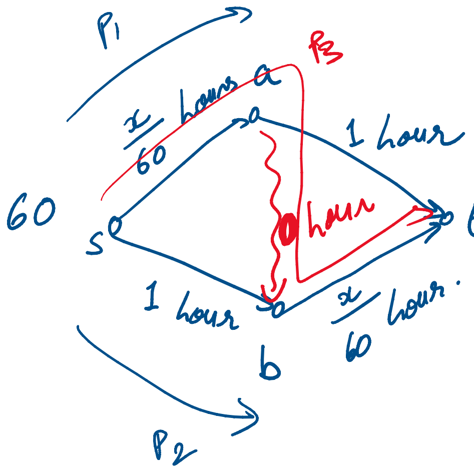
Routing Games: Non-atomic

Thursday, October 5, 2023 10:59 AM

Recall : Braess' Paradox



(# ppl on P_1, # ppl on P_2)

NE = (30, 30) = OPT.

cost (NE) = 1.5 x 60

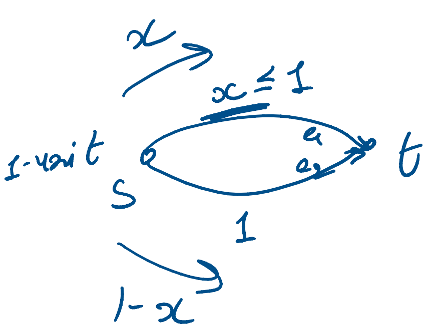
(# ppl on P_1, # ppl on P_2, # ppl on P_3)

OPT = (30, 30, 0) $\xrightarrow{\text{cost}}$ 1.5 x 60

NE = (0, 0, 60) $\xrightarrow{\text{cost}}$ 2 x 60

Price-of-Anarchy (PoA) = $\dfrac{\text{cost at worst NE}}{\text{OPT cost.}} = \dfrac{2 \times 60}{1.5 \times \cancel{60}} = \boxed{\dfrac{4}{3}}$

★ Pigou N/W.



(# units taking e_1, # units taking e_2)

NE = (1, 0)

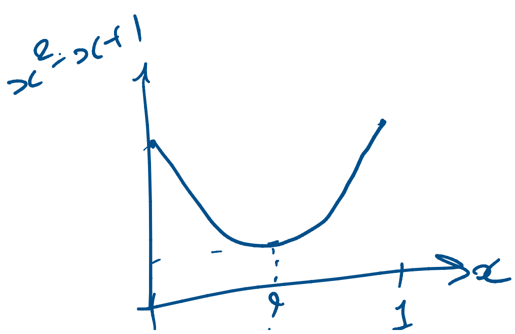
cost (NE) = # unit of flow$_{edge}$ * cost of edge

= 1 * 1 = 1

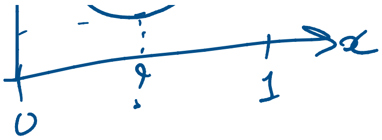
OPT = (x, 1-x) = (1/2, 1/2)

cost (OPT) = $\min_x$ x * x + ((1-x) * 1)

= $\min_x$ (x^2 - x + 1) = 1/4 - 1/2 + 1 = 3/4



$\frac{d}{dx}(x^2 - x + 1) = 2x - 1 = 0$



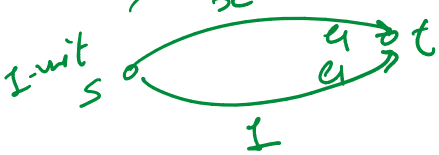
$$\frac{d}{dx}(x^2 - x + 1) = 2x - 1 = 0$$
$$\Rightarrow x = 1/2$$

$$PoA = \frac{\text{NE cost}}{\text{OPT cost}} = \frac{1}{3/4} = \boxed{\frac{4}{3}}\,!$$

