Correlated Equilibrium – (CE)
(Aumann'74)

- **Mediator** declares a joint distribution P over $S=\times_i S_i$
- Tosses a coin, chooses $s = (s_1, \ldots, s_n) \sim P$.
- Suggests s_i to player i **in private**

- P is at equilibrium if each player wants to follow the suggestion when others do.
  - $U_i\left(s_i, P_{(s_i,.)}\right) \geq U_i\left(s_i', P_{(s_i,.)}\right), \forall s_i' \in S_1$

CE for 2-Player Case

- **Mediator** declares a joint distribution $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix}$
- Tosses a coin, chooses $(i, j) \sim P$. (w.p. p_{ij})
- Suggests i to Alice, j to Bob, in private.

- P is a CE if each player wants to follow the suggestion, when the other does.

Given Alice is suggested i, she knows Bob is suggested $j \sim P(i, .)$

$$\langle A(i, .), P(i, .)\rangle \geq \langle A(i', .), P(i, .)\rangle \quad : \forall i' \in S_1$$

$$\langle B(., j), P(., j)\rangle \geq \langle B(., j'), P(., j)\rangle \quad : \forall j' \in S_2$$

Players: {Alice, Bob}

Two options: {Football, Shopping}

	F	S
F	1 2 (0.5)	0 0 (0)
S	0 0 (0)	2 1 (0.5)

Instead they agree on ½(F, S), ½(S, F)

Payoffs are (1.5, 1.5)   Fair!

CE!

Prisoner's Dilemma

	C	NC
C	-5, -5 (1)	0, -6 (0)
NC	-6, 0 (0)	-1, -1 (0)

C strictly dominates NC

Rock-Paper-Scissors (Aumann)

	R	P	S
R	0, 0 (0)	0, 1 (1/6)	1, 0 (1/6)
P	1, 0 (1/6)	0, 0 (0)	0, 1 (1/6)
S	0, 1 (1/6)	1, 0 (1/6)	0, 0 (0)

When Alice is suggested R

Bob must be following $P_{(R,.)} \sim (0,1/6,1/6)$ 2/6

Following the suggestion gives her 1/6 / 2/6

While P gives 0, and S gives 1/6. / 2/6

Computation: Linear Feasibility Problem

Game (A, B). Find, joint distribution $P = \begin{bmatrix} p_{11} & & p_{1n} \\ \vdots & \cdots & \vdots \\ p_{m1} & & p_{mn} \end{bmatrix}$

$$\frac{1}{\sum_j p_{ij}} \sum_j A_{ij} p_{ij} \geq \frac{1}{\sum_j p_{ij}} \sum_j A_{i'j} p_{ij} \quad \forall i, i' \in S_1$$

$$\frac{1}{\sum_i p_{ij}} \sum_i B_{ij} p_{ij} \geq \frac{1}{\sum_i p_{ij}} \sum_i B_{ij'} p_{ij} \quad \forall j, j' \in S_2$$

$$\sum_{ij} p_{ij} = 1; \quad p_{ij} \geq 0, \ \forall (i, j)$$

Computation: Linear Feasibility Problem

Game (A, B). Find, joint distribution $P = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & \vdots & \vdots \\ p_{m1} & \cdots & p_{mn} \end{bmatrix}$

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$$\sum_i B_{ij} p_{ij} \geq \sum_i B_{ij'} p_{ij} \quad \forall j, j' \in S_2$$

$$\sum_{ij} p_{ij} = 1; \quad p_{ij} \geq 0, \ \forall (i, j)$$

N-player game: Find distribution P over $S = \times_{i=1}^{N} S_i$

s.t. $U_i\left(s_i, P_{(s_i,.)}\right) \geq U_i\left(s_i', P_{(s_i,.)}\right), \forall s_i, s_i' \in S_i$

$$\sum_{s \in S} P(s) = 1$$

↑

$$\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) P(s_i, s_{-i})$$

Linear in P variables!

Computation: Linear Feasibility Problem

N-player game: Find distribution P over $S = \times_{i=1}^{N} S_i$

s.t. $U_i\left(s_i, P_{(i,.)}\right) \geq U_i\left(s_i', P_{(s_i,.)}\right)$, $\forall s_i, s_i' \in S_i$

↑ $\sum_{s \in S} P(s) = 1$

$\sum_{s_{-i} \in S_{-i}} U_i(s_i, s_{-i}) P(s_i, s_{-i})$ Linear in P variables!

