum_{e \in P_i^* \setminus P_i} c_e(f_e+1) + \sum_{e \in P_i^* \cap P_i} c_e(f_e)$$

$$\leq \sum_{e \in P_i^*} c_e(f_e+1)$$



$\boxed{\text{Step-2}}$ (trivial) $\quad$ $$\sum_i C_i(P_i^*, P_{-i}) \leq \sum_i \sum_{e \in P_i^*} c_e(f_e+1)$$ $= f_e^*$

$$\boxed{\text{step}} \quad \sum_i c(\dots) \qquad \sum_i \quad \sum_{e \in P_i}$$

(non-trivial)

$$= \sum_{e \in E} c_e(f_e+1) \underbrace{\sum_{\substack{i \in N: \\ e \in P_i^*}} 1}_{= f_e^*}$$

$(\because c_e(x) = a_e x + b_e)$

$$= \sum_{e \in E} f_e^* \left(a_e (f_e+1) + b_e\right)$$

$$= \sum_{e \in E} a_e \underbrace{f_e^* (f_e+1)}_{x = f_e^*,\ y = f_e} + \sum_{e \in E} b_e f_e^*$$

$$\left(\begin{array}{c} x, y \in \{0, 1, 2, \dots\} \\ x(y+1) \le \frac{5}{3} x^2 + \frac{1}{3} y^2 \end{array} \right) \quad \text{Magic!}$$

$$\le \sum_{e \in E} a_e \left(\frac{5}{3} f_e^{*2} + \frac{1}{3} f_e^2 \right) + \frac{5}{3} \sum_{e \in E} b_e f_e^*$$

$$+ \frac{1}{3} \sum_{e \in E} b_e f_e$$

$$= \frac{5}{3} \sum_{e \in E} \left(a_e f_e^{*2} + b_e f_e^* \right) + \frac{1}{3} \sum_{e \in E} \left(a_e f_e^2 + b_e f_e \right)$$

$$= \frac{5}{3} \sum_{e \in E} f_e^* \underbrace{\left(a_e f_e^* + b_e \right)} + \frac{1}{3} \sum_{e \in E} f_e \left(a_e f_e + b_e \right)$$

$$= \frac{5}{3} \sum_e f_e^* c_e(f_e^*) + \frac{1}{3} \sum_e f_e c_e(f_e)$$

$(\because c_e(x) =)$

$\left(\because c_e(x) = a_e x + b_e\right)$

$$= \frac{5}{3} \text{cost}(p^*) + \frac{1}{3} \text{cost}(p)$$

$$\boxed{\sum_{i \in N} C_i(p_i^*, p_{-i}) \le \underbrace{\frac{5}{3}}_{\lambda} \text{cost}(p^*) + \underbrace{\frac{1}{3}}_{\mu} \text{cost}(p)} \rightarrow (2)$$

Step-3

trivial

$$(1),(2) \Rightarrow \text{cost}(p) \overset{\because (1)}{\le} \sum_{i \in N} C_i(p_i^*, p_{-i}) \overset{\because (2)}{\le} \frac{5}{3} \text{cost}(p^*) + \frac{1}{3} \text{cost}(p)$$

$$\Rightarrow PoA = \frac{\text{cost}(p)}{\text{cost}(p^*)} \le \frac{\overset{\lambda}{5/3}}{(1 - \underset{\mu}{1/3})} = \frac{5}{2}$$

$$\frac{\lambda}{(1-\mu)}$$

★ (λ, μ) - Smooth Game if p : NE, $p^* \in \times_i \mathcal{P}_i$

$$\sum_i C_i(p_i^*, p_{-i}) \le \lambda \sum_i C_i(p^*) + \mu \sum_i C_i(p)$$

(step - 2)