★ $p \geq 1$

$x = 0.999999.$

$x^p \leq 1 \ (\because x \leq 1)$



$(1-x) = 0.0000001$

$$\begin{array}{cc} e_1 & e_2 \end{array}$$
$$NE = (1, 0)$$

$$\text{cost (NE)} = 1 * (1)^p = 1.$$

$$OPT = (x, 1-x)$$

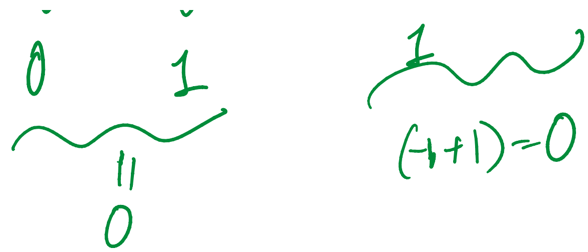
$$\text{s.t. } \text{cost(OPT)} = \min_x \left(x * x^p + (1-x) * 1 \right)$$
$$= \min_x \underline{x^{p+1} - x + 1}$$

$$\frac{d}{dx}\left(x^{p+1} - x + 1\right) = 0 \Rightarrow (p+1)x^p - 1 = 0$$
$$\Rightarrow x = \underline{\left(\frac{1}{p+1}\right)^{1/p}}.$$

$$\lim_{p \to \infty} \text{cost(OPT)} = \left(\frac{1}{p+1}\right)^{\frac{p+1}{p}} - \left(\frac{1}{p+1}\right)^{1/p} + 1$$

$$= \lim_{p \to \infty} \underbrace{\left(\frac{1}{p+1}\right)}_{\to 0} \underbrace{\left(\frac{1}{p+1}\right)^{1/p}}_{\to 1} - \underbrace{\left(\frac{1}{p+1}\right)^{1/p}}_{\to 1} + 1$$

$$0 \quad 1 \quad 1$$



$= 0$

$$\lim_{p\to\infty} PoA = \frac{1}{0} \Rightarrow \infty \,!$$

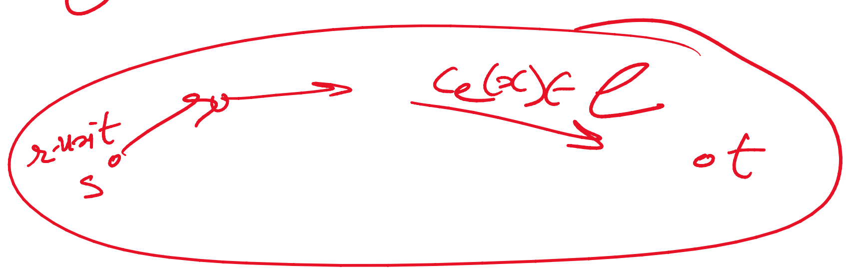
Conclusion: Degree of the cost func matter.

Q: Does n/w structure also matter.

Ans: NO!! Goal

Thm (Informal): Given $G = (V, E)$, $s, t \in V$, $r \geq 0$, $\forall e$, $c_e \in \mathcal{C}$.

$\mathcal{C}$: class of functions.

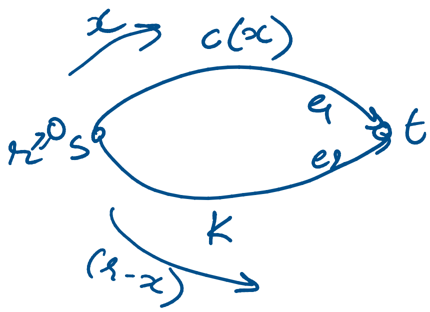


$$\text{PoA of } G \leq \underbrace{\text{PoA of Pigou n/w w/ costs from } \mathcal{C} \cup \{\text{constant func.}\}}_{\alpha(\mathcal{C})}$$

★ $\alpha(\mathcal{C})$:

$\star\ \alpha(\mathcal{C})$:

$c \in \mathcal{C}$ non-neg, non-deg.



$k = c(r)$

x : variable

everything else fixed.

$NE = (r, 0)$

$cost(NE) = r * c(r)$

$OPT = (x, r-x)$

$cost(OPT) = \inf_{0 \le x \le r} \underbrace{(x \cdot c(x)) + ((r-x)\cdot c(r))}_{\#}$

$$\boxed{\begin{aligned} \frac{d}{dx}(\#) &= c(x) + x\cdot c'(x) - c(r) \\ &= \underbrace{(c(x) - c(r))}_{\le 0} + \underbrace{x \cdot c'(x)}_{\ge 0} \\ &= 0 \Rightarrow c(x) \le c(r) \\ &\Rightarrow x \le r \end{aligned}}$$

$= \inf_{x \ge 0} \ \#$

$\alpha(\mathcal{C}) = PoA = \sup_{\substack{c \in \mathcal{C} \\ r \ge 0}} \ \sup_{x \ge 0} \dfrac{r * c(r)}{(x \cdot c(x)) + ((r-x)\cdot c(r))}$

of Pigou w/ cost from $\mathcal{C}$

Thm: Given $G = (V, E)$, $s, t \in V$, $r \ge 0$ units of flow $s \to t$.