Can optimize any convex function as well!

Coarse-Correlated Equilibrium

- After mediator declares P, each player opts in or out.
- Mediator tosses a coin, and chooses s ~ P.
- If player i opted in, then the mediator suggests her s_i in private, and she has to obey.
- If she opted out, then (knowing nothing about s) plays a fixed strategy $t \in S_i$
- At equilibrium, each player wants to opt in, if others are opting in.

$$U_i(P) \geq U_i(t, P_{-i}), \ \forall t \in S_i$$

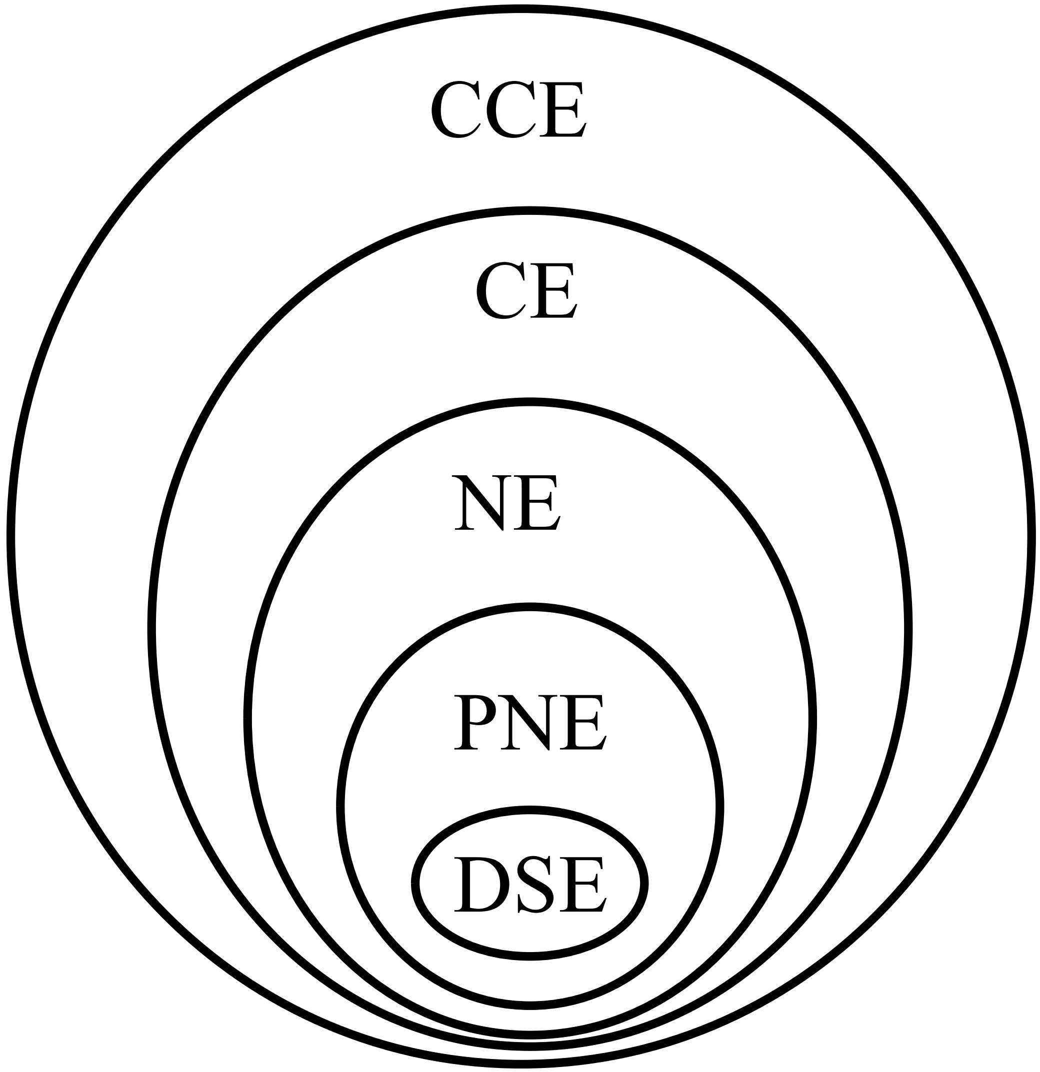
Where P_{-i} is joint distribution of all players except i.

Importance of (Coarse) CE

- Natural dynamics quickly arrive at approximation of such equilibria.
  - No-regret, Multiplicative Weight Update (MWU)

- Poly-time computable in the size of the game.
  - Can optimize a convex function too.

Show the following

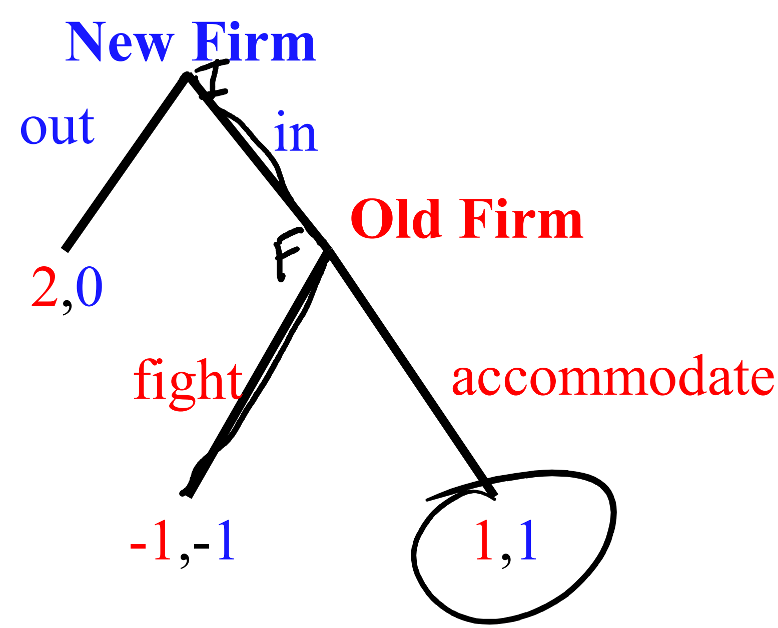


Extensive-form Game

- Players move one after another
  - Chess, Poker, etc.
  - Tree representation.

Strategy of a player:
What to play at each of its node.