Given $\mathcal{C}$: class of cost fun^c, $\forall e \in E$, $c_e \in \mathcal{C}$.

$$PoA(G, r, s \to t) \leq \alpha(\mathcal{C}).$$

★ **Definitions** :

Given directed.

→ n/w $G = (V, E)$

$s, t \in V$

→ r units of flow $s \rightsquigarrow t$

→ $\mathcal{C}$: class of cost func.

$c_e \in \mathcal{C}$, $\forall e \in E$.

→ f : valid $s \to t$ flow of r units.

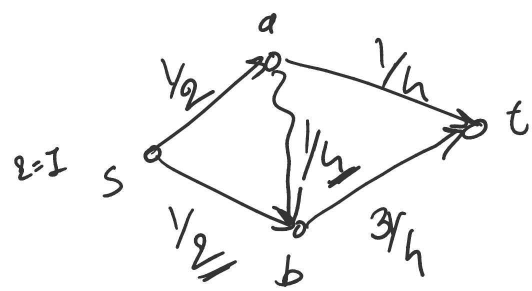
→ f_e : flow on edge e.

$c_e(f_e)$: per unit cost on edge e.

$$Cost(f) = \sum_{e \in E} f_e * c_e(f_e)$$

→ f_p : flow on path $p \in \mathcal{P}$

$$C_p(f) = \sum_{e \in P} c_e(f_e)$$: per unit cost on path P.



$f_{(s,a)} = 1/2$, $f_{(a,b)} = 1/4 = f_{(a,t)}$

$f_{(s,b)} = 1/2$, $f_{(b,t)} = 3/4$

$\mathcal{P} = \{ s\text{-}a\text{-}t,\ s\text{-}b\text{-}t,\ s\text{-}a\text{-}b\text{-}t \}$

$f_{s\text{-}a\text{-}t} = 1/4$, $f_{s\text{-}b\text{-}t} = 1/2$, $f_{s\text{-}a\text{-}b\text{-}t} = 1/4$

$\mathcal{P}$: set of all $s \to t$ paths in G.

$$\boxed{\text{NE: } \forall p \in \mathcal{P},\ f_p > 0 \Rightarrow c_p(f) \leq c_q(f) \quad \forall q \in \mathcal{P}.}$$

$\Rightarrow \sum f_e * c_e(f)$

$$\text{cost}(f) = \sum_{p \in \mathcal{P}} \underline{f_p} * \underline{c_p(f)}$$

$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$ (exe)

$$= \sum_{e \in E} f_e * c_e(f_e)$$

Pf: Let f: worst NE flow, f^*: OPT flow. $\Big\}$ 1 unit $s \to t$ TPT. $\text{PoA} \le \alpha(\mathcal{C})$

$$s \xrightarrow[p]{c_p(f)} t$$

claim 1: $\sum (f_e - f_e^*)\, c_e(f_e) \le 0$

Pf: f is NE $\Rightarrow$ $\displaystyle\sum_{p \in \mathcal{P}} f_p\, c_p(f) \underset{(\because \text{NE})}{=} \Big(\sum_{p \in \mathcal{P}} f_p\Big)\Big(\min_{p \in \mathcal{P}} c_p(f)\Big)$

$$= \Big(\sum_{p \in \mathcal{P}} f_p^*\Big)\underline{\Big(\min_{p \in \mathcal{P}} c_p(f)\Big)}$$

$$\le \sum_{p \in \mathcal{P}} f_p^*\, c_p(f)$$

$$\Rightarrow \sum_{e \in E} f_e\, c_e(f_e) \le \sum_{e \in E} f_e^*\, c_e(f_e)$$

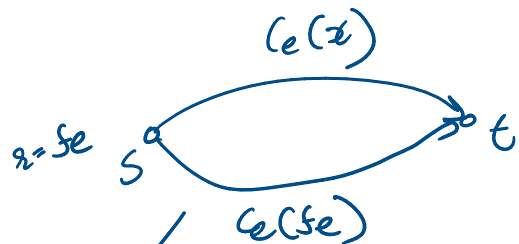
$$\Rightarrow \sum_{e \in E} (f_e - f_e^*)\, c_e(f_e) \le 0 \quad \blacksquare$$