	I	O
F	-1, -1	2, 0
A	1, 1	2, 0

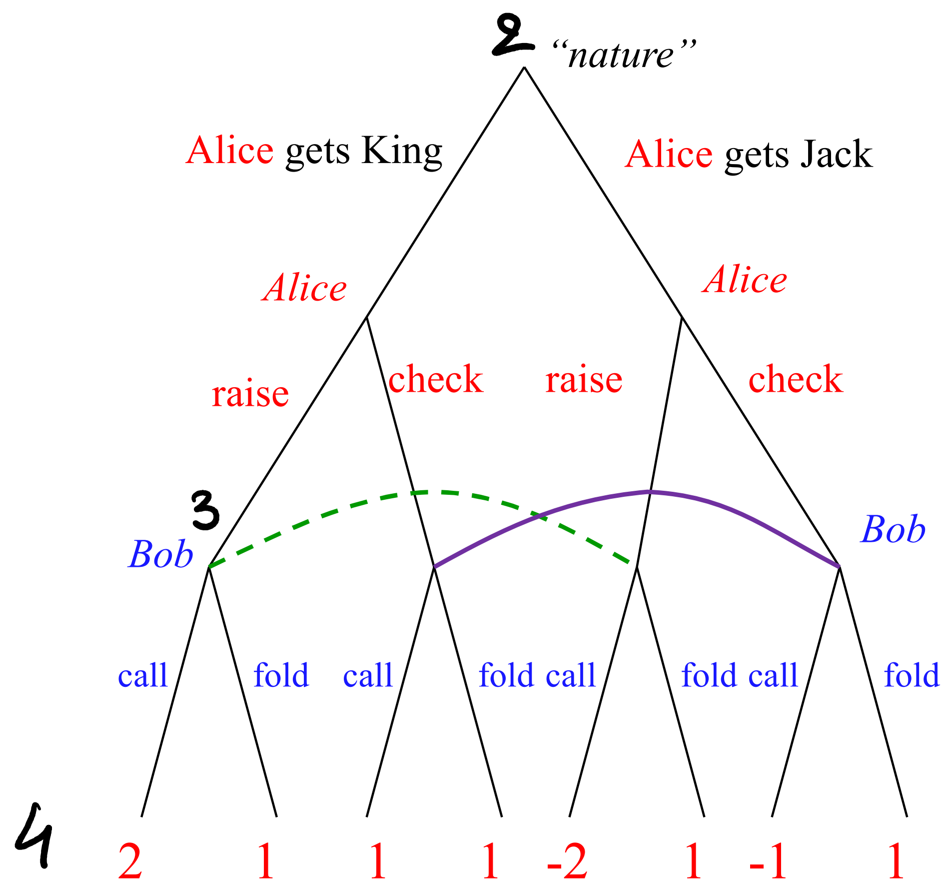


Entry game

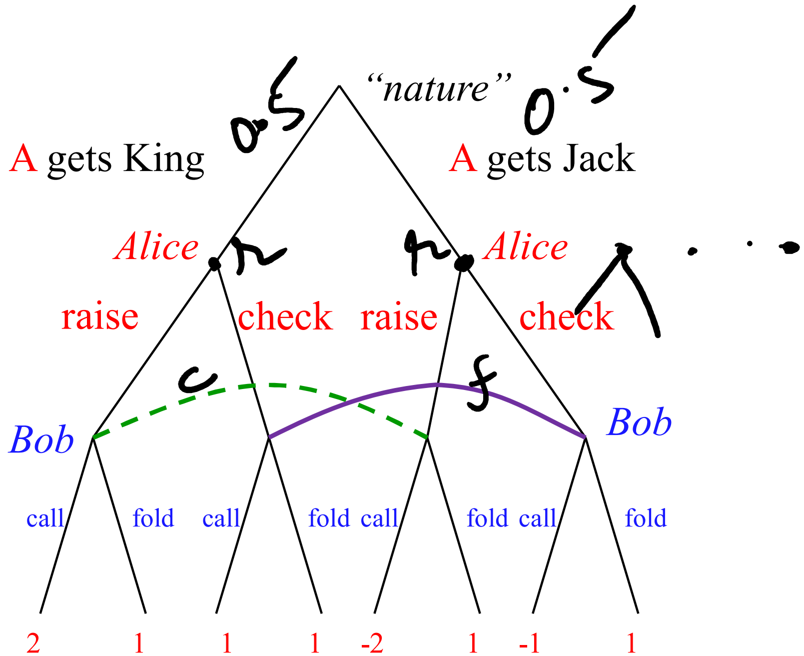
A poker-like game

Zero-sum Game.

- Both players put 1 chip in the pot
- Alice gets a card (King is a winning card, Jack a losing card)
- Alice decides to raise (add one to the pot) or check
- Bob decides to call (match) or fold (Alice wins)
- If Bob called, Alice's card determines pot winner



Poker-like game in normal form

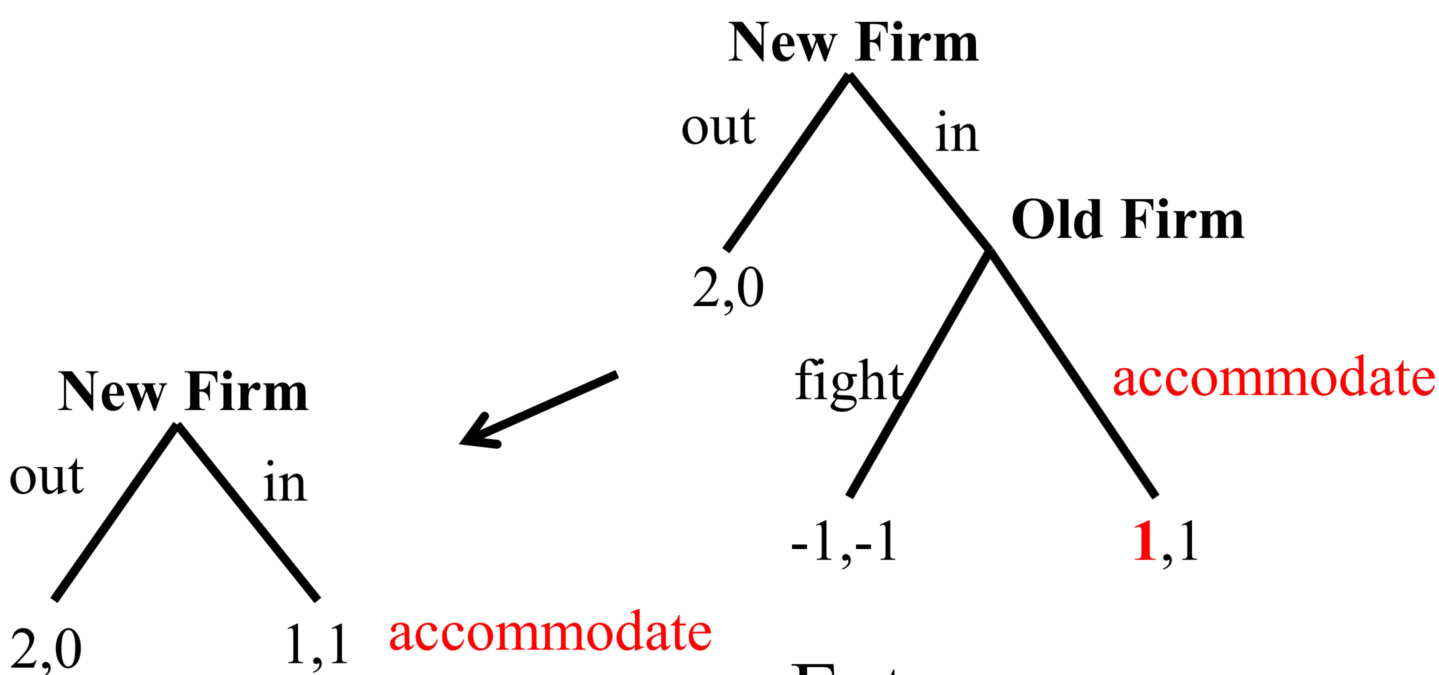


	cc	cf	fc	ff
rr	0, 0	0, 0	1, -1	1, -1
rc	.5, -.5	1.5, -1.5	0, 0	1, -1
cr	-.5, .5	-.5, .5	1, -1	1, -1
cc	0, 0	1, -1	0, 0	1, -1

Can be exponentially big!

Sub-Game Perfect Equilibrium

- Every sub-tree is at equilibrium
- Computation when perfect information (no nature/chance move): Backward induction



Entry game

Sub-Game Perfect Equilibrium

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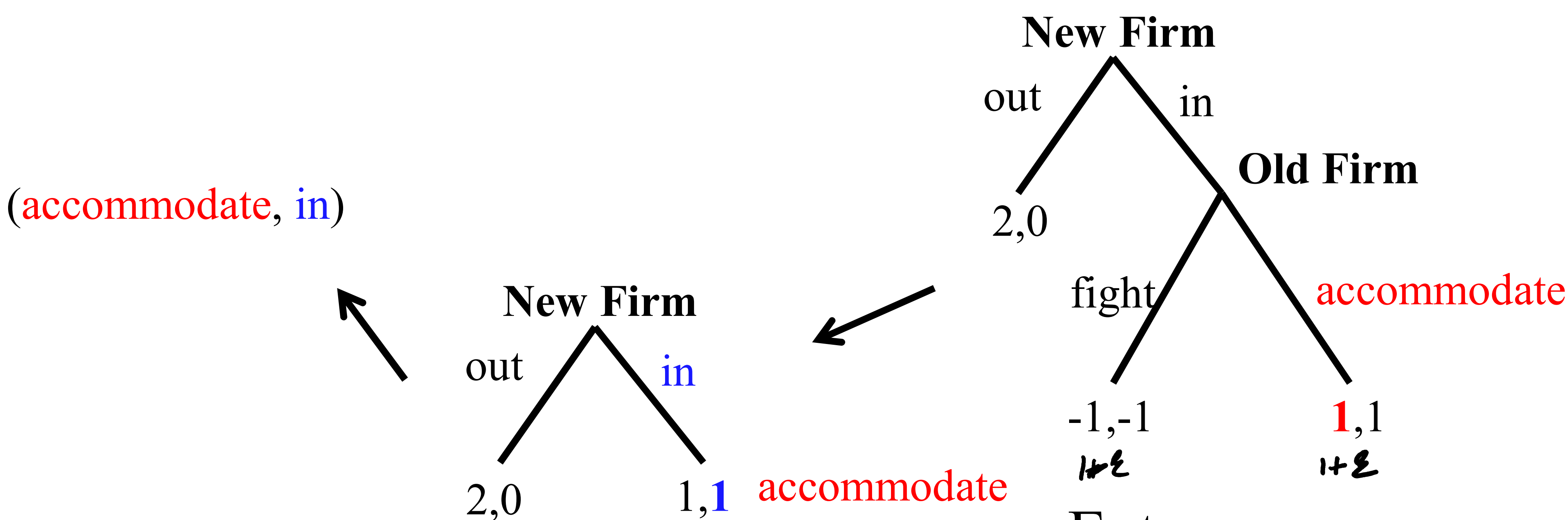


Entry game

Corr. Eq. in Extensive form Game

- How to define?
  - CE in its normal-form representation.
- Is it computable?
  - Recall: exponential blow up in size.
- Can there be other notions?