T.P.T $\quad \text{PoA} = \dfrac{\sum_{e \in E} f_e\, c_e(f_e)}{\sum f_e^*\, c_e(f_e^*)} \le \alpha(\mathcal{C})$

$$\frac{\cdots}{\sum_{e\in E} f_e^* \, c_e(f_e^*)} \le \alpha(e)$$

$$\Leftarrow \quad \forall e\in E, \quad \frac{f_e * c_e(f_e)}{\alpha(e)} \le f_e^* \cdot c_e(f_e^*)$$

$\xrightarrow{\text{imagine}}$



$$\alpha(e) \ge PoA(\quad) = \sup_{x \ge 0} \frac{f_e * c_e(f_e)}{x \cdot c_e(x) + (f_e - x)\, c_e(f_e)}$$

$$\underset{x = f_e^*}{\ge} \frac{f_e * c_e(f_e)}{f_e^* \cdot c_e(f_e^*) + (f_e - f_e^*)\, c_e(f_e)}$$

$$\Downarrow$$

$$\forall e \in E, \quad \sum_{e\in E} \frac{f_e * c_e(f_e)}{\alpha(e)} \le \sum_{e\in E} f_e^* \cdot c_e(f_e^*) + \underline{\sum_{e\in E} (f_e - f_e^*)\, c_e(f_e)}$$

$$\le \sum_{e\in E} f_e^* \cdot c_e(f_e^*)$$

$$\Rightarrow PoA = \sum_{e\in E} f_e \cdot c_e(f_e) \cdots$$

$$\Rightarrow PoA = \frac{\sum_{e\in E} f_e \cdot c_e(f_e)}{\sum_{e\in E} f_e^* \cdot c_e(f_e^*)} \le \alpha(\mathcal{C})$$

G = Chicago n/w.

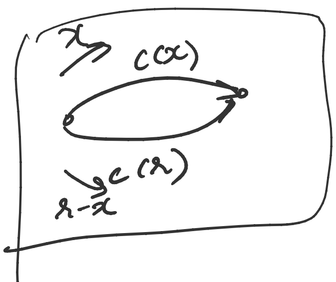
$\forall e: c_e \in \mathcal{C}$.

$$\mathcal{C} = \{ax+b \mid a,b \geq 0\}$$

$r = 100{,}000$ units of flow from s to t.

What is PoA?

$$PoA \le \alpha(\mathcal{C})$$



$$\alpha(\mathcal{C}) = \sup_{\substack{c\in\mathcal{C}\\ r\geq 0}} \sup_{x\geq 0} \frac{r\cdot c(r)}{x\cdot \underset{ax+b}{c(x)} + (r-x)\,c(r)}$$

$$= \sup_{\substack{a,b\geq 0\\ r\geq 0}} \boxed{\frac{r\cdot(ar+b)}{\inf_{x\geq 0}\; x\cdot(ax+b) + (r-x)(ar+b)}}$$

$$\frac{d}{dx}(\;) = (ax+\cancel{b}) + x\cdot a - ar - \cancel{b} = 0$$

$$2ax - ar = 0$$

$$\Rightarrow x = \frac{ar}{2a} = \frac{r}{2} = \arg\inf_{x\geq 0}$$

$$\Rightarrow x = \frac{[illegible]}{2\alpha} = \frac{[illegible]}{2} \quad x \geq 0$$

$$= \sup_{\substack{a,b \geq 0 \\ x \geq 0}} \frac{\cancel{x}\,(ax+b)}{\frac{\cancel{x}}{2}\left(a\frac{x}{2}+b\right)+\frac{\cancel{x}}{2}(ax+b)}$$

$$= \sup_{\substack{a,b \geq 0 \\ x \geq 0}} \frac{1}{\frac{1}{2}\times\frac{1}{2}\left(1+\frac{b}{ax+b}\right)+\frac{1}{2}}$$

→ minimized $b=0,\ a=1,\ x=1$

$$= \frac{1}{\frac{1}{4}(1+0)+\frac{1}{2}} = \frac{1}{\frac{1}{4}+\frac{1}{2}} = \frac{1}{3/4}$$

$$= \boxed{\frac{4}{3}}$$