See "Extensive-Form Correlated Equilibrium: Definition and Computational Complexity" by von Stengel and Forges, 2008.

Commitment
(Stackelberg strategies)

Commitment

Unique Nash equilibrium (iterated strict dominance solution)

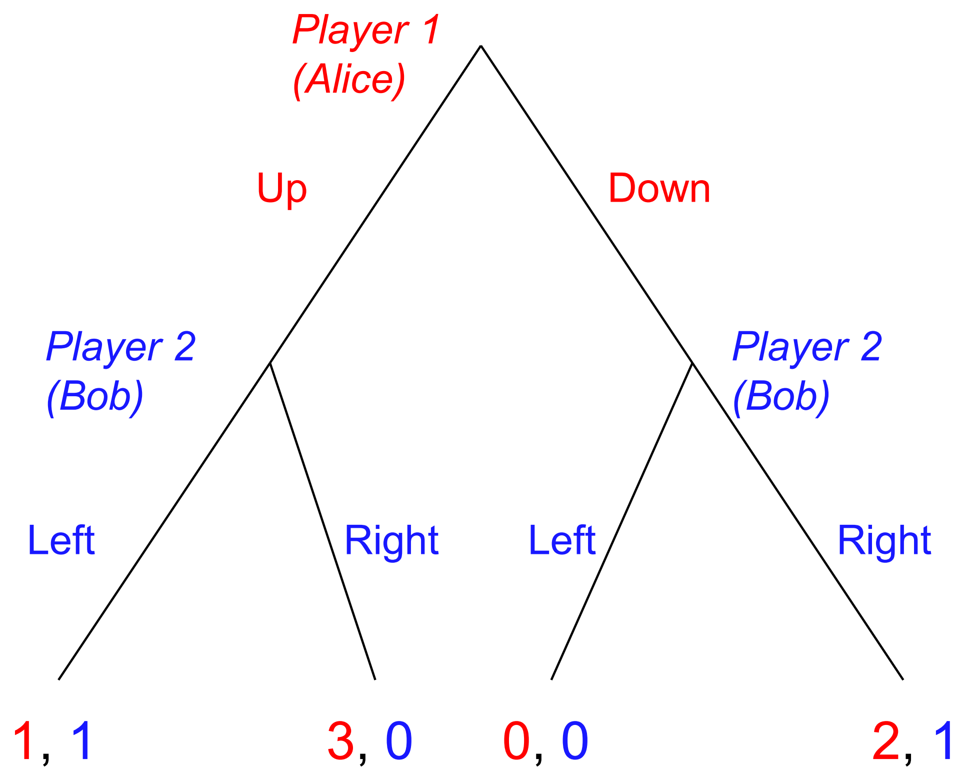
1, 1	3, 0
0, 0	2, 1

von Stackelberg

- Suppose the game is played as follows:
  - Alice **commits** to playing one of the rows,
  - Bob observes the commitment and then chooses a column
- Optimal strategy for Alice: commit to Down

Commitment: an extensive-form game

For the case of committing to a pure strategy:



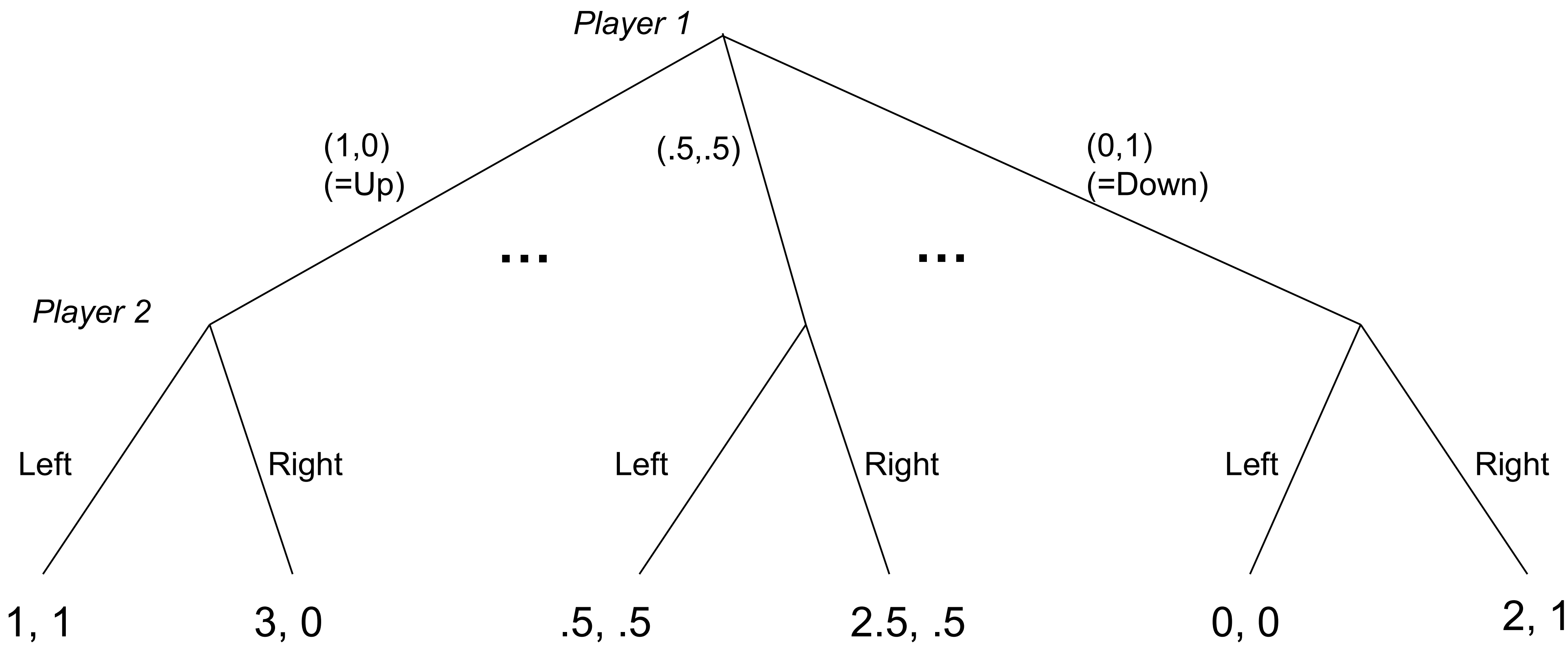
Commitment to mixed strategies

	0	1
.49	1, 1	3, 0
.51	0, 0	2, 1

Also called a Stackelberg (mixed) strategy

Commitment: an extensive-form game

- … for the case of committing to a mixed strategy:



- Economist: Just an extensive-form game, nothing new here
- Computer scientist: Infinite-size game! Representation matters

Computing the optimal mixed strategy to commit to

[Conitzer & Sandholm EC'06]

$\max_x$ ~~$\left(\arg\max_y x^T B y\right)$~~ $x^T A y$

j^*

- Player 1 (Alice) is a leader.
- Separate LP for every column $j^* \in S_2$:

maximize $\sum_i x_i A_{ij^*}$ — Alice's utility when Bob plays j^*

subject to $\forall j, \; \left(x^T B\right)_{j^*} \geq \left(x^T B\right)_j$ — Playing j^* is best for Bob

$x \geq 0, \; \sum_i x_i = 1$ — x is a probability distribution

Among soln. of all the LPs,
pick the one that gives max utility.

On the game we saw before

	L	R
x_1	1, 1	3, 0
x_2	0, 0	2, 1

$$\textit{maximize } \mathbf{1}x_1 + \mathbf{0}\, x_2$$

subject to

$$\mathbf{1}\, x_1 + \mathbf{0}\, x_2 \geq \mathbf{0}\, x_1 + \mathbf{1}\, x_2$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0,\, x_2 \geq 0$$

$$\textit{maximize } \mathbf{3}\, x_1 + \mathbf{2}\, x_2$$

subject to

$$\mathbf{0}\, x_1 + \mathbf{1}\, x_2 \geq \mathbf{1}\, x_1 + \mathbf{0}\, x_2$$

$$x_1 + x_2 = 1$$

$$x_1 \geq 0,\, x_2 \geq 0$$